Dimensional Analysis of Two-Layer Spherical Dielectric Resonator

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***Abstract*—Dimensional analysis is used to study the complex resonance of a two-layer spherical dielectric resonator (DR). In the analysis, the various dimensionless products, or Pi terms, of the system are found. From the Pi theorem, the Pi terms are related to one another by a function, which is found using the curve-fitting technique in this paper. For demonstration, the fundamental TE mode, the TE_{111} mode, of a spherical DR is investigated. Simple formulas for the resonant frequency and Q-factor are obtained. The results are compared with the exact solutions and excellent agreement is obtained.**

***Index Terms*—Dielectric resonators, nonhomogeneous media, resonance.**

I. Introduction

The dielectric resonator (DR) is one of the important elements in microwave engineering. It can be found in filter, oscillator, and even antenna [1]–[3] applications. The shape of a DR can be rectangular, cylindrical, or spherical, but only the last one admits an analytical solution. Moreover, at millimeter-wave frequencies where the DR dimensions are very small, the spherical DR is easier to produce than the others [4]. For these reasons, the spherical DR is studied in this paper.

In DR designs, it is useful to know the complex resonant frequency so that the (real) resonant frequency and Q-factor of the DR can be determined. In this paper, it is assumed that the dielectric is lossless and the finite Q-factor is only attributed to radiation loss. The complex resonances of spherical structures have been studied extensively, such as the single-layer dielectric sphere [5], shielded dielectric sphere [4], immersed dielectric sphere [6], and conducting sphere with lossy coatings [7]. For a two- or multi-layer sphere, however, the analysis has principally concentrated on the antenna application [3] and scattering problems [8]–[10]. In this paper, the complex resonance of a two-layer spherical DR is studied. Simple formulas are obtained by using the dimensional analysis [11], [12] and the curve-fitting technique. To illustrate the method, the fundamental TE mode, the TE_{111} mode, is considered. Nevertheless, the formulation is very general and can be applied to other DR modes.

Dimensional analysis has been used extensively in solid mechanic, fluid dynamic systems, and chemical engineering [11]. Recently, Mah *et al.* [12] have introduced this analysis to the microwave community by considering the design of a microstripline. In this paper, the dimensional analysis is used to analyze the complex resonance of a DR. Use is made of the Pi theorem to establish the dependencies between the dimensionless products found from the analysis. Based on the dependencies, the curve-fitting technique is used to obtain simple design formulas for the resonant frequency and Q-factor. The results can be easily reduced to those for the single-layer DR and should be useful to the DR design engineer.

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	Core Matrix x_p				Residual Matrix x_q			
$[F_1]$	a_{11}	a_{12}	$\cdots$	a_{1p}	b_{11}	b_{12}	$\cdots$	b_{1q}
$[F_2]$	a_{21}	a_{22}	$\cdots$	a_{2p}	b_{21}	b_{22}	$\cdots$	b_{2q}
$\vdots$		$\vdots$				$\vdots$		
$[F_m]$	a_{m1}	a_{m2}	$\cdots$	a_{mp}	b_{m1}	b_{m2}	$\cdots$	b_{mq}

Fig. 1. Dimensional matrix dividing the various parameters into two sets that are linearly independent (x_p) and dependent (x_q) of one another.

II. Theory

This section presents the procedure to obtain the various dimensionless products, which is also known as Pi terms. The form of the theory is somewhat different from that given in [12], though the principles behind them are the same. Suppose a system has n parameters $x_1, x_2, \cdots, x_n$, whose units are denoted by $[x_1]$, $[x_2]$, $\cdots$, $[x_n]$, respectively. To begin with, the units $[x_i]$ are expressed in terms of fundamental units $[F_j](j = 1, 2, \cdots, m)$. The parameters are then divided into two sets, namely the core and residual sets. The former has p parameters of units that are linearly independent of one another, whereas the latter set contains all the other $q = n - p$ parameters. The units of the residual set are linearly dependent on those in the core set. It is recommended [11] that the essential parameters, such as the target parameter (e.g., resonant frequency in this paper), be put in the residual set. The parameters x_p and x_q are listed in a matrix shown in Fig. 1, where $[x_p] = \prod_{j=1}^{m} [F_j]^{a_{jp}}$ and $[x_q] = \prod_{j=1}^{m} [F_j]^{b_{jq}}$. Simple row operations are then performed to convert the core matrix into the unity matrix. This enables us to check the rank of the matrix and to reach a form ready for Pi terms determination. The result is shown in Fig. 2, where r and s are the rank of the matrix and the number of Pi terms, respectively, with $r + s = n$. It can be proved [11] that the Pi terms are easily obtained from Fig. 2 as follows:

