Staircasing Errors in FDTD at an Air–Dielectric Interface

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***Abstract*— An analytical expression is derived for the reflection coefficient of a staircased air/dielectric interface. This expression for the reflection coefficient is then used to determine the attenuation and propagation constants of the wave induced by staircasing. It is demonstrated here that the errors due to staircasing increase as the relative dielectric permittivity is increased and converges to the results for an air–PEC interface.**

***Index Terms*— Air–dielectric interface, FDTD, staircasing errors.**

I. Introduction

THE finite-difference time-domain (FDTD) method for solving Maxwell's equations has become a very popular and widely used technique. The traditional FDTD algorithm is based on a Cartesian coordinate system. A simple and common approach to modeling arbitrary geometries that do not conform to a Cartesian grid is to use a staircased approximation of the curved surface. The issue of staircasing error has been addressed by many in the FDTD community [1]–[3], and several methods to overcome the inaccuracies through modified FDTD algorithms have been proposed (e.g., [4]–[8]). Railton and Schneider [3] compared some of these methods for the treatment of curved PEC surfaces. Cangellaris and Wright [2] presented a numerical approach for the analysis of the staircasing errors present at an air–PEC interface in two dimensions. The reflection coefficients for TM_z and TE_z polarizations were derived in order to find the characteristic equation for surface waves supported by the staircased boundary. In this letter, we present a generalization of the technique originally introduced in [2] that includes the ability to characterize the errors associated with a staircased boundary between free space and a lossless dielectric.

II. Analysis Method

The reflection coefficient for a TE_z plane wave being scattered by a lossless dielectric at the interface can be derived by starting with the FDTD equations for the electric and magnetic fields. For the purpose of this study, we will adopt the same geometry as that originally proposed by Cangellaris and Wright [2], where the boundary is tilted by 45°. The inclination of the interface introduces a new coordinates system, (x', y'), which is a translation of the (x, y) coordinate system by 45°. The discretized form of Maxwell's equation for the magnetic field at a location (l, m) which is in the center of the cell just to the left of the interface, is given by

$$\frac{1}{\delta}\left(H^{n+1}_{z(l,m)} - H^{n}_{z(l,m)}\right) = \frac{1}{\mu_o h}\left[E^{n+1/2}_{x(l,m+1/2)} - E^{n+1/2}_{x(l,m-1/2)} + E^{n+1/2}_{y(l-1/2,m)} - E^{n+1/2}_{y(l+1/2,m)}\right]. \quad (1)$$

The electric fields surrounding the magnetic field at (l, m) may be written in the form

$$E^{n+1/2}_{y(l-1/2,m)} = E^{n-1/2}_{y(l-1/2,m)} - \frac{\delta}{\varepsilon_o h}\left(H^{n}_{z(l,m)} - H^{n}_{z(l-1,m)}\right) \quad (2)$$

$$E^{n+1/2}_{y(l+1/2,m)} = E^{n-1/2}_{y(l+1/2,m)} - \frac{\delta}{\varepsilon_2 h}\left(H^{n}_{z(l+1,m)} - H^{n}_{z(l,m)}\right) \quad (3)$$

$$E^{n+1/2}_{x(l,m+1/2)} = E^{n-1/2}_{x(l,m+1/2)} - \frac{\delta}{\varepsilon_o h}\left(H^{n}_{z(l,m+1)} - H^{n}_{z(l,m)}\right) \quad (4)$$

$$E^{n+1/2}_{x(l,m-1/2)} = E^{n-1/2}_{x(l,m-1/2)} - \frac{\delta}{\varepsilon_2 h}\left(H^{n}_{z(l,m)} - H^{n}_{z(l,m-1)}\right) \quad (5)$$

where δ is the time step and h is the cell size. Equations (2) and (4) represent the electric fields in the dielectric material and differ from equations (1) and (3) by the relative dielectric permittivity ε_{r2}. Substituting (2)–(5) into the equation for the magnetic field (1) results in

