# An Iterative Algorithm for Reducing Dispersion Error on Yee's Mesh in Cylindrical Coordinates

Michał Rewieński and Michał Mrozowski, *Member, IEEE*

***Abstract*—This letter presents an iterative algorithm for reducing the error due to numerical dispersion while using the finite difference frequency domain (FDFD) scheme on Yee's mesh in cylindrical coordinates. It is shown that the algorithm allows one to significantly enhance the accuracy of the results over a limited frequency band. Consequently, the algorithm provides a cost effective way of achieving high accuracy in the finite difference numerical analysis of such problems as computing resonant frequencies of empty and loaded resonators with cylindrical symmetry.**

***Index Terms*—FDFD, numerical dispersion, resonators, Yee's mesh.**

## I. Introduction

THE PROBLEM of numerical dispersion inevitably arises while applying finite difference (FD) schemes to approximate differential operators. In electromagnetic modeling, one of the well known effects of numerical dispersion is the anisotropy of the phase velocity of the simulated wave modes propagating in an FD mesh, which models, for example, a homogeneous medium [1]. The phase velocity of the modes varies with the wavelength and the direction of propagation with respect to the mesh resulting in the phase error which accumulates along the propagation direction. Consequently, in electromagnetic structures whose geometrical dimensions are significantly larger than the wavelength, the numerical dispersion is the main source of error [2], [3].

One approach toward reducing the effects of numerical dispersion is to refine the finite difference mesh. This has a disadvantage of increasing the size of the solved numerical problem. Consequently, particularly for electrically large electromagnetic systems, both memory and computational costs may become unacceptable. A different approach is based on introducing a correction which annihilates or reduces the dispersion error directly into the discrete operator equations. The idea is based on replacing, for example, the standard central difference operator

$$\boldsymbol{D}F(x) = \frac{F(x+\Delta x/2) - F(x-\Delta x/2)}{\Delta x} \qquad (1)$$

Manuscript received February 29, 2000; revised July 5, 2000. This work was supported by the Polish State Committee for Scientific Research under Contract 8T11D 003 18.

M. Rewieński was with Faculty of Electronics, Telecommunications and Informatics at Technical University of Gdańsk, 80-952 Gdańsk, Poland. He is now with the Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, MA 02139 USA.

M. Mrozowski is with Faculty of Electronics, Telecommunications and Informatics at Technical University of Gdańsk, 80-952 Gdańsk, Poland (e-mail: mrewiens@mit.edu, mim@pg.gda.pl).

Publisher Item Identifier S 1051-8207(00)08450-6.

with a modified operator

$$\boldsymbol{D}F(x) = A\frac{F(x+\Delta x/2) - F(x-\Delta x/2)}{\Delta x} \qquad (2)$$

where $A$ ($\neq$1) is a certain constant number. This approach has been developed in [1] for the finite difference time domain (FDTD) method applied to solving Maxwell's equations in rectangular coordinate system. One disadvantage of this approach is that it is frequency selective i.e., the error can be reduced over a limited frequency band. Somewhat different schemes toward reducing numerical dispersion, also referring to rectangular coordinate system and FDTD method, may be found in [3] and [4]. In this letter we derive a modified FDFD scheme on Yee's mesh defined in cylindrical coordinate system based on the simple idea presented above. We also show how it may be used in an iterative algorithm, which allows one to enhance the accuracy of the obtained numerical results. The approach proposed in this letter is particularly attractive for the applications where high accuracy is required at a specific frequency such as the FDFD analysis of large resonators with cylindrical symmetry.

## II. Iterative Computing of the Correction Factor

Assuming we consider a cylindrical TE wave in a homogeneous region, we obtain the following expression for the fields [the $\exp(jn\phi + jk_z z)$ factor was omitted for clarity]:

$$\begin{aligned} H_r &= jH_{r0}B_n'(k_r r) \quad H_\phi = \frac{H_{\phi 0}}{r}B_n(k_r r) \\ H_z &= H_{z0}B_n(k_r r) \\ E_r &= E_{r0}\frac{1}{r}B_n(k_r r) \quad E_\phi = jE_{\phi 0}B_n'(k_r r) \quad E_z = 0 \end{aligned}$$

where $B(k_r r)$ denotes an appropriate Bessel function. Application of the modified FD scheme [analogous to (2)] to Maxwell's curl equations and substitution of the above fields into appropriate discrete equations results in the following dispersion relation in (3), shown at the bottom of the next page, where

