Analysis of Inhomogeneously Filled Waveguides Using a Bi-Orthonormal-Basis Method

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***Abstract*—A general theoretical formulation to analyze inhomogeneously filled waveguides with lossy dielectrics is presented in this paper. The wave equations for the tranverse-field components are written in terms of a nonself-adjoint linear operator and its adjoint. The eigenvectors of this pair of linear operators define a biorthonormal-basis, allowing for a matrix representation of the wave equations in the basis of an auxiliary waveguide. Thus, the problem of solving a system of differential equations is transformed into a linear matrix eigenvalue problem. This formulation is applied to rectangular waveguides loaded with an arbitrary number of dielectric slabs centered at arbitrary points. The comparison with theoretical results available in the literature gives good agreement.**

***Index Terms*—Dielectric-loaded waveguides, eigenvalues, eigenvectors, numerical analysis, waveguide theory.**

I. Introduction

INHOMOGENIOUSLY filled waveguides have received considerable attention in the last decades because of their applications in a variety of waveguide components. The modes of propagation of such waveguides are not, in general, TM or TE modes, but hybrid modes. The boundary value method has been used to calculate the modal solutions for concentric [1], [2] and eccentric [3] dielectric-loaded circular guides, as well as for rectangular guides filled with dielectric slabs [4], [5]. In that method, the electromagnetic field is expanded in terms of analytical functions in the relevant regions of the waveguide, and a linear eigenvalue problem is obtained after imposing the boundary conditions in the corresponding interfaces. When the guide is filled with two or more dielectrics, the determination of the propagation constants and of the mode fields becomes difficult because of the transcendental equations involved. Alternatively, a variational method is used in [6] to calculate the eigenvalues in rectangular waveguides loaded with lossless dielectric slabs. The finite-element method has been extensively applied to find the eigenvalues and modal fields in dielectric loaded guides [7], [8]–[10]. There are, as well, a number of matrix formulation methods to analyze inhomogeneously filled waveguides in which the fields are expanded in a set of basis functions [11]–[16], and based on Laplace and Fourier transforms techniques [17].

In this paper, we develop a rigorous and computationally efficient method to obtain the modal spectrum in inhomogeneously filled waveguides with lossy dielectric of arbitrary profiles. Starting with the differential equations governing the propagation of the tranverse electric and magnetic fields, we identify a pair of linear nonself-adjoint operators, whose eigenvectors satisfy a biorthogonality relationship. The key element of our approach is to transform the system of differential equations into a linear matrix eigenvalue problem by means of the Galerkin method, using the eigenvectors of an auxiliary problem. From a computational point-of-view, this method is very efficient because the integrals involved in the matrix elements are, in principle, frequency independent, so they have to be evaluated only once to obtain the dispersion curves, thus generating a robust and efficient code. This method has been applied to study open dielectric waveguides, as reported in [18] and [19]. Comparisons between our results and the available numerical published data fully validate the theory presented here.

II. Theoretical Formulation

Our starting point are Maxwell's equations for uniform cross-section waveguides partially or totally filled with a lossy dielectric media defined by its dielectric permittivity $\varepsilon(x, y) = \varepsilon_0\varepsilon_r(x, y)$. We assume that the considered media does not have magnetic properties $\mu = \mu_0$. Thus, the solution of the problem can be obtained as a superposition of fields with explicit harmonic dependence on z (we assume that the time dependence is always implicit and has a harmonic form $e^{j\omega t}$ for all vector fields)

$$\begin{aligned} \boldsymbol{E}(x, y, z) &= \boldsymbol{e}(x, y)\exp(-j\beta z) \\ \boldsymbol{H}(x, y, z) &= \boldsymbol{h}(x, y)\exp(-j\beta z) \end{aligned} \tag{1}$$

where β is the propagation constant and $\boldsymbol{e}$ and $\boldsymbol{h}$ represent the transverse-dependent (depending on x and y only) three-dimensional vector amplitudes of the electric and magnetic fields, respectively. We are interested in rewriting Maxwell's equations

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in terms of the transverse components of the electric and magnetic fields $\boldsymbol{e}_t = \begin{bmatrix} e_x \\ e_y \end{bmatrix}$ and $\boldsymbol{h}_t = \begin{bmatrix} h_x \\ h_y \end{bmatrix}$. Following [20], we can obtain a set of equations for them as follows:

$$\left\{\nabla_t^2 + \varepsilon_r(x, y)k_0^2 - \beta^2\right\}\boldsymbol{h}_t = (\nabla_t \times \boldsymbol{h}_t) \times \frac{\nabla_t \varepsilon_r(x, y)}{\varepsilon_r(x, y)} \tag{2a}$$

$$\left\{\nabla_t^2 + \varepsilon_r(x, y)k_0^2 - \beta^2\right\}\boldsymbol{e}_t = \nabla_t \left(\boldsymbol{e}_t \cdot \frac{\nabla_t \varepsilon_r(x, y)}{\varepsilon_r(x, y)}\right) \tag{2b}$$

where $\varepsilon_r(x, y)$ is the relative dielectric permittivity, k_0 is the free-space wavenumber ($k_0 = \omega\sqrt{\mu_0 \varepsilon_0}$), and the operator ∇_t is the transverse gradient operator. The axial components e_z and h_z are determined by $\boldsymbol{e}_t$ and $\boldsymbol{h}_t$ through constraint relations given by Maxwell's equations.

