Inverse of Exact Solution by Synthetic Asymptote—An Example of Stripline

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***Abstract*—The novel technique *synthetic asymptote* is a useful tool in deriving the inverse formula of an exact solution for CAD purposes. As an example, the inverse formula of stripline is derived in this paper. The average error is less than 1%.**

***Index Terms*—Stripline, synthetic asymptote.**

I. Introduction

ANALYSIS in electromagnetics gives the response of a device with a given structure. It is normally quite difficult to find its *inverse*, that is: to design a suitable structure with a specified response. An inverse formula is convenient. It is found that the novel technique of *synthetic asymptote* [1] can derive the inverse formula with little difficulty and with high accuracy. The ease applies even when the exact analytical solution is known. The stripline is chosen as the illustration.

Assume that an unknown function has known asymptotes at the two limits of a parameter of interest. The synthetic asymptote then is a formula that is constructed so that it converges into the known asymptotes. This formula approximates the original function well (maximum error $<$10%) if the function is monotonically increasing or decreasing.

The maximum error naturally occurs at some *point* between the two known asymptotes. If the synthetic asymptote is simply adjusted to include one or two extra numerical data somewhat close to this *point*, the maximum error may easily be reduced to 1 to 2%. The two extra data points can be obtained from 2 *simple runs* of existing numerical software.

Regular asymptotes are frequently simple analytical functions. As a result a regular asymptote can easily be inverted, by exchanging its dependent and independent variables. The synthetic asymptote constructed from the inverted asymptotes is the inverse of the original analysis function. This paper constructs a formula of the characteristic impedance with a given structure of stripline with different substrate dielectrics. Then the inverse of the formula is constructed. The average error of the inverse formula is less than 1%.

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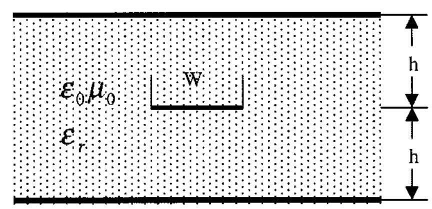

Fig. 1. Cross section of stripline.

II. Derivation of the Inverse Formula of Stripline by Synthetic Asymptote

For the stripline shown in Fig. 1, Cohn [2] gave the exact analysis formula by conformal mapping method, that is

$$C = 4\varepsilon_r \varepsilon_0 \frac{K(k')}{K(k)} \tag{1}$$

where

$$k = \sec h \left(\frac{\pi}{2} \frac{W}{2h} \right) \tag{2}$$

and $k' = \sqrt{1-k^2}$.

After the distributed capacitance is obtained, the characteristic impedance of stripline is

$$Z_0 = \sqrt{\frac{\mu_0 \varepsilon_0}{CC_0}} \tag{3}$$

where C and C_0 are the distributed capacitances with dielectric and air substrates, respectively.

Equation (1) is an exact solution in the complete elliptic integral of the first kind. The complexity of the elliptic integral makes its inverse difficult. On the other hand, if the elliptic integral is separated into two simpler asymptotes, i.e., two simpler analytical functions, we can invert the two asymptotes. Using the two inverted asymptotes, the inverse synthetic asymptote can be obtained. In addition, the asymptotes should give good physical insight.

A. Derivation of the Inverse Near Asymptote $(Z_0 \to 0)$

When $k \to 0$, i.e., $h \to 0$, we can expand the ratio of the elliptic integral of the first kind in (1) by the q-series [3], [4]. Taking the first term, we get the *near* asymptote

$$\underset{k\to 0}{Asym} \frac{K(k')}{K(k)} \approx -\frac{1}{\pi} \ln \left(\frac{k^2}{16} \right). \tag{4}$$

Also, the asymptote of k (when $h \to 0$) can be obtained from (2). After some manipulations, it is

$$\underset{h\to 0}{Asym}\ k = 2e^{-(\pi/2)(W/2h)}. \tag{5}$$

Substituting (4) and (5) into (1), we obtain the *near* asymptote of the distributed capacitance

$$C_1 = \underset{h\to 0}{Asym}\, C = 4\varepsilon_r\varepsilon_0\left(\frac{2}{\pi}\ln 2 + \frac{W}{2h}\right). \tag{6}$$

