An Efficient Series Expansion for the 2-D Green's Function of a Microstrip Substrate Using Perfectly Matched Layers

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***Abstract*— A new efficient technique is proposed to derive a series representation for the two-dimensional (2-D) Green's function of a planar substrate. A perfectly matched layer (PML) is used to turn the original open configuration into a closed one. The resulting structure is regarded as a waveguide and the resulting—analytically known—discrete set of eigenmodes is then used to expand the Green's function. The method turns out to be elegant and efficient for distances larger than 0.1λ away from the source.**

***Index Terms*— Absorbing boundary condition, Green's function, substrate, surface waves.**

I. Introduction

A typical monolithic microwave integrated circuit (MMIC) consists of a layered substrate and distributed and/or lumped components that interact through their interconnected ports. If such a structure is invariant in a given direction, such as a microsrip line, a two-dimensional (2-D) method of moments can be used to study the structure. The core of such a method of moments simulation is the 2-D Green's function of the structure. This 2-D Green's function represents the fields generated by a line source. In [1] it is shown that after spatial Fourier transformation of Maxwell's equations the layered structure can be transformed in an equivalent transmission line cascade which can be easily solved. The 2-D Green's function in the space domain is then found by a numerical inverse Fourier transformation of the solution of this cascade. In this paper a new technique to calculate the 2-D Green's function is proposed by placing a PML [2] above the air-region and expanding the Green's function in the eigenmodes of the resulting closed waveguiding structure.

II. PML Waveguides

In the context of this letter, a perfectly matched layer (PML) will be considered in its stretched coordinate formalism [3], [4]. In [3] and [4] it is shown how the analytic continuation of the frequency-domain Maxwell's equations to complex space achieves the reflectionless absorption of the electromagnetic waves. In Cartesian coordinates (Fig. 1) a PML is formed by using a stretched $\tilde{z}$-coordinate which obeys

$$\tilde{z} = \int^{z} \alpha(z')\, dz' \quad (1)$$

with $\alpha(z) = 1 + (\kappa_0 - 1)f(z) - j(\sigma_0/\omega\epsilon_0)f(z)$ (κ_0, σ_0 and $f(z)$ being the parameters of the PML) while leaving the tangential x- and y-coordinates unchanged. This leads to the conclusion that a layer of air with a PML on top of it can be seen as one layer of air with a complex thickness $\tilde{d}_{air} = d_{air} + \tilde{d}_{PML}$ where $\tilde{d}_{PML} = \int_0^{d_{PML}} \alpha(z')\, dz'$ and where $z' = 0$ is taken on the interface between the air and the PML [5]. In this letter $f(z)$ will be equal to one.

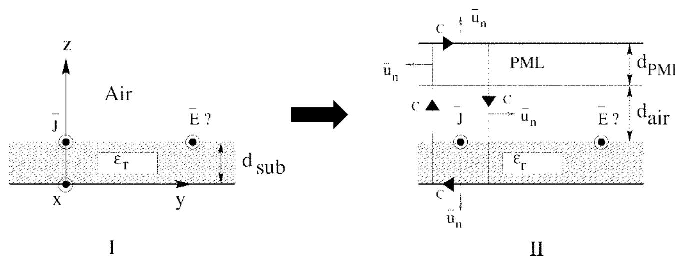

Fig. 1. Configuration without (I) and with (II) the PML. The structure is invariant in the x-direction.

Consider a grounded dielectric substrate with thickness $d_{\rm sub}$ and relative permittivity ϵ_r [Fig. 1(I)]. Above the substrate there is a layer of air with thickness d_{air}. On top of the air we have a PML with thickness d_{PML} [Fig. 1(II)]. We look for the TE-polarized eigenmodes (thus having only E_x, H_y and H_z-components) of this structure which propagate along the y-axis and hence obey an $e^{-jk_y y}$-dependence. As we already stated, the layer of air and the PML can be seen as one layer of air with a complex thickness. Making use of this, it is straightforward to deduce the dispersion relation for TE-polarized eigenmodes [5]

$$\gamma_{\rm sub} \cot(\gamma_{\rm sub} d_{\rm sub}) = -\gamma_{air} \cot(\gamma_{air} \tilde{d}_{air}) \quad (2)$$

where $\gamma_{\rm sub} = \sqrt{\omega^2 \epsilon_r \epsilon_0 \mu_0 - k_y^2}$ and $\gamma_{air} = \sqrt{\omega^2 \epsilon_0 \mu_0 - k_y^2}$. In Fig. 2 the solutions for this dispersion relation are plotted for a configuration with the following parameters: $d_{PML} =$ 3.5 mm, $d_{air} = 5$ mm, $d_{\rm sub} = 9$ mm, $\epsilon_r = 3$, $f = 12$ GHz, $(\sigma_0/\omega\epsilon_0) = 8$ and $\kappa_0 = 10$. The eigenmodes in the waveguide of Fig. 1(II) can be grouped into three categories [5]–[7]. The first category of eigenmodes are propagating modes. These eigenmodes have a real propagation constant. These modes