For our purposes, it is more interesting to rewrite (2) in a different way. We will express the two previous equations in a two-dimensional matrix form. Both $\boldsymbol{e}_t$ and $\boldsymbol{h}_t$ are two-dimensional vector fields that are represented by two component vectors. Thus, the differential operators acting on them can be expressed as 2×2 matrices. Equation (2b) involves tranverse two-dimensional vectors only. However, (2a) includes the three-dimensional axial vector ($\nabla_t \times \boldsymbol{h}_t$) on its right-hand side. On the one hand, it is easy to check that the double three-dimensional vector product in (2a) can be rewritten in terms of a 2×2 matrix acting on $\boldsymbol{h}_t$ (using, e.g., the completely antisymmetric tensor in two-dimensions $\epsilon_{\alpha\gamma}$). On the other hand, for reasons that will become clear later, it is more convenient to rewrite (2b) not in terms of $\boldsymbol{e}_t$, but in terms of the closely related vector field $\bar{\boldsymbol{e}}_t \equiv \begin{bmatrix} e_y \\ -e_x^* \end{bmatrix}^*$ (* is the conjugation operation). After manipulating (2b) in a suitable way, one can obtain the following equivalent set of equations in matrix form:

$$\begin{aligned} L\begin{bmatrix} h_x \\ h_y \end{bmatrix} &= \beta^2 \begin{bmatrix} h_x \\ h_y \end{bmatrix} \\ L^\dagger \begin{bmatrix} e_y^* \\ -e_x^* \end{bmatrix} &= (\beta^*)^2 \begin{bmatrix} e_y^* \\ -e_x^* \end{bmatrix} \end{aligned} \tag{3}$$

where L and $L^\dagger$ are 2×2 matrix differential operators given by

$$\begin{aligned} L = &\begin{bmatrix} \nabla_t^2 + k_0^2\varepsilon_r(x, y) & 0 \\ 0 & \nabla_t^2 + k_0^2\varepsilon_r(x, y) \end{bmatrix} \\ &- \begin{bmatrix} F_y\nabla_y & -F_y\nabla_x \\ -F_x\nabla_y & F_x\nabla_x \end{bmatrix} \end{aligned} \tag{4a}$$

$$\begin{aligned} L^\dagger = &\begin{bmatrix} \nabla_t^2 + k_0^2\varepsilon_r^*(x, y) & 0 \\ 0 & \nabla_t^2 + k_0^2\varepsilon_r^*(x, y) \end{bmatrix} \\ &+ \begin{bmatrix} \nabla_y F_y^* & -\nabla_y F_x^* \\ -\nabla_x F_y^* & \nabla_x F_x^* \end{bmatrix} \end{aligned} \tag{4b}$$

where $\boldsymbol{F}_t \equiv (\nabla_t \varepsilon_r(x, y))/\varepsilon_r(x, y)$, and the derivative appearing in the second matrix of $L^\dagger$ acts both on the components of $\boldsymbol{F}_t^*$ and on the vector field $\bar{\boldsymbol{e}}_t$. The operator $L^\dagger$ is called the *formal adjoint* to the linear operator L, and is defined as follows (see [21]):

$$\begin{aligned} \langle \boldsymbol{v}, L\boldsymbol{u}\rangle = \langle L^\dagger \boldsymbol{v}, \boldsymbol{u}\rangle \Leftrightarrow &\int_S \boldsymbol{v}^*(\boldsymbol{r}) \cdot L\boldsymbol{u}(\boldsymbol{r})\, dS \\ &= \int_S \left(L^\dagger \boldsymbol{v}(\boldsymbol{r})\right)^* \cdot \boldsymbol{u}(\boldsymbol{r})\, dS \end{aligned} \tag{5}$$

where $\boldsymbol{u}(\boldsymbol{r})$ and $\boldsymbol{v}(\boldsymbol{r})$ are two-dimensional vector complex functions defined on a two-dimensional closed and bounded region S, with boundary C. They are members of a Hilbert space $\mathcal{H}$ with inner product

$$\begin{aligned} \langle \boldsymbol{v}, \boldsymbol{u}\rangle &= \int_S \boldsymbol{v}^*(\boldsymbol{r}) \cdot \boldsymbol{u}(\boldsymbol{r})\, dS \\ &= \int_S \left(v_x^*(\boldsymbol{r})u_x(\boldsymbol{r}) + v_y^*(\boldsymbol{r})u_y(\boldsymbol{r})\right) dS \end{aligned} \tag{6}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{H}$. When $L^\dagger = L$, we say that L is *formally self-adjoint*.

The set of (3) is a system of eigenvalue equations for the nonself-adjoint operator L and its adjoint $L^\dagger$

$$L\boldsymbol{u}_n = \beta_n^2 \boldsymbol{u}_n \quad L^\dagger \boldsymbol{v}_m = (\beta_m^*)^2 \boldsymbol{v}_m \tag{7}$$

where $\boldsymbol{u} \equiv \boldsymbol{h}_t$ and $\boldsymbol{v} \equiv \bar{\boldsymbol{e}}_t$.