$$\begin{aligned} \alpha &= k_r(I+1/2)\Delta r, \quad \beta = k_r(I-1/2)\Delta r, \\ \gamma &= k_r(I+1)\Delta r, \quad \delta = k_r I\Delta r \end{aligned} \qquad (4)$$

and $I$ denotes the index for the discretization along the $r$ coordinate. In the limit ($\Delta r$, $\Delta z \to 0$) the above relation reduces to Bessel equation satisfied iff $k_r^2 + k_z^2 = k^2 = \omega^2/c^2$. The above dispersion relation may be used to compute $k_z$ for given values of $A$, $\Delta r$, $\Delta z$, $k_r$ and $\omega$. If $A = 1$, the value of $k_z$ computed in this way will differ (will be larger) from its theoretical value $k_{zt} = \sqrt{(\omega/c)^2 - k_r^2}$, where $k_r = k\cos(\theta)$ and $\theta$ is the angle of wave propagation with respect to the mesh. Still, one may optimize the value of $A$, so that the difference between $k_z$ computed using the above relation and the theoretical value of this wavenumber ($k_{zt}$) is minimal for a given angle or a given range of angles $\theta$ of wave propagation with respect to the mesh.

The procedure proposed above allows one to obtain a corrected FDFD scheme, provided the mode frequency $\omega$ ($k$) and wavenumber $k_r$ are known. However, in many practical situations, the eigenfrequency of a given mode in a modeled electromagnetic system is unknown and has to be found, for example, by solving an operator eigenproblem. In this case, we may still use the above optimization procedure within the following iterative scheme. In the first step we find the approximate eigenfrequency $\omega_0$ by solving the problem for $A = 1$. In the next step, we use $\omega_0$ to find the optimized value of $A$. This step may be repeated to obtain better approximations of the eigenfrequency $\omega$.

## III. Modeling Hemispherical Resonators—Numerical Results

The iterative scheme presented above has been applied to compute numerically the resonant frequency $f_{0,0,20}$ of the quasi-$\mathrm{TEM}_{0,0,20}$ mode (Gaussian beam notation) in a hemispherical resonator shown in Fig. 1. Both homogeneous (no dielectric sample) and inhomogeneous structure has been considered with mirror separation of 51.3 mm and the spherical mirror radius of 76.2 mm. In this letter, we give only the results for the homogeneous case. One may note that the length of the modeled structure equals $\approx 10\lambda$.

The resonant frequency of the quasi-$\mathrm{TEM}_{0,0,20}$ in a homogeneous structure has been computed for various mesh sizes using the FDFD solver described in [2], [5]. In the first series of tests, the standard FDFD scheme has been used ($A = 1$). The results are shown in Table I. The error is computed relative to the analytical value of $f_{0,0,20}$ obtained from paraxial approximation.

Table II presents the reduction of the error of computing the desired resonant frequency using the proposed iterative scheme. It is apparent that just after three iterations the error due to numerical dispersion has been substantially reduced. The efficiency of the iterative scheme is assessed in Table I. The last row shows the execution time for the two iterations of the FDFD solver with dispersion correction. One may note that by applying the iterative scheme we obtain a significantly more ac-

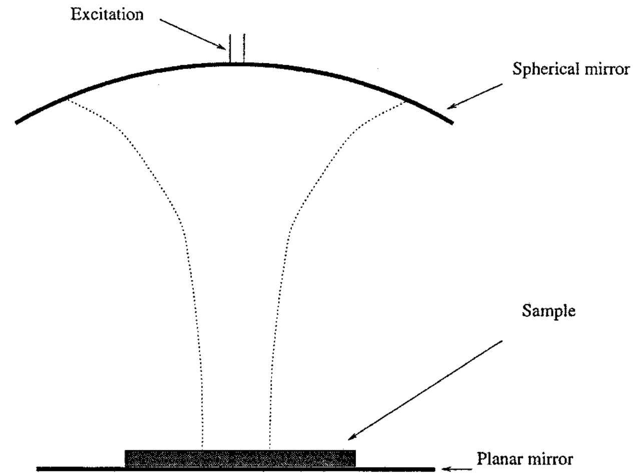


Fig. 1. Open resonator structure loaded with a dielectric disc.