$$\frac{\mu h^2}{\delta^2}\left(H^{n+1}_{z(l,m)} - 2H^{n}_{z(l,m)} + H^{n-1}_{z(l,m)}\right) = \frac{1}{\varepsilon_o}\left(H^{n}_{z(l,m+1)} - 2H^{n}_{z(l,m)} + H^{n}_{z(l-1,m)}\right) + \frac{1}{\varepsilon_2}\left(H^{n}_{z(l+1,m)} - 2H^{n}_{z(l,m)} + H^{n}_{z(l,m-1)}\right). \quad (6)$$

Next, we introduce the following expressions for the incident, reflected, and transmitted magnetic fields, for the interface at a 45° inclination

$$H^i = H_o \exp[-j(k_x(lh) + k_y(mh) - \omega(n\delta)] \quad (7)$$

$$H^r = RH_o \exp[-j(k_y(lh) + k_x(mh) - \omega(n\delta)] \quad (8)$$

$$H^t = TH_o \exp[-j(k_{x2}(lh) + k_{y2}(mh) - \omega(n\delta)] \quad (9)$$

where the total field in the free space region is represented by the sum of the incident and reflected field, given in (7) and (8), respectively.

The propagation constants in the dielectric, denoted by k_{x2} and k_{y2}, are related to the propagation constants in the free

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space region by

$$k_{x2} = \frac{k_x + k_y}{2} + \sqrt{\frac{k_x^2}{2} - \left(\frac{k_x + k_y}{2}\right)^2} \tag{10}$$

$$k_{y2} = \frac{k_x + k_y}{2} - \sqrt{\frac{k_2^2}{2} - \left(\frac{k_x + k_y}{2}\right)^2}. \tag{11}$$

The continuity of the tangential magnetic fields at the interface, $x = y$, leads to the phase matching condition

$$k_{x2} + k_{y2} = k_x + k_y. \tag{12}$$

The term, k_2, appearing in (10) and (11) is given by

$$k_2 = \omega\sqrt{\varepsilon_{r2}}\sqrt{\mu_o\varepsilon_o} = \sqrt{\varepsilon_{r2}}k_1 = \sqrt{\varepsilon_{r2}}\sqrt{k_{x'}^2 + k_{y'}^2} \tag{13}$$

which represents the propagation constant in the dielectric. The x- and y-components of the propagation constant in the (x', y') and (x, y) coordinate systems can be related to each other in the following way:

$$\begin{aligned} k_{x'} &= (k_x + k_y)/\sqrt{2}, \quad k_{y'} = (k_y - k_x)/\sqrt{2} \\ k_x &= (k_{y'} + k_{x'})/\sqrt{2}, \quad k_y = (k_{y'} - k_{x'})/\sqrt{2}. \end{aligned} \tag{14}$$

Following the notation introduced in [2], we assume $\alpha > 0$ and let

$$k_x = \beta - j\alpha, \qquad k_y = \beta + j\alpha. \tag{15}$$

Once the appropriate magnetic fields from (7), (8), and (9) are substituted into (6) and condition (12) is enforced, the following expression for the reflection coefficient results, shown in (16) at the bottom of the page, where

$$A_1 = 4\frac{\mu h^2}{\delta^2}\ \sin^2\left(\frac{\omega\delta}{2}\right). \tag{17}$$

In the limit as $h \to 0$ (i.e., for well-resolved waves), this expression reduces to

$$\lim_{h\to 0} R = \frac{1/\varepsilon_o(k_x - k_y) + 1/\varepsilon_2(k_{y2} - k_{x2})}{1/\varepsilon_o(k_y - k_x) + 1/\varepsilon_2(k_{y2} - k_{x2})} \tag{18}$$

which leads to the well-known result for parallel polarization given by

$$R = \frac{-\cos\theta_i + (1/\sqrt{\varepsilon_{r2}})\sqrt{1 - (1/\varepsilon_{r2})\sin^2\theta_i}}{\cos\theta_i + (1/\sqrt{\varepsilon_{r2}})\sqrt{1 - (1/\varepsilon_{r2})\sin^2\theta_i}}. \tag{19}$$

