# Multilevel Fast Multipole Algorithm for Analysis of Large-Scale Microstrip Structures

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***Abstract*— An efficient algorithm combining multilevel fast multipole method and the discrete complex image method is presented for analyzing large-scale microstrip structures. The resulting algorithm has the memory requirement and the CPU time per iteration proportional to $O(N \log N)$, where $N$ denotes the number of unknowns. Numerical results for microstrip antennas are presented to demonstrate the efficiency and accuracy of this method.**

***Index Terms*—Green's function, fast multipole, method of moments, microstrip, multilevel algorithm.**

## I. Introduction

THE method of moments (MoM) has been widely used for the analysis of microstrip structures, such as microstrip antennas, microwave integrated circuits, and microstrip interconnects. To simulate large-scale complex microstrip structures, it is often necessary to employ a large number of unknowns. For the conventional MoM, the memory requirement is proportional to $O(N^2)$. The computing time can also become very excessive because direct matrix inversion solvers require $O(N^3)$ floating-point operations. When an iterative solver is employed for solving the MoM matrix equation, the operation count is $O(N^2)$ per iteration because of the need to evaluate the matrix-vector multiplication. This operation count is too high for an efficient simulation.

To make the iterative method more efficient, it is necessary to speed up the matrix-vector multiplication. There are several techniques developed for this purpose, including the adaptive integral method (AIM) [1], the fast multipole method (FMM) [2]–[4], and the singular value decomposition-based algorithm [5]. Recently, effort has been made to extend these fast algorithms to microstrip problems. The AIM is successfully adapted in [6] with the aid of the discrete complex image method (DCIM) [7]. Extension of FMM is difficult because of its dependence on the Green's function. One approach is to combine FMM with DCIM [8]–[10]. In [8] and [9], which treat the static and two-dimenionsl (2-D) problems, the equivalent problem is set up by adding $N_c$ images at the corresponding complex coordinates, and therefore, represented by $N(N_c+1)$ basis functions. In the FMM implementation, the translation is different for different images. In [10], both the 2-D and three-dimensional (3-D) FMM's are employed because the surface-wave poles are extracted in DCIM, which makes the implementation complicated. The multilevel algorithm is not implemented in those analyzes. The other FMM approach is to express the Green's function in terms of a rapidly converging steepest descent integral and then to evaluate the Hankel function arising in the integrand by FMM [11]. This approach is good for thin-stratified media.

In this letter, the multilevel fast multipole algorithm (MLFMA) [2], [4] combined with DCIM is presented for efficient analysis of microstrip structures. Instead of being treated separately, the image sources are grouped with the original source. By the use of the multilevel algorithm, the complexity is reduced to $O(N \log N)$. The algorithm requires little extra computation compared with that applied to free-space problems. Numerical results for microstrip antennas are presented to demonstrate the efficiency and accuracy of this method.

## II. Formulations

Consider a general microstrip structure residing on an infinite substrate having relative permittivity $\epsilon_r$ and thickness $h$. The microstrips are in the $x$-$y$ plane and excited by an applied field $\mathbf{E}^a$. The induced current on the microstrips can be found by solving the well-known mixed potential integral equation (MPIE) [12]. First, the microstrips are divided into triangular elements and then the current is expended using triangular rooftop basis functions. Applying Galerkin's method results in a matrix equation

$$\overline{\mathbf{Z}} \cdot \mathbf{I} = \mathbf{V} \tag{1}$$

in which the impedance matrix $\overline{\mathbf{Z}}$ has the elements given by

$$Z_{ij} = j\omega \int_{T_i} \int_{T_j} \left[ \mathbf{f}_i(\mathbf{r}) \cdot \mathbf{f}_j(\mathbf{r}') G_a(\mathbf{r}, \mathbf{r}') - \frac{1}{\omega^2} \nabla \cdot \mathbf{f}_i(\mathbf{r}) \nabla' \cdot \mathbf{f}_j(\mathbf{r}') G_q(\mathbf{r}, \mathbf{r}') \right] d\mathbf{r}' \, d\mathbf{r} \tag{2}$$

where $\mathbf{f}_i$ and $\mathbf{f}_j$ represent the testing and basis function, respectively, $T_i$ and $T_j$ denote their supports, $G_a$ is the $xx$-component of the Green's function for vector potential, and $G_q$ is the Green's function for scalar potential. In general, both $G_a$ and $G_q$ can be expressed as an inverse Hankel transform of their spectral domain counterparts, which is commonly known as the Sommerfeld integral (SI). The analytical solution of the SI is generally not available, and the numerical integration is