TABLE I
COMPARISON OF EXECUTION TIMES FOR THE FDFD ALGORTIHM, FOR DIFFERENT PROBLEM SIZES. THE LAST ROW SHOWS DATA FOR THE TWO REPETITIONS OF THE FDFD ALGORITHM, PERFORMED IN ORDER TO REDUCE THE DISPERSION ERROR. THE TESTS HAVE BEEN PERFORMED IN THE CRAY T3E SYSTEM USING FOUR PROCESSORS

| Problem size $N$ | Grid size | Execution time [s] | Resonant frequency [GHz] | Relative error [%] |
|---|---|---|---|---|
| – | $\lambda/\infty$ | – | 59.375168 | 0.000 |
| 32770 | $\lambda/15$ | 1091.43 | 58.913500 | -0.777 |
| 73780 | $\lambda/20$ | 4690.47 | 59.168571 | -0.348 |
| 165100 | $\lambda/30$ | 8884.84 | 59.266262 | -0.183 |
| 32770 2-iter. | $\lambda/15$ | 2197.78 | 59.386013 | 0.018 |

TABLE II
ITERATIVE REDUCTION OF THE ERROR DUE TO NUMERICAL DISPERSION FOR THE HOMOGENEOUS RESONATOR PROBLEM. THE RESONANT FREQUENCIES FOR THE QUASI-$\mathrm{TEM}_{0,0,20}$ MODE COMPUTED IN ONE STEP ARE USED IN THE NEXT STEP TO FIND THE OPTIMIZED VALUE OF $A$. $\Delta r = \Delta z = \lambda/15$ ($N = 32\,770$)

| Optimization frequency [GHz] | Value of $A$ | Resonant frequency [GHz] | Relative error [%] |
|---|---|---|---|
| - | 1.00000000 | 58.913500 | -0.777 |
| 58.913500 | 1.00814384 | 59.386013 | 0.018 |
| 59.386013 | 1.00800627 | 59.378032 | 0.005 |

$$\begin{vmatrix} -\omega\mu & 0 & 0 & 0 & \frac{2A\sin(k_z\Delta z/2)}{\Delta z} \\ 0 & \omega\mu & 0 & \frac{2A\sin(k_z\Delta z/2)}{\Delta z} & 0 \\ 0 & 0 & \omega\mu B_n(\alpha) & \frac{-nB_n(\alpha)}{((I+1/2)\Delta r)^2} & \frac{A\left[B_n'(\gamma)(I+1) - B_n'(\delta)I\right]}{(I+1/2)\Delta r} \\ 0 & -\frac{2A\sin(k_z\Delta z/2)}{\Delta z} & n & -\omega\epsilon & 0 \\ -\frac{2AB_n'(k_r I\Delta r)\sin(k_z\Delta z/2)}{\Delta z} & 0 & -A\frac{B_n(\alpha) - B_n(\beta)}{\Delta r} & 0 & \omega\epsilon B'(k_r I\Delta r) \end{vmatrix} = 0 \qquad (3)$$

curate result than by grid refinement. Moreover, a considerable gain both in computation time and memory cost (directly proportional to the problem size) is observed.

## IV. Conclusions

In this letter, we have presented an iterative algorithm which allows one to reduce the error due to numerical dispersion on Yee's mesh in cylindrical coordinates. We have shown that this method may be effectively applied to improve the accuracy of the computed resonant frequencies of high order modes in electrically large resonators. The obtained performance results indicate that the proposed algorithm is very cost-efficient.

## Acknowledgment

The authors wish to thank A. Ćwikła and J. Mielewski for providing the code of the FDFD solver of the hemispherical resonator problem, which was used to validate the presented method of reducing the numerical dispersion error. All numerical tests have been carried out at the facilities of the Academic Computer Centre TASK in Gdańsk and the Interdisciplinary Centre for Mathematical and Computational Modeling of the University of Warsaw.

## References

[1] A. Taflove, *Computational Electrodynamics: The Finite-Difference Time-Domain Method*. Norwood, MA: Artech House, 1995.

[2] A. Ćwikła, J. Mielewski, M. Mrozowski, and J. Wosik, "Accurate full wave analysis of open hemispherical resonators loaded with dielectric layers," *IEEE MTT-S Symp. Dig.*, vol. 3, pp. 1265–1268, 1999.

[3] M. F. Hadi and M. Piket-May, "A modified FDTD (2,4) scheme for modeling electrically large structures with high-phase accuracy," *IEEE Trans. Antennas Propagat,*, vol. 45, pp. 254–264, Feb. 1997.

[4] J. W. Nehrbass, J. O. Jevtić, and R. Lee, "Reducing the phase error for finite-difference methods without increasing the order," *IEEE Trans. Antennas Propagat.*, vol. 46, pp. 1194–1201, Aug. 1998.

[5] J. Mielewski, A. Ćwikła, and M. Mrozowski, "Accelerated FD analysis of dielectric resonators," *IEEE Microwave Guided Wave Lett.*, vol. 8, pp. 375–377, Nov. 1998.