The eigenvectors of a nonself-adjoint operator do not satisfy a orthogonality relation. The same applies to the eigenvectors of its adjoint. In our case, this means that $\langle \boldsymbol{u}_n, \boldsymbol{u}_{n'}\rangle \neq \delta_{nn'}$ and $\langle \boldsymbol{v}_m, \boldsymbol{v}_{m'}\rangle \neq \delta_{mm'}$, δ_{mn} being the Kronecker delta symbol. Apparently, the impossibility of using the standard orthogonality relations associated to a self-adjoint operator would prevent us from expanding arbitrary functions in terms of its eigenvectors. However, since we are considering a $\{L, L^\dagger\}$ system, this is not so because we can take advantage of what it is called the *biorthogonality relation* [22]

$$\langle \boldsymbol{v}_n, \boldsymbol{u}_m\rangle = \delta_{nm}. \tag{8}$$

The biorthogonality relations were successfully used by Paiva and Barbosa to analyze inhomogeneous biisotropic planar guides [23]. Despite its apparently formal character, this relation has a very clear physical meaning. If we write the inner product in its integral from and restore the original three-dimensional notation, (8) reads

$$\int_S (\boldsymbol{e}_n \times \boldsymbol{h}_m) \cdot \hat{\boldsymbol{z}}\, dS = \delta_{nm} \tag{9}$$

where $\hat{\boldsymbol{z}}$ represents the unitary vector along the z-direction. In the waveguide literature, the relation (9) is known as the orthogonality condition for the waveguide modes [20].

The previous relation allows us to expand any vector function $\boldsymbol{f}$ of $\mathcal{H}$ in terms of either the L eigenvectors, $\{\boldsymbol{u}_n\}$, or those of its adjoint $L^\dagger$, $\{\boldsymbol{v}_m\}$

$$\boldsymbol{f} = \sum_n c_n \boldsymbol{u}_n = \sum_m d_m \boldsymbol{v}_m \tag{10}$$

where the complex expansion coefficients are given by the inner products $c_n = \langle \boldsymbol{v}_n, \boldsymbol{f}\rangle$ and $d_m = \langle \boldsymbol{f}, \boldsymbol{u}_m\rangle$. Notice the c's and d's coefficients are not trivially related, unless when L is self-adjoint, in that case, $c_n = d_n^*$ and $\boldsymbol{u}_n = \boldsymbol{v}_n$.

The previous results define the framework in which our method is developed. Our aim is to find the propagation modes of a realistic waveguide characterized by a complex relative dielectric permittivity $\varepsilon_r(x, y)$. As we have proven, the electromagnetic propagation in this waveguide is described by the system of eigenvalue (7). Let L be the matrix differential operator (4a) representing the waveguide, we are interested in $\boldsymbol{u}_n$ and $\boldsymbol{v}_m$, the eigenmodes of L and $L^\dagger$, respectively, and β_n, the propagation constant of the nth mode. The system of equations describing this waveguide and that we want to solve is then

$$L\boldsymbol{u}_n = \beta_n^2 \boldsymbol{u}_n \quad L^\dagger \boldsymbol{v}_m = (\beta_m^*)^2 \boldsymbol{v}_m. \tag{11}$$

Now we define an *auxiliary problem* as a waveguide characterized by a relative dielectric permittivity $\tilde{\varepsilon}_r(x, y)$ and with the same boundary conditions as the waveguide described by (11). The eigenmodes of the auxiliary problem $\{\tilde{\boldsymbol{u}}_p, \tilde{\boldsymbol{v}}_q\}$ and, thus, their respective propagation constants, are supposed to be perfectly known. The equations describing the auxiliary problem constitute another $\{\tilde{L}, \tilde{L}^\dagger\}$ system

$$\tilde{L}\tilde{\boldsymbol{u}}_p = \tilde{\beta}_p^2 \tilde{\boldsymbol{u}}_p, \tilde{L}^\dagger \tilde{\boldsymbol{v}}_q = (\tilde{\beta}_q^*)^2 \tilde{\boldsymbol{v}}_q \tag{12}$$

and, consequently, all the properties of a biorthogonal-basis apply to the $\{\tilde{\boldsymbol{u}}_p, \tilde{\boldsymbol{v}}_q\}$ set. In particular, the $\{\tilde{\boldsymbol{u}}_p, \tilde{\boldsymbol{v}}_q\}$ modes can be used as a basis to represent any arbitrary vector. Thus, following (10)

$$\boldsymbol{u}_n = \sum_p c_{(n)p} \tilde{\boldsymbol{u}}_p \quad \boldsymbol{v}_m = \sum_q d_{(m)q} \tilde{\boldsymbol{v}}_q. \tag{13}$$