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time consuming. This problem can be alleviated using DCIM [7], which yields closed-form expressions as

$$G(\mathbf{r},\mathbf{r}') = \sum_{p=0}^{N_c} a_p \frac{e^{-jkr_p}}{4\pi r_p}, \qquad r_p = |\mathbf{r} - (\mathbf{r}' + \hat{z}b_p)| \quad (3)$$

where $a_p$ and $b_p$ are the complex coefficients obtained from DCIM.

To use FMM, we first divide the entire structure into groups denoted by $G_m (m = 1, 2, \cdots, M)$. Letting $\mathbf{r}_i$ be the field point in group $G_m$ centered at $\mathbf{r}_m$, and $\mathbf{r}_j$ be the source point in group $G_{m'}$ centered at $\mathbf{r}_{m'}$, we have

$$\begin{aligned}\mathbf{r}_{ij} &= \mathbf{r}_i - (\mathbf{r}_j + \hat{z}b_p)\\ &= (\mathbf{r}_i - \mathbf{r}_m) + (\mathbf{r}_m - \mathbf{r}_{m'}) + (\mathbf{r}_{m'} - \mathbf{r}_j) - \hat{z}b_p\\ &= \mathbf{r}_{im} + \mathbf{r}_{mm'} - \mathbf{r}_{jm'} - \hat{z}b_p. \end{aligned} \quad (4)$$

Employing the addition theorem and the elementary identity [3], we can rewrite the Green's function in (3) as

$$\begin{aligned}G(\mathbf{r}_i,\mathbf{r}_j) \approx \frac{k}{j16\pi^2} &\oint \sum_{p=0}^{N_c} a_p e^{j\mathbf{k}\cdot\hat{z}b_p}\\ &\times e^{-j\mathbf{k}\cdot(\mathbf{r}_{im}-\mathbf{r}_{jm'})} T(\hat{r}_{mm'}\cdot\hat{k})\, d^2\hat{k} \end{aligned} \quad (5)$$

where

$$T(\hat{r}_{mm'}\cdot\hat{k}) = \sum_{l=0}^{L} (-j)^l (2l+1) h_l^{(2)}(kr_{mm'}) P_l(\hat{r}_{mm'}\cdot\hat{k}). \quad (6)$$

Substituting (5) into (2), we obtain

$$\begin{aligned}Z_{ij} = \frac{\omega k}{16\pi^2}\Bigg[&\oint S_a(\hat{k})\mathbf{U}_{im}(\hat{k})\cdot T(\hat{r}_{mm'},\hat{k})\mathbf{U}^*_{jm'}(\hat{k})\, d^2\hat{k}\\ &- \frac{1}{\omega^2}\oint S_q(\hat{k})V_{im}(\hat{k})T(\hat{r}_{mm'},\hat{k})V^*_{jm'}(\hat{k})\Bigg]\, d^2\hat{k} \end{aligned} \quad (7)$$

