An Algorithm with Error Bounds for Calculating Intermodulation Products

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***Abstract*—We make use of newly obtained theoretical results concerning systems driven by asymptotically almost periodic inputs to obtain a useful algorithm with error bounds for calculating intermodulation products. The method is applicable to systems described by Volterra integral equations of the second kind that meet the circle criterion and satisfy some additional constraints. In this paper we show that an interesting family of circuits of practical interest can be analyzed using the algorithm.**

***Index Terms*—Almost periodic inputs, circle criterion, error bounds, intermodulation products.**

I. Introduction

A FUNCTION f given by

$$f(t) = \sum_m A_m \sin(\omega_m t + \theta_m), \qquad t \in (-\infty, \infty) \qquad (1)$$

in which the sum is finite and the A_m, ω_m, and θ_m are real constants, is called a *trigonometric polynomial* (because of its exponential-form representation). In (1) the frequencies $\{\omega_m\}$ need not be integrally related. The set of real-valued *almost periodic functions* consists of these trigonometric polynomials together with all limits, with respects to the usual uniform norm, of sequences of trigonometric polynomials.

Asymptotically almost periodic functions (AAP) are defined only for $t \geq 0$. They are sums of the restriction to $[0, \infty)$ of an almost periodic function and a continuous function that approaches zero as $t \to \infty$.

The calculation of intermodulation products ordinarily requires the calculation of the spectral coefficients of the output of systems driven by AAP inputs. Here we give an analytical basis for evaluating the spectral coefficients using a convergent iterative process and we give bounds on the errors incurred in truncating the process.

II. Definitions and Known Results

In the following, $\mathbb{R}$ denotes the real numbers, $\mathbb{Z}$ stands for the set of all integers, and $\mathbb{N}$ denotes $\{1, 2, 3, \cdots\}$. We use j for $\sqrt{-1}$.

$L_\infty(0, \infty)$ denotes the space of bounded (Lebesgue) measurable real-valued functions defined on $[0, \infty)$ and $L_p(0, \infty)$ for $p \in \{1, 2\}$ stands for the space of $\mathbb{R}$-valued p-th power integrable functions defined on $[0, \infty)$.

For an AP function x the spectral coefficient A_k^x at frequency ω_k is defined by

$$A_k^x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t) e^{-j\omega_k t}\, dt.$$

It is known that this is well defined for $x \in AP$ and that it is nonzero for only a countable number of frequencies ω_k [1].[1] We define the module $\overline{\Lambda}_x$ of x by $\overline{\Lambda}_x = \{q \in \mathbb{R}\colon q = \sum_{l=1}^{m} k_l \omega_l,\, k_l \in \mathbb{Z},\, m \in \mathbb{N}\}$.

We use $\|x\|_B$ to denote $(\sum_{k=1}^{\infty} |A_k^x|^2)^{1/2}$ where $\{\omega_1, \omega_2, \cdots\}$ is an enumeration of all the nonzero spectral coefficients of x. Using material in [1], it follows that this defines a norm on AP. For $x \in$ AAP we define A_k^x to be the corresponding spectral coefficient of its AP part.

III. Theorems

Many systems of practical interest are described by integral equations of the form

$$u(t) = y(t) + \int_0^t k(t-\tau)\psi[y(\tau)]\, d\tau, \qquad t \geq 0 \qquad (2)$$

where t denotes time, u is the input (or a modified input that takes into account initial conditions) and y is the output. We are interested in the case where $u = u_1 + u_2$ where u_1 is the restriction to $[0, \infty)$ of an AP function g and u_2 takes into account the initial conditions.

In connection with questions concerning the long-time response of systems governed by (2), one is often not interested in transients and it is natural to consider the integral equation

$$g(t) = f(t) + \int_{-\infty}^t k(t-\tau)\psi[f(\tau)]\, d\tau, \qquad t \in \mathbb{R}. \qquad (3)$$

We make the following assumptions.

i) $u_2 \in L_\infty(0, \infty)$ and $\lim_{t\to\infty} u_2(t) = 0$.
ii) ξ given by $\xi(t) = t^p k(t), t \geq 0$ belongs to $L_1(0, \infty) \bigcap L_2(0, \infty)$ for $p \in \{0, 1, 2\}$.
iii) $\psi(0) = 0$ and there exists positive constants α and β such that

$$\alpha \leq \frac{\psi(b) - \psi(a)}{b - a} \leq \beta$$

for all $a \neq b$.