We are certainly concerned with the matrix representation of a linear operator of the real problem L. The matrix elements of the L operator in the $\{\tilde{\boldsymbol{u}}_p, \tilde{\boldsymbol{v}}_q\}$ basis will then easily be obtained by applying the standard Galerkin moment method [21]. By inserting the first equation of (13) into the first equation of (11), and applying the linear properties of L, we find

$$\sum_p c_{(n)p} L\tilde{\boldsymbol{u}}_p = \beta_n^2 \sum_p c_{(n)p} \tilde{\boldsymbol{u}}_p. \tag{14}$$

The next step in the application of the Galerkin procedure is to choose a set of weighting functions $\{\tilde{\boldsymbol{v}}_q\}$, and to take the inner product for each $\tilde{\boldsymbol{v}}_q$ yielding

$$\sum_p c_{(n)p} \langle \tilde{\boldsymbol{v}}_q, L\tilde{\boldsymbol{u}}_p \rangle = \beta_n^2 \sum_p c_{(n)p} \langle \tilde{\boldsymbol{v}}_q, \tilde{\boldsymbol{u}}_p \rangle = \beta_n^2 c_{(n)q}. \tag{15}$$

The above system can be written in matrix form as

$$\begin{bmatrix} L_{11} & L_{12} & \cdots & L_{1q} & \cdots \\ L_{21} & L_{22} & \cdots & L_{2q} & \cdots \\ \vdots & \vdots & & \vdots & \\ L_{p1} & L_{p2} & \cdots & L_{pq} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} c_{(n)1} \\ c_{(n)2} \\ \vdots \\ c_{(n)p} \\ \vdots \end{bmatrix} = \beta_n^2 \begin{bmatrix} c_{(n)1} \\ c_{(n)2} \\ \vdots \\ c_{(n)p} \\ \vdots \end{bmatrix} \tag{16}$$

where the elements of the matrix $[L]$ are obtained as

$$L_{pq} = \langle \tilde{\boldsymbol{v}}_q, L\tilde{\boldsymbol{u}}_p \rangle. \tag{17}$$

For practical purposes, it is convenient to introduce the difference operator $\Delta \equiv L - \tilde{L}$

$$\Delta = k_0^2 \Big(\varepsilon_r(x, y) - \tilde{\varepsilon}_r(x, y)\Big) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} (F_y - \tilde{F}_y)\nabla_y & -(F_y - \tilde{F}_y)\nabla_x \\ -(F_x - \tilde{F}_x)\nabla_y & (F_x - \tilde{F}_x)\nabla_x \end{bmatrix}. \tag{18}$$

Thus, the elements of the operator $L = \tilde{L} + \Delta$ in the auxiliary basis $\{\tilde{\boldsymbol{u}}_p, \tilde{\boldsymbol{v}}_q\}$ are trivially obtained by means of (17)

$$L_{pq} = \tilde{\beta}^2 \delta_{pq} + \langle \tilde{\boldsymbol{v}}_p, \Delta \tilde{\boldsymbol{u}}_q \rangle \tag{19}$$

where the first term is diagonal because the operator $\tilde{L}$ is expressed in its own biorthogonal basis. At this point, it is important to remark that we have tranformed the differential operator system (11) into a linear matrix eigenvalue problem defined in (16). An analog equation for its adjoint matrix $[L^\dagger][d_{(m)}] = (\beta_m^*)^2 [d_{(m)}]$ can also be derived. Thus, the information contained in the above matrix equations is the same as in the differential equations for the L and $L^\dagger$ operators (11). Diagonalization of the $[L]$ matrix yields the squared of the nth-mode propagation constant—the nth eigenvalue of $[L]$—and also its transverse magnetic amplitude $\boldsymbol{h}_t$ through the knowledge of the nth eigenvector $[c_{(n)}]$ (recall that its components constitute the expansion coefficients of the unknown mode $\boldsymbol{u}_n(\boldsymbol{r})$ in terms of the auxiliary modes $\{\tilde{\boldsymbol{u}}_p(\boldsymbol{r})\}$). It is important to note that the diagonalization of $[L]$ not only provides us with the propagation constants and tranverse magnetic amplitudes of the modes, but also with their whole three-dimensional magnetic- and electric-field structure. Both the axial component of the magnetic field and the transverse and axial components of the electric field are related to $\boldsymbol{h}_t$ through constraints given by Maxwell's equations [20]. This fact is very important from a computational point-of-view because only the diagonalization process for the $[L]$ matrix is requested in the numerical implementation of this method.

However, the matrix $[L]$ is infinitely dimensional. In order to develop a realistic method, we have to work with a finite set of auxiliary fields. Unfortunately, there are no general conditions that guarantee the convergence of the expansions. This convergence will depend on both the nature of the L operator and the auxiliary problem chosen to define the biorthogonal basis. In general, we observe that the real modes are better described by increasing the number of auxiliary modes. In the same way, auxiliary basis encompassing the most relevant features of the real problem produce faster convergence. In any case, numerical convergence tests must be done by sweeping the number of auxiliary modes over meaningful ranges and studying the stability of the solutions.

