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IDENTIFICATION OF DAMPING: PART 4, ERROR ANALYSIS

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In previous papers (S. ADHIKARI and J. WOODHOUSE 2001 *Journal of Sound and Vibration* **243**, 43–61; 63–88; S. ADHIKARI and J. WOODHOUSE 2002 *Journal of Sound and Vibration* **251**, 477–490) methods were proposed to obtain the coefficient matrix for a viscous damping model or a non-viscous damping model with an exponential relaxation function, from measured complex natural frequencies and modes. In all these works, it has been assumed that exact complex natural frequencies and complex modes are known. In reality, this will not be the case. The purpose of this paper is to analyze the sensitivity of the identified damping matrices to measurement errors. By using numerical and analytical studies it is shown that the proposed methods can indeed be expected to give useful results from moderately noisy data provided a correct damping model is selected for fitting. Indications are also given of what level of noise in the measured modal properties is needed to mask the true physical behaviour.

1. INTRODUCTION

The presence of noise of some kind is inevitable in any experimental analysis. In this paper, the effects of measurement noise are considered on the damping identification procedures developed in references [1–3]. These damping identification procedures rely on complex natural frequencies and mode shapes. For lightly damped structures, complex natural frequencies can be expressed in terms of undamped natural frequencies and modal Q-factors. Natural frequencies, Q-factors and mode shapes, collectively called the *modal parameters* or *modal data*, are commonly measured from vibration experiments. Typically, one measures time histories of the responses at different degrees of freedom together with time histories of the input forces. The transfer functions of a system are obtained by taking the ratios of the Fourier transforms of the output time histories to those of the corresponding input time histories. The modal parameters of a structure can be extracted from a set of transfer functions obtained in this way, by finding the poles and residues [4].

In the presence of noise, the time histories and consequently the measured transfer functions become noisy. This in turn makes the natural frequencies, Q-factors and mode shapes erroneous as they are extracted from the transfer functions. The effect of errors in the modal data on identification of viscous and non-viscous damping will be discussed using numerical simulations and analytical perturbation analysis. In order to simulate the effect of noise, strictly speaking, one should start by perturbing the time histories with random noise and then treat them as if they were obtained from a real experiment. Here, however, a more direct approach is adopted by perturbing the modal data directly. There are interesting issues associated with this. Recall that for lightly damped systems the imaginary parts of the complex modes are small compared with their corresponding real parts. Thus, the presence of random errors is likely to affect the imaginary parts more than the real parts. Clearly, if the identified matrix is very sensitive to errors in the imaginary parts of complex modes then these methods may turn out not to be very useful in practice.

2. ERROR ANALYSIS FOR VISCOUS DAMPING IDENTIFICATION

In reference [1], the method for viscous damping identification was developed after assuming that the complex natural frequencies and modes were known exactly. In this section, the influence of errors in the modal data on the identified viscous damping matrix is investigated. This can be done best by considering the numerical examples used in the earlier studies. Figure 1 shows the model system together with the numerical values used. The damping elements are associated between the 8th and 17th masses. Again, two non-viscous damping models, namely, exponential (model 1) and Gaussian (model 2), are used. The nature of both damping models is controlled by a single parameter γ, a non-dimensional characteristic time constant. The quantities are all defined in equation (28) of reference [1]. As noted in the earlier papers, if γ is close to zero the damping behaviour will be near-viscous, and *vice versa*.

2.1. NUMERICAL STUDY

2.1.1. *Noise assumptions*

In order to simulate the effect of noise, the modal data were perturbed by adding zero-mean Gaussian random noise to them. Numerical experiments have been performed by adding different levels of noise to the following four quantities (a list of nomenclature is given in Appendix A):

(1) real parts of the complex natural frequencies (noise level r_ω);
(2) imaginary parts of the complex natural frequencies (noise level r_ζ);
(3) real parts of the complex modes (noise level r_u);
(4) imaginary parts of the complex modes (noise level r_v).

Levels of random noise, denoted by the quantities r_ω, r_ζ, r_u and r_v, are specified by the standard deviations of the corresponding Gaussian distributions. These quantities are expressed as a percentage of their corresponding original (mean) values. The following cases are considered regarding the noise levels r_ω, r_ζ, r_u and r_v:

Case (a): $r_\omega = 2\%$, $r_\zeta = 0\%$, $r_u = 0\%$ and $r_v = 0\%$;
Case (b): $r_\omega = 0\%$, $r_\zeta = 10\%$, $r_u = 0\%$ and $r_v = 0\%$;

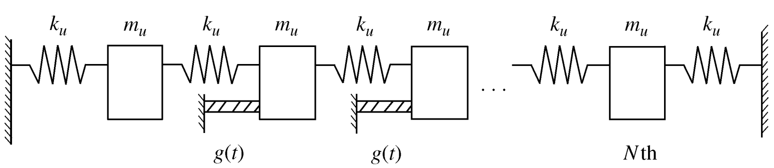

Figure 1. Linear array of N spring–mass oscillators used for numerical studies, $N = 30$, $m_u = 1$ kg, $k_u = 4 \times 10^5$ N/m.