$$\pi_q = \frac{x_q}{\prod_{p=1}^{r} x_p^{c_{pq}}}. \tag{1}$$

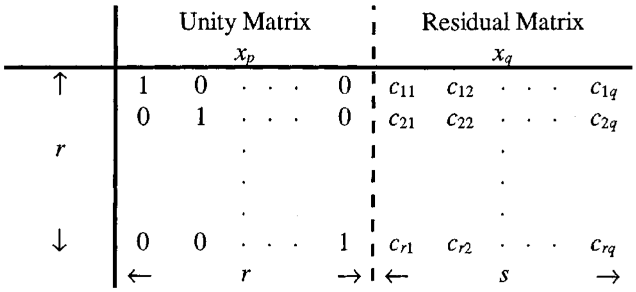

Fig. 2. Core matrix is changed to the unity matrix after row operations and the new matrix is ready for Pi terms determination.

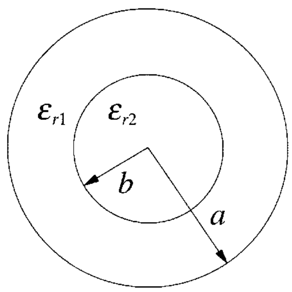

Fig. 3. Geometry of a two-layer spherical DR.

By Pi theorem, the Pi terms are related to one another via a function ϕ

$$\pi_s = \phi(\pi_1, \pi_2, \cdots, \pi_{s-1}). \tag{2}$$

In this paper, the function ϕ is determined by virtue of curve-fitting technique.

III. Analysis of the Two-Layer Spherical DR

Fig. 3 shows the geometry of a two-layer spherical DR. The outer layer of the DR has radius a and permittivity $\varepsilon_1 = \varepsilon_{r1}\varepsilon_0$, whereas the inner layer has radius b and permittivity $\varepsilon_2 = \varepsilon_{r2}\varepsilon_0$. As in [12], a dimensional constant c, the speed of light in vacuum, is also included in the parameter list. The dimensional matrix of the problem is given as follows:

	ε_2	a	c	b	ε_1	f_r
$[M]$	−1	0	0	0	−1	0
$[L]$	−3	1	1	1	−3	0
$[T]$	2	0	−1		2	−1
$[Q]$	2	0	0	0	2	0

where $[M]$, $[L]$, $[T]$, and $[Q]$ are fundamental units of mass, length, time, and electric charge, respectively. Note that the rank of the matrix is $r = 3$, as the first and last rows are linearly dependent on each other. Thus, the last row is removed from the analysis. To obtain the Pi terms, simple row operations are performed so that the core matrix is changed to the unity matrix. The result is shown as follows:

ε_2	a	c	b	ε_1	f_r
1	0	0	0	1	0
0	1	0	1	0	−1
0	0	1	0	0	1

Using (1), the Pi terms are given by

$$\pi_1 = \frac{f_r}{\varepsilon_2^0 a^{-1} c^1} = \frac{a f_r}{c}, \quad \pi_2 = \frac{\varepsilon_1}{\varepsilon_2^1 a^0 c^0} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}$$
$$\pi_3 = \frac{b}{\varepsilon_2^0 a^1 c^0} = \frac{b}{a}. \tag{3}$$

From the Pi theorem, we have

$$f_r = \frac{c}{a}\phi_1\left(\frac{\varepsilon_{r1}}{\varepsilon_{r2}}, \frac{b}{a}\right). \tag{4}$$

For a single-layer DR, we have $b = 0$ and $f_r = (c/a)\phi_1(\varepsilon_{r1})$, where we have arbitrarily put $\varepsilon_{r2} = 1$. The result tells that the resonant frequency is directly proportional to the reciprocal of the radius, a fact well-known for a single-layer spherical DR. Direct application of the analysis to the single-layer DR was carried out and the same result was obtained, as expected.