The method we have just presented involves no restriction on the vector character of the electromagnetic field. The key eigenvalue (7) are completely general and involve the nonself adjoint operator L. It is remarkable that the nonself-adjoint character of L is present even when the medium is lossless ($\varepsilon_r(x, y)$ is a real function)] showing the inherent nonself-adjoint character of electromagnetic propagation. It is also interesting to note that it is precisely the nonself-adjoint part of L, the second matrix in (4a), the one responsible for polarization mixing. The diagonal

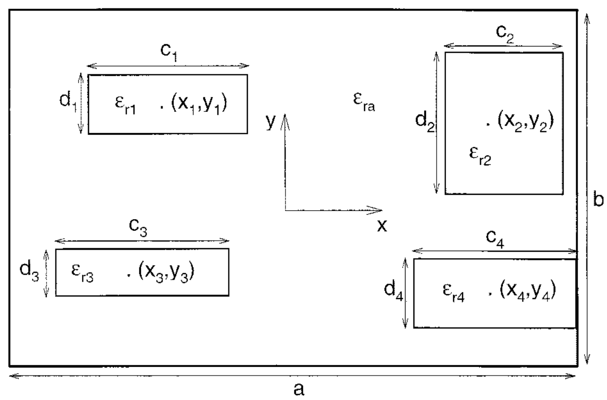

Fig. 1. Cross section of an inhomogeneously filled rectangular waveguide with arbitrarily placed lossy dielectric slabs.

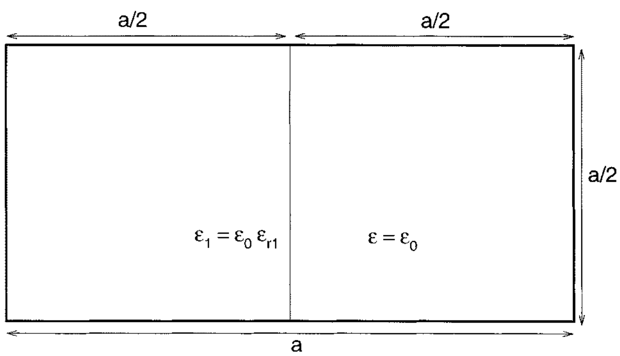

Fig. 2. Rectangular waveguide loaded with a dielectric slab along the sidewall.

TABLE I
NORMALIZED PROPAGATION CONSTANT β/k_0 OF A RECTANGULAR WAVEGUIDE LOADED WITH A DIELECTRIC SLAB ALONG THE SIDEWALL

Mode order	Exact solution	Ref. [9]	This method	Rel. difference (%)
1	1.27576	1.27327	1.27574	0.002
2	0.97154	0.97101	0.97159	0.005
3	0.72865	0.72539	0.72863	0.003
4	0.59390	0.59280	0.59388	0.003

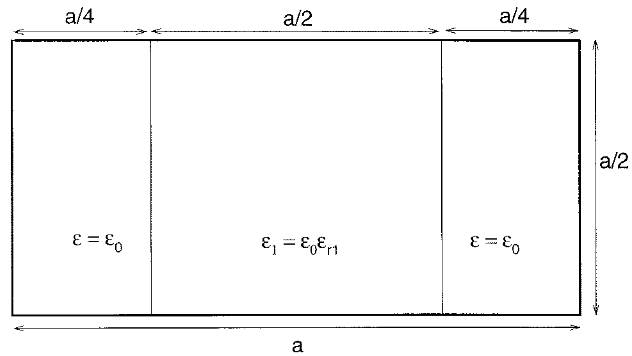

Fig. 3. Rectangular waveguide loaded with a centered dielectric slab.

TABLE II
NORMALIZED PROPAGATION CONSTANT βa OF A RECTANGULAR WAVEGUIDE LOADED WITH A CENTERED DIELECTRIC SLAB

Mode	Ref. [6]	This method	Rel. difference (%)
LSE_{10}	17.127	17.127	¡ 0.006
LSE_{20}	15.147	15.147	¡ 0.007
LSE_{30}	11.821	11.820	0.008
LSE_{40}	7.602	7.602	¡ 0.013
LSE_{11}	15.933	15.932	0.006
LSE_{21}	13.783	13.782	0.007
LSE_{31}	10.013	10.012	0.01
LSE_{41}	4.280	4.278	0.05
LSE_{12}	11.637	11.637	¡ 0.009
LSE_{22}	8.457	8.456	0.01
LSM_{11}	15.739	15.723	0.1
LSM_{21}	13.203	13.161	0.3
LSM_{31}	9.450	9.491	0.4
LSM_{41}	4.989	4.858	2.6
LSM_{12}	11.369	11.348	0.2
LSM_{22}	7.476	7.401	1.0

and, thus, nonpolarization mixing, part is self-adjoint in lossless media. This fact makes evident the close relation between the nonself-adjoint character of L and the full-vector description given by its eigenvectors. Indeed, a cylindrical waveguide uniformly loaded with a homogeneous dielectric is described by the first matrix of (4a) (the second matrix is zero), and only TM and TE modes appear. However, when it is filled with an inhomogeneous dielectric, due to the second matrix of (4a), most of the modes are hybrid. To end, we emphasize the unambiguous and rigorous character of the matrix construction and of the mode expansions presented here, which are based on the biorthogonality property (8) satisfied by the auxiliary basis.