where

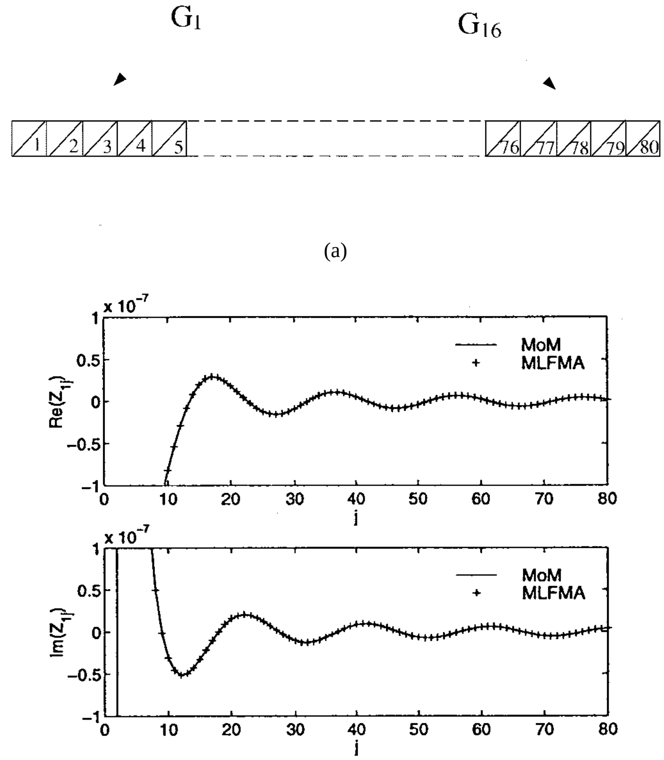
$$\mathbf{U}_{im}(\hat{k}) = \int_{T_i} e^{-j\mathbf{k}\cdot\mathbf{r}_{im}} \mathbf{f}_i \, d\mathbf{r} \quad (8)$$

$$V_{im}(\hat{k}) = \int_{T_i} e^{-j\mathbf{k}\cdot\mathbf{r}_{im}} \nabla\cdot\mathbf{f}_i d\mathbf{r} \quad (9)$$

$$S(\hat{k}) = \sum_{p=0}^{N_c} a_p e^{j\mathbf{k}\cdot\hat{z}b_p}. \quad (10)$$

When an iterative method is used to solve (1), the matrix-vector multiplication can be performed in such a way that the contributions from nearby groups are calculated directly and the far interactions are calculated using (7).

The concept of FMM can be extended to multilevel using MLFMA [4]. To implement MLFMA, the entire solution region is first enclosed in a large box, which is divided into four smaller boxes. Each subbox is then recursively subdivided into smaller boxes until the edge length of the finest box is less than half a wavelength. For two elements in the same or nearby finest boxs, their interaction is calculated in a direct manner. However, when the two elements reside in different nonnearby boxes, their interaction is calculated by FMM, as described above. The level of boxes on which FMM is applied depends on the distance between the two elements.




(b)

Fig. 1. Matrix element $Z_{1j}$ $(j = 1, 2, \cdots, 80)$ calculated by the conventional MoM and MLFMA.

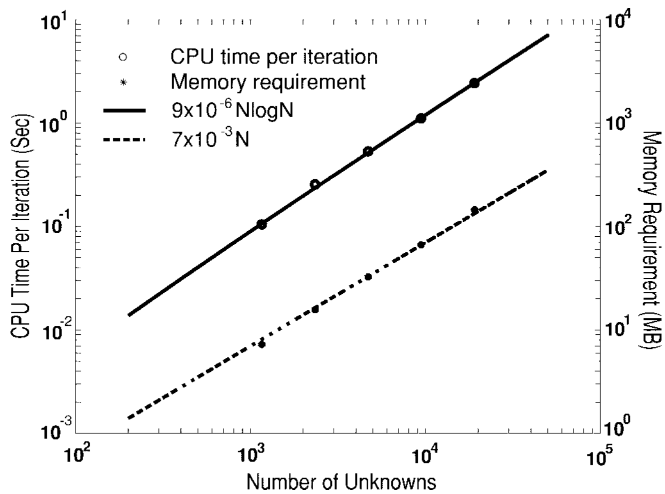
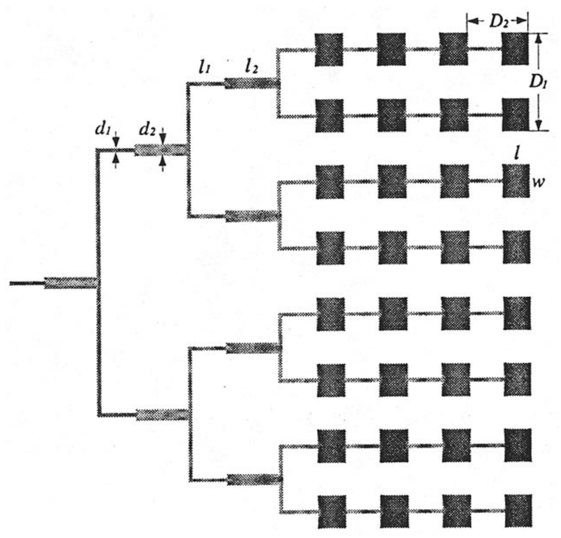