Finally, by substituting (10) and (11) into (16) and expressing the result in terms of α and β yields

$$R = -\left[\frac{\left(A_1 + \left[\frac{1}{\varepsilon_o}\ (A_2e^{\alpha h} - 2) + \frac{1}{\varepsilon_2}\ (A_2A_3 - 2)\right]\right)}{\left(A_1 + \left[\frac{1}{\varepsilon_o}\ (A_2e^{-\alpha h} - 2) + \frac{1}{\varepsilon_2}\ (A_2A_3 - 2)\right]\right)}\right] \tag{20}$$

where

$$\begin{aligned} A_2 &= 2\cos(\beta h) \\ A_3 &= \exp(-j\sqrt{(\beta h)^2(\varepsilon_{r2} - 1) - \varepsilon_{r2}(\alpha h)^2}). \end{aligned} \tag{21}$$

At this point, we turn our attention to finding expressions for the propagation constant β and attenuation constant α associated with the surface wave supported by the staircased boundary. A surface wave is supported as the reflection coefficient, R, goes to infinity. Hence, setting the denominator of (20) to zero and rearranging terms results in

$$e^{-\alpha h} + \frac{1}{\varepsilon_{r2}}\ A_3 = \frac{\left(1 + \frac{1}{\varepsilon_{r2}}\right) - 2A_4}{A_2/2} \tag{22}$$

where

$$A_4 = \left(\frac{h}{c\delta}\right)^2 \sin^2\left(\frac{\omega\delta}{2}\right) \tag{23}$$

and c is the speed of light in free space. Since this equation contains two unknown quantities, α and β, a second equation is required in order to solve for these parameters. A numerical dispersion analysis of the wave equation propagating on the FDTD grid toward the boundary provides a second expression relating α and β, which is given by [2]

$$\cos(\beta h) = \frac{1 - A_4}{\cosh(\alpha h)}. \tag{24}$$

$$R = -\frac{A_1 + \frac{1}{\varepsilon_o}\ (e^{-jk_yh} + e^{jk_xh} - 2) + \frac{1}{\varepsilon_2}\ (e^{-jk_{x2}h} + e^{jk_{y2}h} - 2)}{A_1 + \frac{1}{\varepsilon_o}\ (e^{jk_yh} + e^{-jk_xh} - 2) + \frac{1}{\varepsilon_2}\ (e^{-jk_{x2}h} + e^{jk_{y2}h} - 2)} \tag{16}$$

$$\left[A_1 + \frac{1}{\varepsilon_o}\left(e^{-j\alpha h}\left(\frac{2(1 - A_4)}{\cosh(\alpha h)}\right) - 2\right) + \frac{1}{\varepsilon_2}\left(2\exp\left(-j\sqrt{\left(\cos^{-1}\left[\frac{A_4}{\cosh(\alpha h)}\right]\right)^2(\varepsilon_{r2} - 1) - \varepsilon_{r2}(\alpha h)^2} - 2\right)\right)\right] = 0 \tag{25}$$