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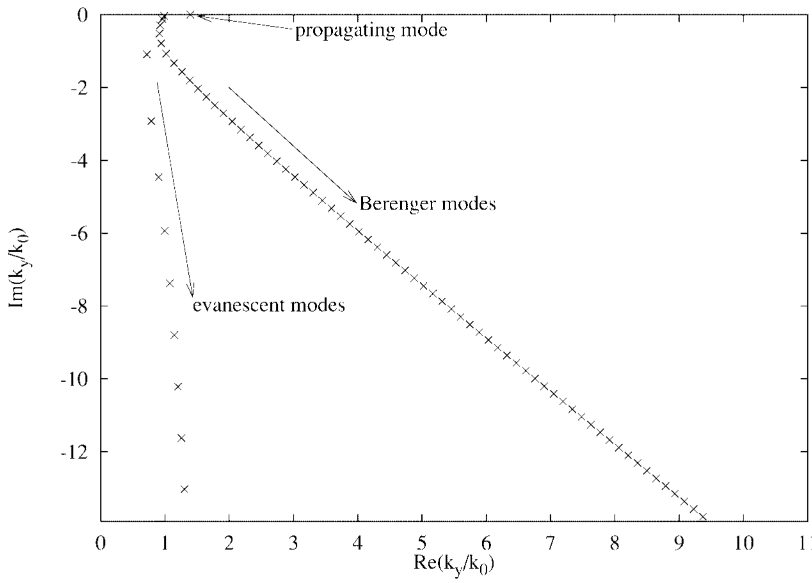

Fig. 2. Normalized propagation constants for the waveguiding structure depicted in Fig. 1 (II).

also exist in the original open waveguide without the PML layers [Fig. 1(I)]. The propagation constants of these modes are the same (with a relative difference $<10^{-7}$) for both the waveguide with and without the PML's. We call the second category of eigenmodes Berenger eigenmodes. For these modes the field is mainly concentrated in the PML. For higher order modes more of the field is concentrated in the PML. The last category of eigenmodes can be labeled as evanescent eigenmodes. They behave as evanescent eigenmodes in a closed waveguide. These modes have substantial fields inside the substrate. When the frequency increases these modes will turn into propagating eigenmodes with real propagation constants.

III. Green's Function

On the interface between the substrate and the air we place a Dirac line current $\overline{J} = \delta(z - d_{\rm sub})\delta(y)\overline{u}_x$ (Fig. 1). The aim is to calculate the E_x-component of the electrical field on the interface. One can easily prove that E_y and E_z equal zero for this problem. We will do this in two ways. For the configuration of Fig. 1(I) this can be done by Fourier transforming Maxwell's equations along the y-direction for the given structure. Solving the equations in the spectral domain yields $E(k_y, z = d_{\rm sub})$. $E_x(y, z = d_{\rm sub})$ can then be found by a numerical inverse Fourier transformation. This yields the exact 2-D G_{xx}-component of the Green's function for the electrical field on the interface. It is straightforward to show that $E_x(y, z = d_{\rm sub})$ is given by

$$\begin{aligned} E_x(y, z = d_{\rm sub}) &= -\frac{j\omega\mu_0}{2\pi} \int_{-\infty}^{\infty} \frac{\tan(\gamma_{\rm sub} d_{\rm sub})}{\gamma_{\rm sub} + j\gamma_{air}\tan(\gamma_{\rm sub} d_{\rm sub})} \\ &\quad \cdot e^{-jk_y y}\, dk_y. \end{aligned} \tag{3}$$

A second and new way to calculate $E_x(y, z = d_{\rm sub})$ is to use an expansion in the TE-eigenmodes of the modified configuration [Fig. 1(II)]

$$E_x(y, z = d_{\rm sub}) = \sum_{n=1}^{N} K_n E_{xn}(z = d_{\rm sub}) e^{-jk_y y} \tag{4}$$

where K_n are the expansion coefficients and E_{xn} the modal fields. Details on E_{xn} will be given in [5]. The method to determine E_{xn} is found in [8]. The TM-polarized eigenmodes must not be taken into account as their E_x-component is identically zero and hence they do not contribute to the E_x-field on the substrate–air interface. We apply the Lorentz reciprocity theorem [9]

$$\begin{aligned} &\oint_C (\overline{E}_a \times \overline{H}_b - \overline{E}_b \times \overline{H}_a) \cdot \overline{u}_n\, dl \\ &\quad = \iint_S (\overline{J}_a \cdot \overline{E}_b - \overline{J}_b \cdot \overline{E}_a)\, dS \end{aligned} \tag{5}$$

to the contour depicted on Fig. 1(II). As a field we choose the fields generated by the line source $\overline{J}_a = \delta(z - d_{\rm sub})\delta(y)\overline{u}_x$ and as b field the field of an eigenmode with index $n = i$. The eigenmodes are normalized such that

$$\tfrac{1}{2} \int_0^d (\overline{E}_i \times \overline{H}_i) \cdot \overline{u}_y\, dz = 1 \tag{6}$$

where $d = d_{\rm sub} + d_{air} + d_{PML}$ is the total thickness of the resulting closed waveguiding structure. In this way the expansion coefficients are found to obey the following simple and elegant formula

$$K_n = -\frac{E_{xn}(z = d_{\rm sub})}{4} \tag{7}$$

IV. Results

We apply the two methods described in the previous section to the following configuration: $d_{\rm sub} = 9$ mm, $\epsilon_r = 3$, and $f = 12$ GHz ($\lambda = 25$ mm). At this frequency the substrate only supports one surface wave with $n_{eff} =$