When $h \to 0$, the second term in (6) represents the parallel plate capacitance and it is more dominant than the first term. Therefore, we can rewrite (6) as that of two parallel plate capacitors, from the strip to the top and the bottom ground plates

$$C_1 = \underset{h\to 0}{Asym}\, C = 2\varepsilon_r\varepsilon_0\frac{W}{h}. \tag{7}$$

By defining $\overset{0}{h} = h/W$, we get the inverse *near* asymptote of Z_0 from this asymptote of capacitance (7) and (3):

$$\boxed{\overset{0}{h}_1 = \underset{Z_0\to 0}{Asym}\ \overset{0}{h} = 2Z_0\sqrt{\frac{\varepsilon_r\varepsilon_0}{\mu_0}}} \tag{8}$$

which is the inverse *near* asymptote of characteristic impedance Z_0.

B. *Derivation of the Inverse Far Asymptote* $(Z_0 \to \infty)$

When $k \to 1$, i.e., $h \to \infty$, the ratio for the complete elliptic integrals of the first kind in (1) can also be expanded by q-series [3], [4]. Taking the first term, we now have

$$\underset{k\to 1}{Asym}\,\frac{K(k')}{K(k)} \approx -\frac{\pi}{\ln\left(\dfrac{1-k^2}{16}\right)}. \tag{9}$$

From (2) and (9), and after some manipulations, we get the *other* asymptote of k as

$$\underset{h\to\infty}{Asym}\,(1-k^2) = \left(\frac{\pi}{2}\frac{W}{2h}\right)^2. \tag{10}$$

Substituting (9) and (10) into (1), we can get the *far* asymptote of the distributed capacitance

$$C_1 = \underset{h\to\infty}{Asym}\, C = -\frac{4\varepsilon_r\varepsilon_0\pi}{\ln\left(\dfrac{\left(\dfrac{\pi}{2}\dfrac{W}{2h}\right)^2}{4}\right)} = 2\pi\varepsilon_0\varepsilon_r\left[\ln\left(\frac{8}{\pi}\frac{2h}{W}\right)\right]^{-1}. \tag{11}$$

Like the parallel plate interpretation of (7), the near asymptote, there is an interpretation of (11), the far asymptote. That is, the infinite series of images of the strip (from the top and the bottom ground plates) is equivalent to just two equivalent images (each of half the negative charge and at the same distance $4h/\pi$, above or below the strip).

From (11) and (3), we get the inverse *far* asymptote of Z_0

$$\overset{0}{h}_2 = \underset{Z_0\to\infty}{Asym}\ \overset{0}{h} = \frac{\pi}{16}\,e^{2\pi Z_0\sqrt{\varepsilon_0\varepsilon_r/\mu_0}}. \tag{12}$$

Synthetic asymptote is frequently constructed by summing the regular asymptotes at the two parameter's limits. This requires an "asymptote consistency condition." The condition is that the *far* asymptote will approach zero or a small number at the near parameter limit and vice versa for the *near* asymptote.

Under this condition, the inverse *far* asymptote (12) reduces to $\pi/16$ at the near limit of $Z_0 \to 0$. Still, this $\pi/16$ is too large for the inverse *near* asymptote (8) of zero at $Z_0 \to 0$. The inverse *far* asymptote would have to drop to zero. Without disturbing its value at the far limit, we may modify the inverse *far* asymptote (12) to

$$\boxed{\overset{0}{h}_2 = \underset{Z_0\to\infty}{Asym}\ \overset{0}{h} = \frac{\pi}{16}\left(e^{2\pi Z_0\sqrt{\varepsilon_0\varepsilon_r/\mu_0}} - 1 - 2\pi Z_0\sqrt{\frac{\varepsilon_0\varepsilon_r}{\mu_0}}\right)} \tag{13}$$

which is the inverse *far* asymptote of characteristic impedance Z_0.