III. NUMERICAL RESULTS

The present method can be applied to a large variety of waveguides. In fact, its suitability has already been demonstrated to deal with open guides like optical fibers [18], [19]. We will now focus on dielectric-loaded rectangular waveguides. We have developed a FORTRAN code to analyze a rectangular waveguide partially loaded with arbitrarily placed lossy dielectrics of rectangular cross section, as can be shown in Fig. 1. Thus, the relative dielectric permittivity $\varepsilon_r(x, y)$ of this guide can be expressed as follows:

$$\begin{aligned}\varepsilon_r(x, y) = {} & \varepsilon_{ra} + \sum_{i=1}^{N} (\varepsilon_{ri} - \varepsilon_{ra}) \\ & \times \left(H\left(x - x_i + \frac{c_i}{2}\right) - H\left(x - x_i - \frac{c_i}{2}\right)\right) \\ & \times \left(H\left(y - y_i + \frac{d_i}{2}\right) - H\left(y - y_i - \frac{d_i}{2}\right)\right)\end{aligned} \tag{20}$$

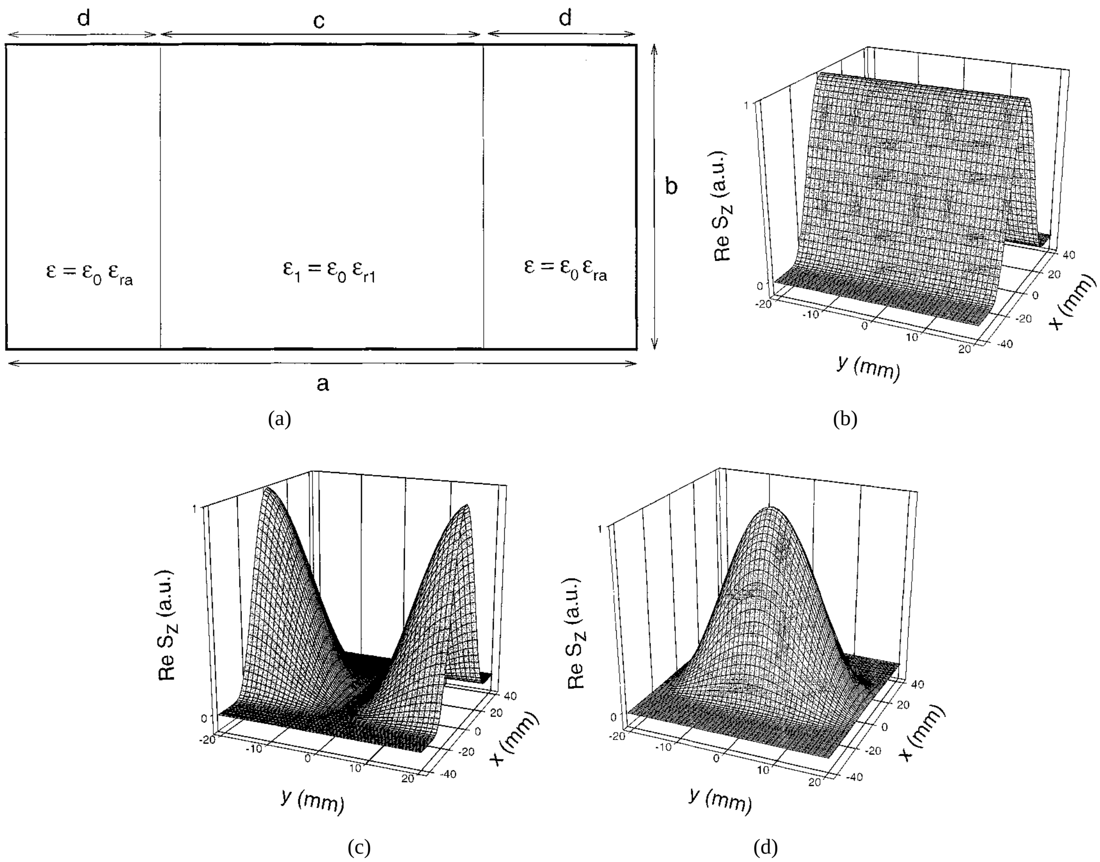

Fig. 4. (a) Rectangular waveguide loaded with three dielectric slabs. Plots of the real part of the Poynting vector (z-component) for the: (b) TE_{01}, (c) TE_{02}, and (d) TE_{03} modes, respectively.