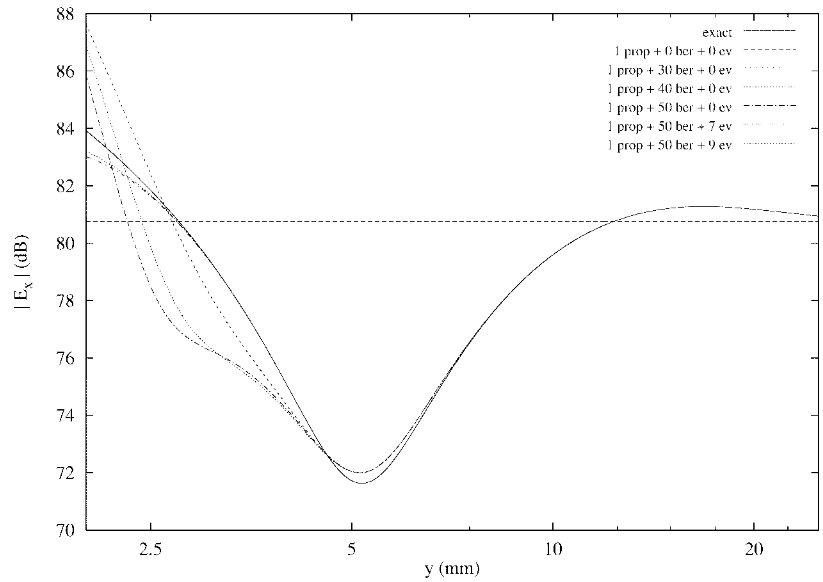

Fig. 3. Magnitude for $E_x(y, z = d_{\rm sub})$ as a function of y; $\rm prop$ stands for the propagating mode, $\rm ber$ for the number of Berengermodes, and $\rm ev$ for the number of evanescent modes.

TABLE I
NUMBER OF MODES AS A FUNCTION OF DISTANCE FOR AN ACCURACY OF 0.1%

y (mm)	$N_{\rm evan}$	$N_{\rm ber}$	$N_{\rm tot}$
3	7	60	67
5	1	50	51
10	0	20	20
20	0	10	10
30	0	3	3
40	0	1	1

$(k_y/k_0) = 1.393\,79$. In Fig. 3 we plotted the magnitude of $E_x(y, z = d_{\rm sub})$ as a function of y on a logaritmic scale (the source is located at $y = 0$). The full line represents $E_x(y, z = d_{\rm sub})$ for the original open structure. The other curves are the results of our PML-waveguide expansion technique. Our numerical results are obtained for $d_{air} = d_{PML} = (d_{\rm sub}/2)$. Similar results are obtained for the phase of $E_x(y, z = d_{\rm sub})$. However, numerous numerical experiments prove that he precise values of d_{air} and d_{PML} are not critical with respect to the final results. Given the set of propagation constants, E_x can be calculated everywhere almost instantaneously. In the expansion, the three types of eigenmodes described in Section II can be used. It is observed that for $y > 8$ mm only the single propagating and the Berenger modes must be taken into account to calculate $E_x(y, z = d_{\rm sub})$. This is of course not surprising because the propagation constants of the evanescent modes have a large imaginary part and hence will not contribute for large y. For $y < 8$ mm there is a visible difference between the exact and the series expansion result when no evanescent modes are used in the expansion. When one also takes these modes into account a very good approximation is obtained for $y > 2$ mm (a relative error smaller than 0.01%). In Table I we show, for a given y, how many Berenger and evanescent modes one has to use to obtain an approximation for $E_x(y, z = d_{\rm sub})$ with a relative error smaller than 0.1%.

V. CONCLUSIONS

A new method is proposed to calculate the 2-D Green's function for a planar grounded substrate. We place a PEC-backed PML on top of the structure thus turning the original open structure into a closed one. The discrete set of eigenmodes (propagating, Berenger- and evanescent modes) is then used to derive a series expansion for the Green's function. It turns out that we recover both the magnitude and the phase of the exact Green's function with a relative error smaller than 0.01% for distances larger than 0.1λ away from the source. The generalization to multilayered substrates is easy to perform by using a scattering matrix technique to find the dispersion relation for the eigenmodes of the multilayered substrate backed by a PML.

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