Efficient Computation of Double Series for the Planar Circuit Analysis via APA-E Algorithm

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***Abstract*—A novel efficient numerical algorithm for computation of double series for the planar circuit analysis is proposed in this paper. The method is based on the abstract Pade approximate and extrapolation algorithm. In order to demonstrate its advantages, the classical multivariable rational Pade approximate is also constructed for comparison. It is shown by the calculation results that the method presented here can provide fast convergence and high accuracy, so it is much better than the classical Pade approximate and original analytical formula for computing planar circuit parameters with the summation of two-dimensional infinite modes.**

***Index Terms*—Classical Pade approximate, multivariable approximate, planar circuit.**

I. Introduction

In microwave and high-speed circuits, the planar circuit is widely used in many places: for example, the planar resonator in microwave circuits and the power/ground-plane structure in high-speed circuits. Generally, the formulas of the planar circuit parameters are the summations of two-dimensional (2-D) infinite modes. In order to analyze the electric performance of the planar circuit system, it is necessary to take the summation with large enough modes. It would need long computation time and large memory. For improving the computation efficiency, several researchers have proposed some numerical techniques [1]–[3]. But these algorithms here have been applied to the problems with a summation of one-dimensional (1-D) infinite terms.

Not long ago, a novel algorithm based on the concept of the abstract Pade [4] approximate and the extrapolation technique—the η-algorithm [5], whose advantages are simpler and convergence speed faster than other techniques—has been proposed. Then the summation of the 2-D infinite modes in a planar structure discussed in [7] can be transformed to double power series, and finally represented by a double variable rational approximation. In order to demonstrate the advantages of the new method, multivariable rational Pade approximate [6] based on the classical single variable Pade approximate is also developed. By analyzing the high-frequency characterization of power/ground plane structures, this method can provide an accurate approximate of the planar circuit parameters within a wide frequency range, requiring minimum storage and CPU time when compared with the classical Pade approximate and the original analytic formula.

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II. Theory

In prior years, the classical Pade approximate is widely used with the form of single variable rational approximate, especially in circuit order reduction. In the following, at first, we construct the multivariable Pade approximate algorithm based on classical single variable Pade approximate algorithm in order to compare with the new proposed algorithm.

A. Classical Pade Approximate Algorithm with Multivariables

Suppose $\sum_{i=0}^{\infty} c_i z^i$ is a given power series and $f(z) \equiv \sum_{i=0}^{\infty} c_i z^i$. Then the classical Pade approximate formula of $f(z)$ is a rational approximate $Q_m/P_n \in R(m, n)$ in $R(m, n)$ domain, noted by $(m/n)_f$, and its Taylor expansion would have as many identical terms of the power series as possible. The (m, n) approximate

$$f_{m,n}(z) \equiv \sum_{q=0}^{m} a_q z^q \Big/ \sum_{p=0}^{n} b_p z^p \tag{1}$$

is defined by the formal identification

$$\sum_{i=0}^{\infty} c_i z^i - Q_m(z)/P_n(z) = O(z^{m+n}). \tag{2}$$

Because the case of the multivariable Pade approximant is similar to that of two variables, here, the case of two variables is only considered in order to avoid the complexity in symbol indication. Assuming

$$f(x, y) = \sum_{i,j=0}^{\infty} C_{\rm ij} x^i y^j \tag{3}$$

and setting

$$z = (x, y) \in R^2 c_k z^k = \sum_{i+j=k} C_{\rm ij} x^i y^j$$

$$C_{\rm ij} = \frac{1}{i!j!} \frac{\partial^k f(x, y)}{\partial x^i \partial y^j}$$

the $[m/m]$ approximant is thus of the form

$$f_{m.m}(x, y) \equiv \sum_{\mu,\nu=0}^{m} a_{\mu\nu} x^\mu y^\nu \Big/ \sum_{\sigma,r=0}^{m} b_{\sigma r} x^\sigma y^r. \tag{4}$$

If x and y are supposed to be one, then (4) it is changed to solve the summation of finite serial terms

$$f_{\rm mm} = \sum_{\mu,\nu=0}^{m} a_{\mu\nu} \Big/ \sum_{\sigma,\gamma=0}^{m} b_{\sigma\gamma}. \tag{5}$$

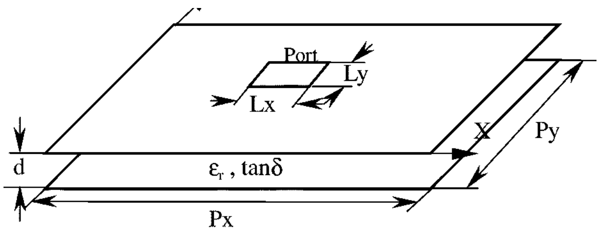


Fig. 1. Power/ground-plane structure

When the algorithm is used to accelerate convergence of the double series for the planar circuit analysis, it is inefficient, as shown in Fig. 2. Therefore, a new algorithm is proposed as follows.