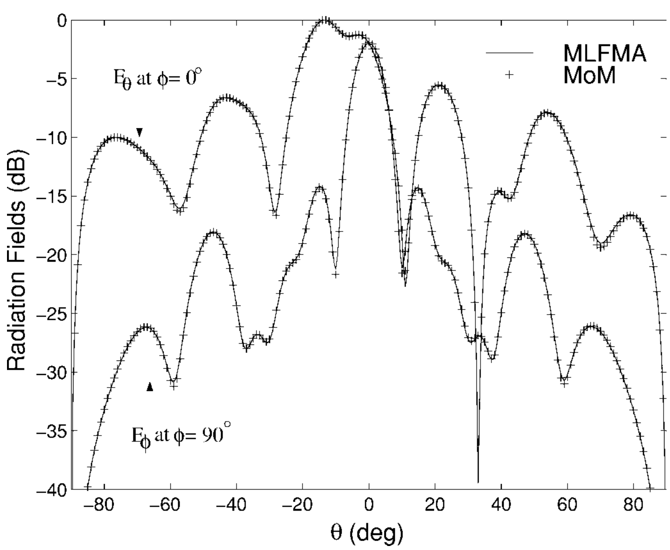

Fig. 2. Complexity of MLFMA. The CPU time per iteration is close to $9 \times 10^{-6} N \log N$ and the memory requirement is close to $7 \times 10^{-3} N$.

## III. Numerical Results

Before the proposed method is applied to realistic problems, the accuracy of this algorithm is examined. Consider a microstrip line on a substrate with relative permittivity $\epsilon_r = 2.17$ and thickness $h = 1.58$ mm. The frequency is 3.0 GHz. The line is 5 mm wide and 400 mm long. As shown in Fig. 1, the line is discretized into triangular elements with edge length 5 mm. Fig. 1 shows the values of matrix elements $Z_{1j} (j = 1, 2, \cdots, 80)$ obtained by using two different approaches. One approach is to use the original




(a)




(b)

Fig. 3. Current distribution and radiation patterns of the corporate-fed microstrip antenna array. $\epsilon_r = 2.2$, $h = 1.59$ mm, $l = 10.08$ mm, $w = 11.79$ mm, $d_1 = 1.3$ mm, $d_2 = 3.93$ mm, $l_1 = 12.32$ mm, $l_2 = 18.48$ mm, $D_1 = 23.58$ mm, $D_2 = 22.40$ mm, $f = 9.42$ GHz.

formulation (2). The other approach is to use MLFMA, where the group size $d$ is $0.25\lambda_0$ with $\lambda_0$ being the wavelength in free space. Note that $Z_{1j} (j = 1, 2, \cdots, 14)$ is considered as the near interaction and is calculated directly, and $Z_{1j}$ $(j = 15, 16, \cdots, 80)$ is considered as the far interaction and is calculated by MLFMA. As seen from Fig. 1, these two approaches agree well. In this calculation, the number of modes $L$ is chosen to be $kd + 3\ln(\pi + kd)$.

Next, the complexity of this algorithm is evaluated. The CPU time per iteration and the memory requirement versus the number of unknowns are plotted in Fig. 2 for solving the problem of a rectangular microstrip patch antenna. It is seen that the CPU time per iteration is scaled as $O(N \log N)$ and the memory requirement is scaled as $O(N)$.