where $H(x)$ is the Heaviside function, N is the number of dielectric slabs, and the ith dielectric is centered at point $(x_i,\, y_i)$, its size being c_i by d_i. In our case, the auxiliary problem is chosen to be a homogeneously filled rectangular waveguide, characterized by $\tilde{\varepsilon}_r(x,\, y) = \varepsilon_{ra}$, whose eigenvalues and eigenvectors are well known (see, e.g., [24]). Following the theoretical formulation, the matrix elements of $[L]$, derived from (19), are given by

$$L_{pq} = \tilde{\beta}^2 \delta_{pq} + k_0^2 \int_S \Big(\varepsilon_r(x,y) - \tilde{\varepsilon}_r(x,y)\Big)\big(\tilde{\boldsymbol{e}}_p \times \tilde{\boldsymbol{h}}_q\big) \cdot \hat{\boldsymbol{z}}\, dS + \int_S \left[\tilde{\boldsymbol{e}}_p \times \left(\left(\frac{\boldsymbol{\nabla}_t \varepsilon_r(x,\, y)}{\varepsilon_r(x,y)} - \frac{\boldsymbol{\nabla}_t \tilde{\varepsilon}_r(x,\, y)}{\tilde{\varepsilon}_r(x,\, y)}\right) \times \big(\boldsymbol{\nabla}_t \times \tilde{\boldsymbol{h}}_q\big)\right)\right] \cdot \hat{\boldsymbol{z}}\, dS. \tag{21}$$

After some algebraic manipulations, these integrals have been analytically calculated. As a consequence, only a numerical diagonalization process has to be performed for each frequency point, thus resulting in fast code implementation.

We have compared the results of our approach with existing ones for five different dielectric-slab-loaded rectangular guides. The first case is a rectangular waveguide of width a and height $a/2$, with a dielectric slab along the sidewall, as shown in Fig. 2. Half of the waveguide is filled with dielectric material whose relative permittivity is $\varepsilon_{r1} = 2.25$ and the other half is vacuum. This case is particularly interesting because the analytical solution does exist [20]. To solve this simple case, we have properly located one dielectric rectangular slab, and we have taken the vector mode functions of an empty rectangular guide as the auxiliary basis. In Table I, we compare our results with the exact solution, and also with results calculated in [9] using the finite-element method. For the comparison, we present the normalized propagation constant of the first modes β/k_0, calculated for the working frequency given by $k_0 a = 3$. Only 200 auxiliary modes have been used, taking 31 s on a CrayOrigin2000 machine. The results agree with each other accurately.

The second case is again a rectangular waveguide of width a and height $a/2$ with a centered dielectric slab, as shown in Fig. 3. In Table II, we present the normalized propagation constant βa for the LSE and LSM modes calculated for the operating frequency defined by $k_0 a = 4\pi$. We compare our results with those provided by a variational approach in [6]. The dielectric region size is $a/2$ by $a/2$ with relative permittivity $\varepsilon_{r1} = 2$. As in the previous case, the auxiliary problem is an empty rectangular guide. The results of Table II have been obtained using 400 basis functions, the computation time being 103 s on a CrayOrigin2000 machine. The agreement between both methods is good.

The third example is a rectangular waveguide loaded with three dielectric slabs [see Fig. 4(a)]. This structure is used to model a microwave cure applicator in [5]. In this problem, we locate a dielectric slab of relative permittivity $\varepsilon_{r1} = 10.0$ at the center of a standard WR-340 guide, which was homogeneously filled with a dielectric of relative permittivity $\varepsilon_{ra} = 1.5$, whose modes are the auxiliary basis. In Table III, we give the cutoff

TABLE III
CUTOFF FREQUENCIES (GHz) OF A RECTANGULAR WAVEGUIDE LOADED WITH THREE DIELECTRIC SLABS FOR DIFFERENT VALUES OF THE RELATIVE WIDTH OF THE CENTRAL DIELECTRIC REGION (c/a)

$c/a = 0.2$			
Mode	Ref. [5]	This method	Rel. difference (%)
TE_{01}	0.778	0.775	0.4
TE_{02}	2.362	2.360	0.08
TE_{03}	3.372	3.371	0.03
$c/a = 0.4$			
Mode	Ref. [5]	This method	Rel. difference (%)
TE_{01}	0.625	0.624	0.2
TE_{02}	1.550	1.553	0.2
TE_{03}	2.668	2.670	0.007
$c/a = 0.6$			
Mode	Ref. [5]	This method	Rel. difference (%)
TE_{01}	0.569	0.570	0.2
TE_{02}	1.224	1.221	0.2
TE_{03}	1.984	2.001	0.9
$c/a = 0.8$			
Mode	Ref. [5]	This method	Rel. difference(%)
TE_{01}	0.556	0.558	0.4
TE_{02}	1.121	1.122	0.1
TE_{03}	1.681	1.701	1.2

frequencies for four different widths of the central dielectric region, and we compare our results with those obtained by solving a transcendental equation [5]. The number of basis functions employed in this case is 300. The distribution of the electromagnetic field is important in the applications of this kind of structures in order to focus the energy in the central region of the guide. Our method provides the propagation constants and the fields of the modes. As an example, we show the Poynting vector profiles for the first propagating modes in Fig. 4(b)–(d). In these plots, the operating frequency is 5 GHz, and the relative width of the central dielectric region is $c/a = 0.6$. One of the most attractive features of our method is the versatility and flexibility for the analysis and design of complex dielectric structures filling a rectangular guide, thus becoming a powerful and effective computer-aided design (CAD) tool. That is to say, the inclusion of other slabs inside the guide is a very simple task employing this algorithm, while other techniques require to recalculate the full problem in order to find the new transcendental equation.