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[1] For other references concerning almost periodic functions, see e.g., [2] and [3].

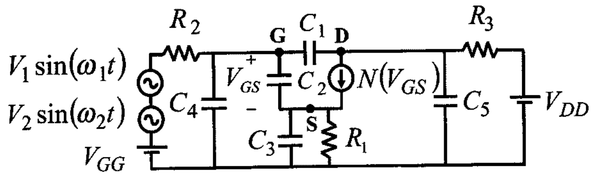


Fig. 1. Simple amplifier circuit.

The Fourier transform K of k is defined by

$$K(\omega) = \int_0^{\infty} k(t) \exp(-j\omega t)\, dt, \qquad \omega \in \mathbb{R}.$$

We assume that K satisfies the *circle criterion*, by which we mean that the locus of $K(\omega)$ for all ω avoids the disk in the complex plane centered on the real axis of the complex plane at $(-1/2(\alpha^{-1} + \beta^{-1}), 0)$ with radius $1/2(\alpha^{-1} - \beta^{-1})$ and does not encircle it.

Let c_0 denote $1/2(\beta + \alpha)$ and define

$$r = \frac{1}{2}(\beta - \alpha) \sup_{\omega} \left| \frac{K(\omega)}{1 + c_0 K(\omega)} \right|.$$

If K satisfies the circle criterion then $r < 1$. This fact is used in connection with Theorem 2, below.

Our first result, Theorem 1 below, establishes an important connection between (2) and (3).

Theorem 1: Let the conditions indicated be met, and let $g \in$ AP. Then

a) there is a unique $f \in$ AP such that

$$g(t) = f(t) + \int_{-\infty}^{t} k(t - \tau)\psi[f(\tau)]\, d\tau, \qquad t \in \mathbb{R} \qquad (4)$$

and

b) with u_1 the restriction to $[0, \infty)$ of g, and y the solution of

$$u_1(t) + u_2(t) = y(t) + \int_0^t k(t - \tau)\psi[y(\tau)]\, d\tau, \qquad t \geq 0 \quad (5)$$

we have

$$\lim_{t \to \infty} |y(t) - f(t)| = 0.$$

Our next theorem leads directly to an algorithm for numerically evaluating the spectral coefficients (i.e., Fourier coefficients) of f, where f is the AP solution of (4) corresponding to a given $g \in$ AP.

In our theorem, $\mathcal{B}$ stands for the space of $\mathbb{R}$-valued bounded continuous functions on $\mathbb{R}$, I denotes the identity operator on $\mathcal{B}$, and $\Psi: \mathcal{B} \to \mathcal{B}$ and $L: \mathcal{B} \to \mathcal{B}$ are defined by

$$(\Psi x)(t) = \psi[x(t)], \qquad t \in \mathbb{R}$$

and

$$(Lx)(t) = \int_{-\infty}^{t} k(t - \tau)x(\tau)\, d\tau, \qquad t \in \mathbb{R}.$$

Theorem 2: Under the conditions described, $(I + c_0 L)$ is an invertible map of $\mathcal{B}$ onto itself, and for any $g \in$ AP and any $f_0 \in$ AP satisfying $\overline{\Lambda}_{f_0} \subset \overline{\Lambda}_g$ the sequence $\{f_n\}_{n=0}^{\infty}$ given by

$$f_{n+1} = (I + c_0 L)^{-1} g - (I + c_0 L)^{-1} L(\Psi - c_0 I) f_n, \qquad n \geq 0$$

belongs to AP and satisfies

$$\|f - f_n\|_B \leq \frac{r^n}{1 - r} \|f_1 - f_0\|_B$$

as well as $\overline{\Lambda}_{f_n} \subset \overline{\Lambda}_g$ for $n \geq 0$, where f is the associate of g via (4).

Theorems 1 and 2 are proved in [4].[2]

IV. Example of an Application

Consider the model of a simple amplifier shown in Fig. 1, and assume zero initial conditions.

The equations governing this circuit may be cast in the form

$$y + LNy = L_2 v_L + L_3 v_R \qquad (6)$$

in which y is the gate-source voltage, L, L_2 and L_3 are convolutions and v_L and v_R are the time functions corresponding to the Thévenin equivalent sources on the left and right.

In the case where N (over its domain of interest) is a polynomial our algorithm for finding y reduces to algebraic manipulation of the sum of complex exponentials.