Now consider a corporate-fed microstrip antenna array, which involves 6569 facets and 8668 unknowns. At frequency $f = 9.42$ GHz, the current distribution and the radiation patterns in the two principal planes are given in Fig. 3, which shows excellent agreement between this method and the conventional MoM. For the conventional MoM, the memory requirement is over 600 MB and the CPU time per iteration is 15.8 s. However, it takes only 36.3 MB and 3.0 s for the five-level MLFMA. The MLFMA also yields an over 70% reduction in the CPU time for the matrix fill comparing to the conventional MoM.

## IV. Conclusion

This letter presents a fast algorithm to deal with large-scale microstrip problems. The MLFMA originally developed for free-space problems is extended to microstrip problems with the aid of DCIM. The complexity of this algorithm is scaled as $O(N \log N)$. The efficiency and accuracy of this method is demonstrated by numerical results.

## References

[1] E. Bleszynski, M. Bleszynski, and T. Jaroszewicz, "AIM: Adaptive integral method for solving large-scale electromagnetic scattering and radiation problems," *Radio Sci.*, vol. 31, pp. 1225–1251, Sept./Oct. 1996.

[2] V. Rokhlin, "Rapid solution of integral equations of scattering in two dimensions," *J. Comput. Phys.*, vol. 86, pp. 414–439, Feb. 1990.

[3] R. Coifman, V. Rokhlin, and S. Wandzura, "The fast multipole method for the wave equation: A pedestrian prescription," *IEEE Antennas Propagat. Mag.*, vol. 35, pp. 7–12, June 1993.

[4] J. M. Song, C. C. Lu, and W. C. Chew, "Multilevel fast multipole algorithm for electromagnetic scattering by large complex objects," *IEEE Trans. Antennas Propagat.*, vol. 45, pp. 1488–1493, Oct. 1997.

[5] S. Kapur and D. E. Long, "IES$^3$: Efficient electrostatic and electromagnetic simulation," *IEEE J. Comput. Sci. Eng.*, pp. 60–67, Oct./Dec. 1998.

[6] F. Ling, C. F. Wang, and J. M. Jin, "An efficient algorithm for analyzing large-scale microstrip structures using adaptive integral method combined with discrete complex image method," in *IEEE APS Int. Symp. Dig.*, vol. 3, 1998, pp. 1778–1781.

[7] Y. L. Chow, J. J. Yang, D. G. Fang, and G. E. Howard, "A closed-form spatial Green's function for the thick microstrip substrate," *IEEE Trans. Microwave Theory Tech.*, vol. 39, pp. 588–592, Mar. 1991.

[8] V. Jandhyala, E. Michielssen, and R. Mittra, "Multipole-accelerated capacitance computation for 3-D structures in a stratified dielectric medium using in a closed form Green's function," *Int. J. Microw. Millim.-Wave Comput. Aided Eng.*, vol. 5, pp. 68–78, May 1995.

[9] L. Gurel and M. I. Aksun, "Electromagnetic scattering solution of conducting strips in layered media using the fast multipole method," *IEEE Microwave Guided Wave Lett.*, vol. 6, pp. 277–279, Aug. 1996.

[10] P. A. Macdonald and T. Itoh, "Fast simulation of microstrip structures using the fast multipole method," *Int. J. Numer. Modeling: Electron. Networks, Devices, Fields*, vol. 9, pp. 345–357, 1996.

[11] J. S. Zhao, W. C. Chew, C. C. Lu, E. Michielssen, and J. M. Song, "Thin-stratified medium fast-multipole algorithm for solving microstrip structures," *IEEE Trans. Microwave Theory Tech.*, vol. 46, pp. 395–403, Apr. 1998.

[12] J. R. Mosig, "Arbitrarily shaped microstrip structures and their analysis with a mixed potential integral equation," *IEEE Trans. Microwave Theory Tech.*, vol. 36, pp. 314–323, Feb. 1988.