The fourth example is a shielded rectangular dielectric waveguide. In Fig. 5, we present the dispersion curves for the first two modes, comparing our results with those obtained with a finite-difference method [25]. The relative permittivity of the core is $\varepsilon_{r1} = 2.22$, and the dimensions of the rectangular rod are c_1 and d_1, $c_1/d_1 = 0.99$. The dielectric rectangular rod is shielded by a metallic rectangular guide of dimensions $a = 1.88c_1$ and

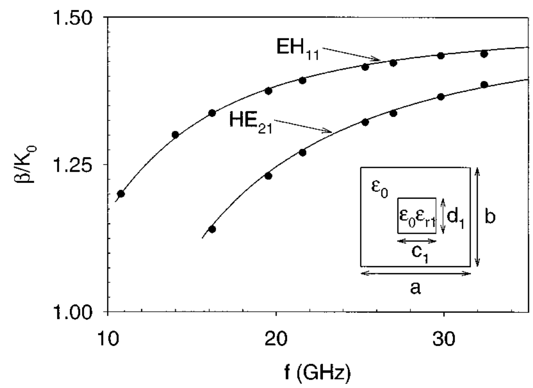

Fig. 5. Dispersion curves for the EH_{11} and HE_{21} modes of a shielded rectangular dielectric waveguide. Comparison between our results (solid lines) and the results obtained with the finite-difference method [25] (dots).

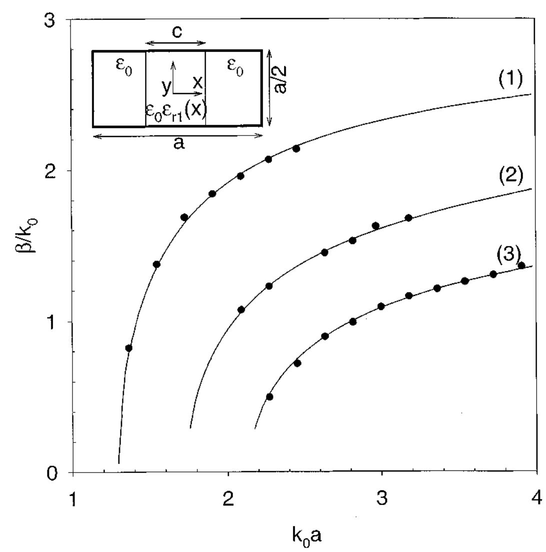

Fig. 6. Dispersion curves for the fundamental mode of a rectangular guide loaded with a centered inhomogeneous dielectric slab, as a function of the thickness: (1) $c/a = 0.5$, (2) $c/a = 0.2$, and (3) $c/a = 0.1$. Comparison between our results (solid lines) and the results obtained with a technique based on a variational formulation [6] (dots).

$b = 1.88d_1$. Only 200 basis functions are necessary to obtain the first modes. The computation time required to obtain the results shown in Fig. 5 is 12 s per frequency point on a CrayOrigin2000 machine. We find a good agreement with previous results.

Finally, we want to show how our method can also be applied to analyze waveguides loaded with inhomogeneous dielectric slabs. The fifth example is a rectangular waveguide loaded with a centered inhomogeneous dielectric slab. The relative permittivity of the inhomogeneous slab depends on the coordinate x in the form (see Fig. 6)

$$\varepsilon_{r1}(x) = 1 + \left(\varepsilon_{r1(\max)} - 1\right)\left(1 - \left(\frac{x}{c/2}\right)^2\right) \tag{22}$$

with $\varepsilon_{r1(\max)} = 9$ and where the origin of the x-axis is at the center of the waveguide. We have computed the dispersion curves of the fundamental mode for three different values of

c/a, using 300 auxiliary modes. In Fig. 6, we compare our results with those presented in [6] using a variational formulation.

IV. CONCLUSIONS

In this paper, we have developed a method for the analysis of inhomogeneously filled waveguides with lossy dielectrics. Once Maxwell's equations are written in terms of the transverse components of the fields of guided modes, we have shown that they can be rewritten as a system of eigenvalue equations for a nonself-adjoint operator and its adjoint L and $L^{\dagger}$, respectively. The eigenvectors of the system $\{L, L^{\dagger}\}$ define a biorthonormal basis and allow to transform the differential operator system into a linear matrix eigenvalue problem, using the eigenvectors of an auxiliary problem to expand the modes of the original problem.

We have developed a FORTRAN code to obtain the modal spectrum of rectangular waveguides filled with dielectric slabs. In principle, our program can deal with any number of lossy dielectric slabs with arbitrary size and location within the rectangular waveguide. We have tested it by comparison with theoretical results found in the technical literature. We demonstrate that it can be used to work out the modal spectrum of a large variety of dielectric guides.

Finally, we showed that our method can deal with inhomogeneous dielectrics. Furthermore, our method can be easily used to analyze dielectric rods with nonrectangular cross section and inhomogeneously magnetic-media-filled waveguides.

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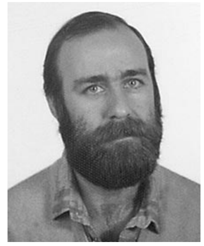

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