Similarly, the functional form of the Q-factor was found which is given by

$$Q = \phi_2\left(\frac{\varepsilon_{r1}}{\varepsilon_{r2}}, \frac{b}{a}\right). \tag{5}$$

The result can be reduced to $Q = \phi_2(\varepsilon_{r1})$ for a single-layer DR. In this case, the Q-factor is independent of the DR size, which is also well known for a single-layer spherical DR. It should be mentioned that the single-layer results are not assumed *a priori*, but are consequences of the analysis. Note that the functional forms (4) and (5) are very general and are not limited to the DR TE_{111} mode.

IV. Results and Discussion

In the previous formulation, a general case is considered in which the dielectric constant ε_{r2} of the inner layer is arbitrary. For simplicity, however, we will only consider the special case of $\varepsilon_{r2} = 1$ in the numerical solutions, and the structure becomes a hollow DR. The results should be useful to the design of the hemispherical DR antenna with an airgap for broadband applications [3]. Based on the knowledge of (4), the following formula is obtained by using the least square curve-fitting technique:

$$f_r(\text{GHz}) = \frac{A_1(t)}{a(\text{mm})}[A_2(t) + \varepsilon_{r1}]^{A_3(t)} \tag{6}$$

where

$$A_1(t) = 139.674 + 28.875t - 156.641t^2 + 235.366t^3$$
$$A_2(t) = 0.3229 + 0.9927t - 5.0132t^2 + 7.1661t^3$$
$$A_3(t) = -0.4868 + 0.0015t - 0.0108t^2 + 0.0262t^3$$

and $t = b/a$. The formula is valid over the ranges of $3 \le \varepsilon_{r1} \le 100$ and $0 \le t \le 0.8$, with error less than 1.2%. It is worth mentioning that for $t = 0$, $A_3 = -0.4868 \sim -0.5$ and f_r is roughly proportional to $1/\sqrt{\varepsilon_{r1}}$, as expected.

Similarly, the following curve-fitting formula is obtained from the information of (5):

$$Q = B_1(t)[B_2(t) + \varepsilon_{r1}]^{B_3(t)} \tag{7}$$

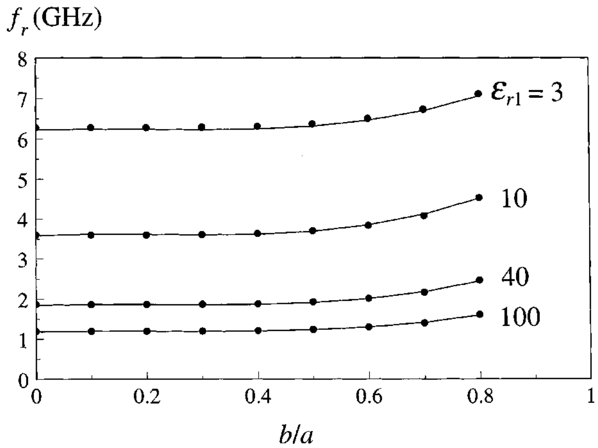

Fig. 4. Resonant frequency of a hollow spherical DR as a function of normalized inner radius for different ε_{r1}: $a = 12.5$ mm and $\varepsilon_{r2} = 1$. — Present theory. • Exact solution.

where

$$B_1(t) = 0.2341 + 0.0233t - 0.0164t^2 - 0.2648t^3$$
$$B_2(t) = 2.9634 + 0.2586t - 2.2012t^2 + 6.0003t^3$$
$$B_3(t) = 1.425 - 0.0245t + 0.1679t^2 - 0.3473t^3$$

with $t = b/a$. The ranges of the formula are $7 \leq \varepsilon_{r1} \leq 100$ and $0 \leq t \leq 0.7$, with error less than 1.9%. Note that putting $t = 0$ in (6) and (7) gives very simple formulas for the single-layer DR.

Fig. 4 shows the TE_{111}-mode resonant frequency as a function of b/a for different ε_{r1}. For ease of comparison, the rigorous results obtained by solving the characteristic equation of the DR are also shown in the same figure. Excellent agreement between the present and rigorous solutions is observed. With reference to the figure, f_r increases monotonically with b. Moreover, the higher the ε_{r1}, the lower the f_r is obtained, which is to be expected. The TE_{111}-mode Q-factor as a function of b/a for different ε_{r1} was also studied. Again, excellent agreement between the present and rigorous results was found. It was observed that the Q-factor decreased conversely with increasing b, as reducing the dielectric volume would lessen the stored energy. It was also seen that using a higher ε_{r1} led to a higher Q-factor, as expected.

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