B. APA-E Algorithm (Abstract Pade Approximant—Extrapolation Algorithm)

The algorithm is based on the concept of abstract Pade approximant algorithm and extrapolation technique.

Suppose there exists an expression (3), similar to the classical multivariable Pade approximant algorithm, setting

$$z=(x,\,y)\in R^2,\; c_k z^k=\sum_{i+j=k} C_{\mathrm{ij}}x^i y^j$$
$$C_{\mathrm{ij}}=\frac{1}{i!j!}\,\frac{\partial^k f(x,\,y)}{\partial x^i \partial y^j}$$

then a new 1-D function is constructed

$$f(z)=\sum_{k=0}^{\infty} c_k z^k. \tag{6}$$

Setting $z=1$, then a series is produced. In order to accelerate the series convergence, we combine with an extrapolation technique η-algorithm.

Given a series $\sum_{k=0}^{\infty} c_k$, the η-algorithm is to construct an η-table by one fixed way so that the original series is transformed to a new series. The procedure is described as follows:

$$\eta_{-1}^{(k)}=\infty \qquad k=0,\,1,\,2\ldots \quad \text{(the first column)} \tag{7}$$
$$\eta_{0}^{(k)}=c_k \qquad k=0,\,1,\,2\ldots \quad \text{(the second column)} \tag{8}$$
$$\frac{1}{\eta_{2i+1}^{(k)}}=\frac{1}{\eta_{2i-1}^{(k+1)}}+\frac{1}{\eta_{2i}^{(k+1)}}-\frac{1}{\eta_{2i}^{(k)}} \qquad (i=0,\,1,\,2\ldots)$$
$$\eta_{2i}^{(k)}=\eta_{2i-2}^{(k+1)}+\eta_{2i-1}^{(k+1)}-\eta_{2i-1}^{(k)} \qquad (i=1,\,2,\,3\ldots)$$
$$\text{(the other columns).} \tag{9}$$

Thus, the η-algorithm is realized by transformation from one given series

$$\sum_{k=0}^{\infty} c_k=\sum_{k=0}^{\infty}\eta_0^{(k)}$$

to a new series $\sum_{i=0}^{\infty}\eta_i^{(0)}$.

III. Numerical Results

To test the algorithm, we have studied the high-frequency characterization of power/ground-plane structure by the planar circuit analytic formula [7]. The power and ground-plane system is a 0.5-in square planar structure with a dielectric thickness of 0.006 in, dielectric constant $\varepsilon_{\mathrm{r}}=4$, and loss tangent $\delta=0.018$. One port locating at the exact center of the plane is 0.013-in square, shown in Fig. 1.

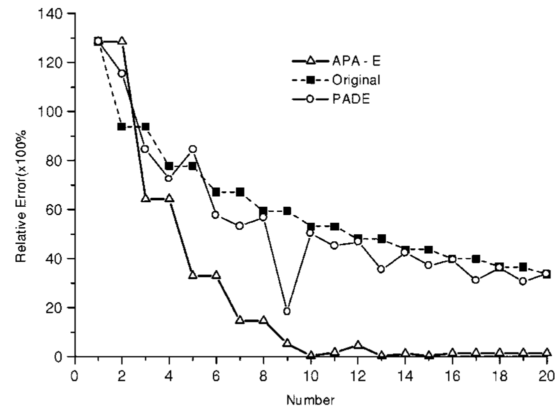


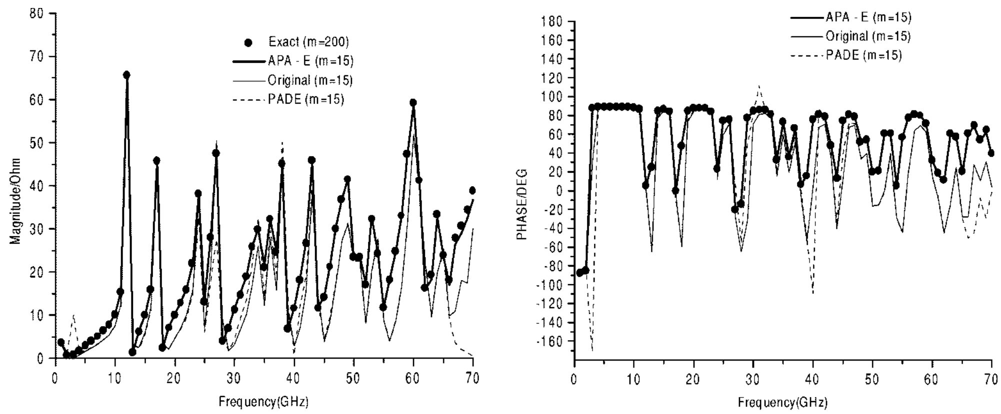
Fig. 2. Comparison of convergence speed

The full-mode representation of the Z-parameters of the power and ground-plane structures is a summation of infinite modes. Namely