C. *Derivation of the Inverse Synthetic Asymptote*

As mentioned earlier, the inverse synthetic asymptote can be simply the sum of the inverse *near* and *far* asymptotes (8) and (13). We get the inverse synthetic asymptote

$$\boxed{\overset{0}{h} = \left\{\left[2Z_0\sqrt{\frac{\varepsilon_r\varepsilon_0}{\mu_0}}\right]^n + \left[\frac{\pi}{16}\left(e^{2\pi Z_0\sqrt{\varepsilon_0\varepsilon_r/\mu_0}} - 1 - 2\pi Z_0\sqrt{\frac{\varepsilon_0\varepsilon_r}{\mu_0}}\right)\right]^n\right\}^{1/n}.} \tag{14}$$

With $n = 1$, the inverse synthetic asymptote converges smoothly at the two limit of Z_0, but still has a maximum error of 10% at the intermediate values of Z_0. A match with *one data point* with numerical computation at an arbitrary Z_0, enables us to set the exponent

$$n = 1.05. \tag{15}$$

This match reduces the maximum error to 1.5%.

III. Results

Fig. 2 shows and compares the results calculated by (14) and Cohn's exact formula (1) from [2] for $\varepsilon_r = 2.55$, 12.9, and 24.0. The average error is less than 1%. As expected, the maximum error is 1.5% at the middle range of characteristic impedance. The curves cover the *practical range of dielectric constant and the shape ratio W/h of stripline.*

IV. Conclusion

Synthetic asymptote is a useful CAD tool in deriving inverse formula for practical structure of MIC lines, even those with exact solutions like the stripline of this paper. The synthetic asymptote formulas are simple and provide good physical insight unlike other curve fitted formulas with many arbitrary constants. The synthetic asymptote, from the regular asymptotes which may be exact, is naturally accurate at the asymptote limits. However, it may not be accurate at some midpoints in between, where the maximum error can be up to 10% and beyond. If one or more arbitrary constant in the middle is used to found the power n in the form of the "root of sum of powers" as in (14), the error may be reduced to about 2%. Also, we can use the synthetic asymptote to derive the inverse formula of exact solution for other practical structures, such as coplanar waveguide with infinite substrate (CPW). It is easy to see that similar inverse formula can also be derived for approximate solutions. An example of ours is that for the microstrip line [5].

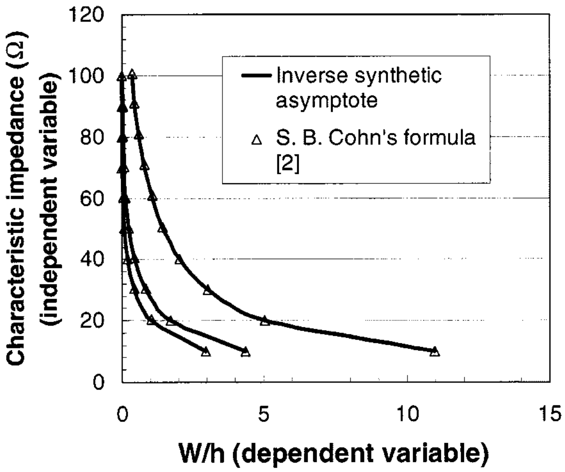

Fig. 2. Comparison of the W/h versus the characteristic impedance Z_0 of stripline calculated by inverse synthetic asymptote formula (15) and S. B. Cohn's formula [2] with $\varepsilon_r = 2.55$, 12.9, and 24.

Formulas of more complicated cases of crosstalk (coupling) between strip lines and between microstrip lines were given by Rainal [6]. They were derived through simplification of grounded substrate effects to just one image per source, with obvious error. We believe that such drastic simplifications are not necessary with synthetic asymptote. As a start, we have just derived the analysis formula for the coupling of two microstrip lines. Unlike the one isolated line case of this paper, for coupling lines we have to apply the synthetic asymptote technique twice, once for the substrate thickness and once for the separation between the lines. The details and the inverse formula will be submitted as a separate paper.

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