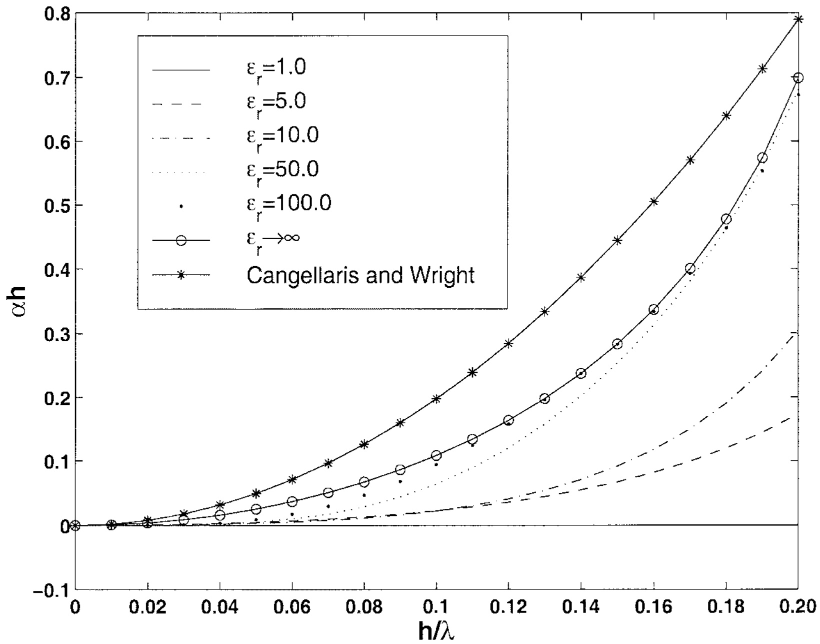

Fig. 1. Attenuation constant αh versus h/λ for various values of ε_{r2} obtained using (26). Plots are made for an assumed Courant number of 0.85.

Equation (24) can be used to eliminate the βh terms that appear in (22), without making any approximations for "well-resolved" waves. This gives rise to a transcendental equation that only depends on αh, shown in (25) at the bottom of the previous page. This equation can be further simplified by using the expression for the Courant number $C_n - \sqrt{2}c\delta/h$, so that

$$\left[A_1 + \frac{1}{\varepsilon_o}\left(e^{-j\alpha h}(2A_5) - 2\right) + \frac{1}{\varepsilon_2}(2\exp \cdot \left(-j\sqrt{(\cos^{-1}(A_5)^2(\varepsilon_{r2}-1) - \varepsilon_{r2}(\alpha h)^2} - 2\right)\right)\right] = 0 \tag{26}$$

where

$$A_5 = \frac{1 - \left(\frac{2}{C_n^2}\right)\sin^2\left(\frac{C_n \pi}{\sqrt{2}}\ \frac{h}{\lambda}\right)}{\cosh(\alpha h)}. \tag{27}$$

The parameter h/λ represents the size of the discretized cells in FDTD as a function of wavelength. The cell size for a conventional FDTD code should be one tenth of a wavelength (i.e., $h/\lambda = 0.1$) or less at the highest frequency of interest. The attenuation constant, αh, is plotted in Fig. 1 as a function of h/λ for various values of ε_{r2}. The case of a PEC from [2] is compared with the dielectric model for $\varepsilon_{r2} \to \infty$. There is a discrepancy between the two plots which is attributed to using different expressions for βh. For our analysis, the exact form for βh derived from (24) is used. On the other hand, Cangellaris and Wright make an approximation for βh in [2], which assumes "well-resolved" waves.

III. Results

The plots of αh as a function of h/λ shown in Fig. 1 contain information about the strength of the artificially induced surface wave on the interface and serve as a direct indication of the level of staircasing error present. These plots illustrate the fact that as the grid resolution decreases (i.e., h/λ increases), the effects of dispersion errors inherent in the difference scheme correspondingly increase. When the permittivity for the material to the right of the interface is set to that of free space (i.e., $\varepsilon_{r2} = 1$), we expect that the attenuation constant identifying the artificial surface wave should be zero, since there is no material discontinuity and the effects of staircasing should not be seen. On the other hand, as ε_{r2} approaches infinity, the values of αh for the surface wave at the air–dielectric interface asymptotically approach those of the air–PEC interface. The plots shown in Fig. 1 indicate that αh values for a surface wave existing on an air–dielectric interface are less than the values for an air–PEC interface, which suggests that the associated staircasing error will also be less. A similar type of analysis can be conducted on interfaces between more general materials.

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