$$\begin{aligned}Z_{\mathrm{in}}(f) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{j\omega\mu d C_m^2 C_n^2}{P_x P_y(k_{\mathrm{xm}}^2+k_{\mathrm{yn}}^2-k^2)}\cos(k_{\mathrm{yn}}T_{\mathrm{yi}})\\ &\quad\times\cos(k_{\mathrm{xm}}T_{\mathrm{xi}})\cos(k_{\mathrm{yn}}T_{\mathrm{yj}})\cos(k_{\mathrm{xm}}T_{\mathrm{xj}})\\ &\quad\times\left[\frac{\sin\left(k_{\mathrm{yn}}\frac{L_{\mathrm{yi}}}{2}\right)}{k_{\mathrm{yn}}\frac{L_{\mathrm{yi}}}{2}}\right]\left[\frac{\sin\left(k_{\mathrm{xm}}\frac{L_{\mathrm{xi}}}{2}\right)}{k_{\mathrm{xm}}\frac{L_{\mathrm{xi}}}{2}}\right]\\ &\quad\times\left[\frac{\sin\left(k_{\mathrm{yn}}\frac{L_{\mathrm{yj}}}{2}\right)}{k_{\mathrm{yn}}\frac{L_{\mathrm{yj}}}{2}}\right]\left[\frac{\sin\left(k_{\mathrm{xm}}\frac{L_{\mathrm{xj}}}{2}\right)}{k_{\mathrm{xm}}\frac{L_{\mathrm{xj}}}{2}}\right]\end{aligned} \tag{10}$$

where

- m — mth mode associated with the x-dimensions;
- n — nth mode associated with the y-dimension;
- P_x and P_y — metal plane widths in the x- and y-directions, respectively;
- T_{xi}, T_{xj}, T_{yi}, and T_{yj} — coordinates of the center of the ith and jth ports in the x- and y-directions, respectively;
- L_{xi}, L_{xj}, L_{yi}, and L_{yj} — much less than the wavelengths of interest and represent the ith and jth port widths in the x- and y-directions, respectively;
- k — real wave number for the lossless case, $k=\omega\sqrt{\mu\varepsilon}$, $k_{\mathrm{xm}}=m\pi/p_x$, and $k_{\mathrm{yn}}=n\pi/P_y$;
- d — dielectric thickness between the power and ground planes;
- μ and ε — permeability and permittivity of the material between power and ground plane, respectively;
- ω — radian frequency;
- j — $=\sqrt{-1}$.



(a) (b)

Fig. 3. Comparison of four cases at wide spectral range. (a) Comparison of impedance magnitude. (b) Comparison of impedance phase.

The constant $C_m = 1$ if $m = 0$, and $C_m = \sqrt{2}$ if $m \neq 0$. Similarly, $C_n = 1$ if $n = 0$, and $C_n = \sqrt{2}$ if $n \neq 0$

When considering a low-loss case, k is complex: $k = k_r - jk_i$ and $k_i = (k_r/2)(\tan(s) + r/d)$. Where k_r is the real part of k, and k_i is the imaginary part of k, $\tan(s)$ is the loss tangent in the dielectric and r is the skin depth in the metal plane. The model has also been employed to calculate complex permittivity of low-loss dielectric [8].

According to the second part, based on the double series, a converged double variable function is firstly constructed in terms of the variables x and y

$$Z(x, y) = \sum_{m, n=0}^{\infty} C_{\text{mn}} x^m y^n. \tag{11}$$

Second, the function is rationally approximated by the classical Pade approximant and APA-E algorithm, respectively. Fig. 2 shows the relative percentage errors of the two techniques and the original analytic formula in a same frequency range, where the vertical axis indicates the relative error and the horizontal axis indicates the number of terms in the series. From this figure, the relative error of the APA-E algorithm is nearly zero, but the relative error of the classical Pade approximant algorithm is 50% when the number of terms is ten. It is evident that the convergence speed of the APA-E algorithm is much faster than that of the classical Pade approximant. Comparing with the original analytic formula, the classical Pade approximant is only of a little help for convergence acceleration of double power series. It is because the convergence speed of the original double series is very slow. Consequently, it is inefficient to directly approximate the original double infinite series.

Next, we examine the spectral range of the new algorithm. Fig. 3(a) and (b) shows the frequency behavior of the magnitude and phase of the impedance, respectively. The "exact" is approximately represented by the summation of $m = 200$ and $n = 200$ modes in formula (10). The "Original" is represented by the summation of $m = 15$ and $n = 15$ modes in (10). From this figure, it is evident that the new algorithm is more efficient than that of the classical Pade approximant. In the frequency range between 0–70 GHz, the result of the new algorithm is very close to the exact value, while the computation time of the new algorithm is only 1% of that of the "exact."

IV. Conclusion

A novel effective numerical algorithm for the double series computation of the planar circuit formula is proposed. The computing efficiency is much better than that of the analytic formula and the classical double variable Pade approximant. According to our experiences, the new method can be extended to solve many similar problems with high efficiency, such as 2-D Green's function of the multilayer dielectric media.

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