We use v_L as the initial estimate for y and then run the algorithm

$$\begin{aligned} y_{n+1} = &(I + c_0 L)^{-1}(L_2 v_L + L_3 v_R) \\ &- (I + c_0 L)^{-1} L(\Psi - c_0 I) y_n, \qquad n \geq 0 \end{aligned}$$

a number of times, each time constraining the order[3] of the intermodulation products to keep the algorithm efficient.

We then perform one last iteration in which we keep intermodulation products of all orders. We know that the B-norm of the error is less than

$$\frac{r}{1 - r} \|y_{q+1} - y_q\|_B$$

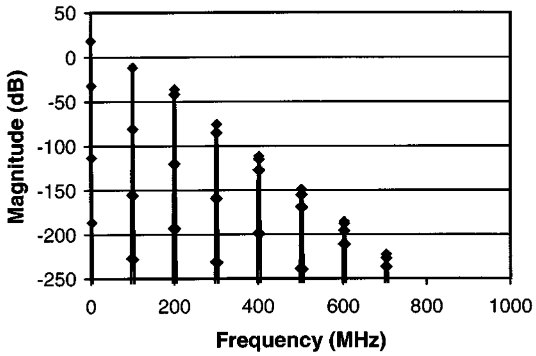
where y_{q+1} is the final (unconstrained) iterate and y_q is the last constrained iterate.

Once we have our estimate y_{q+1} of y, we can obtain an estimate of the drain voltage v_D by applying linear operators to y_{q+1}, v_R and v_L. The dependence of v_D on y (which is approximated by y_{q+1}) leads directly to a bound on the error in estimating the spectral coefficients of v_D.

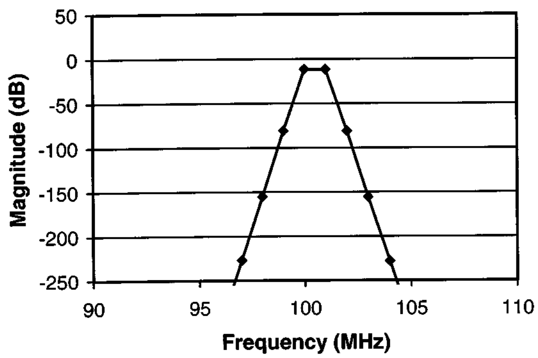
To give an indication of the numbers involved, consider the model shown in Fig. 1 with parameters $f_1 = \omega_1/(2\pi) = 100$ MHz, $f_2 = \omega_2/(2\pi) = 101$ MHz, $V_1 = V_2 = 0.5$ V, $R_1 = 10$ Ω, $R_2 = R_3 = 4$ kΩ, $C_1 = 35$ fF, $C_2 = 320$ fF, $C_3 = 410$ fF, $C_4 = 2$ fF, $C_5 = 150$ fF, and $N(\cdot) = 1.1 \times 10^{-4}(\cdot - 1)^2$ A. One can verify that all conditions for applying Theorem 2 are

[2] For related background material in the context of feedback systems, see [5].

[3] In this case, where the input consists of two sinusoids of frequencies f_1 and f_2, we define the order of the intermodulation product at frequency $mf_1 + nf_2$ as $|m| + |n|$. To constrain the order we simply discard all terms with order greater than the set limit after applying the operator $(\Psi - c_0 I)$ to y_n.



(a)



(b)

Fig. 2. Some of the spectral coefficients of the drain voltage.

met with these values if one assumes that the gate-source voltage never exceeds 10 V.

Using 20 eighth-order iterations before the unconstrained iteration we find that, with y the exact solution to (6), $\|y - y_{n+1}\|_B \leq 6.4 \times 10^{-16}$ V.

This means that we can calculate intermodulation products for v_D down to -263 dB V. These figures take into account possible numerical errors. (It also means e.g., that an intermodulation product calculated to be at a level of e.g., -200 dB V must be within 10^{-2} dB of -200 dB V. At realistic levels of -65 dB V the error is totally insignificant.)

Errors in this sort of range are at any rate of little practical interest. The entire calculation on a 300 MHz processor takes about 8 seconds. Some of the calculated intermodulation products are shown in Fig. 2.

V. Conclusion

We have given a convergent iterative process that can be used to calculate intermodulation products. Since bounds on the errors incurred in truncating the process are also given, this method should be useful in comparing results obtained with techniques that typically do not provide error bounds (e.g., harmonic balance) to determine how well these methods truly perform.

References

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