Case (c): $r_\omega = 0\%$, $r_\zeta = 0\%$, $r_u = 10\%$ and $r_v = 0\%$;
Case (d): $r_\omega = 0\%$, $r_\zeta = 0\%$, $r_u = 0\%$ and $r_v = 10\%$;
Case (e): $r_\omega = 2\%$, $r_\zeta = 10\%$, $r_u = 5\%$ and $r_v = 10\%$.

In cases (a)–(d), noise in all but one quantity is assumed to be zero. These cases are considered to reveal the sensitivity of the fitted damping matrix when any one of the parameters is considered in isolation. It is assumed $r_\omega = 2\%$ while noise in all other quantities is assumed to be 10% in recognition of the fact that natural frequencies are generally the easiest quantity to measure accurately. Case (e) is a combination of noise in all the four parameters intended to represent physically realistic noise levels. In practice one would expect to obtain the real parts of complex natural frequencies and complex mode shapes more accurately than the corresponding imaginary parts. For this reason, we assume $r_\omega = 2\%$ and $r_u = 5\%$ while $r_\zeta = 10\%$ and $r_v = 10\%$.

2.1.2. *Results*

Figures 2 and 3 show the fitted viscous damping matrix for damping model 2 corresponding to noise cases (a) and (b) respectively. The value $\gamma = 0{\cdot}02$ is assumed. Observe that for both cases, the fitted viscous damping matrix is not very different from the exact coefficient matrix. In particular, the fitted viscous damping matrix for noise case (b) is very close to the noise-free case. These results show that 2% noise in the real part of the natural frequencies, and 10% noise in the modal damping factors do not influence the fitting procedure significantly. Numerical studies were also conducted with higher level noise in these quantities. It was observed that higher level of noise ($>5\%$) in the real part of the natural frequencies invalidates the fitting procedure. This, however, is not a serious limitation in practice. Interestingly, it was observed that the fitting procedure is rather insensitive to the noise in the modal damping factors. In this study, it was observed that with as much as 40% noise in the damping factors, the fitting procedure still produces acceptable results.

The fitted viscous damping matrices for damping model 2 in the presence of 10% noise in the real and imaginary parts of complex modes (noise cases (c) and (d)) are shown in

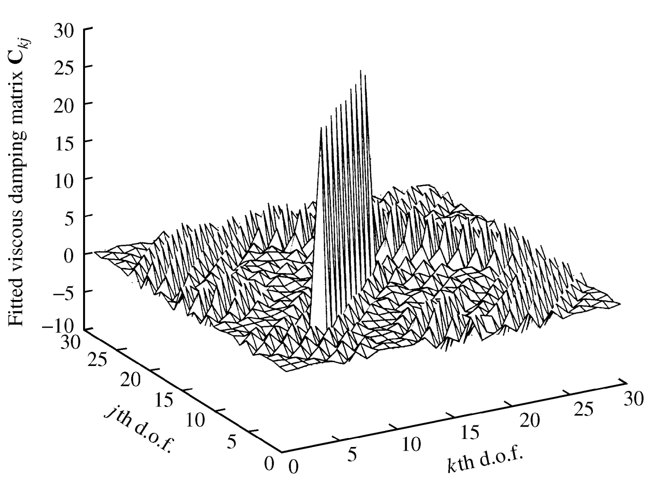

Figure 2. Fitted viscous damping matrix for $\gamma = 0{\cdot}02$, damping model 2, noise case (a).

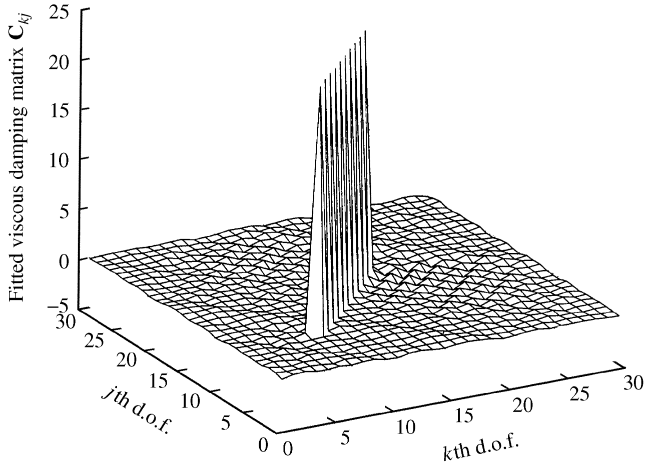

Figure 3. Fitted viscous damping matrix for $\gamma = 0{\cdot}02$, damping model 2, noise case (b).

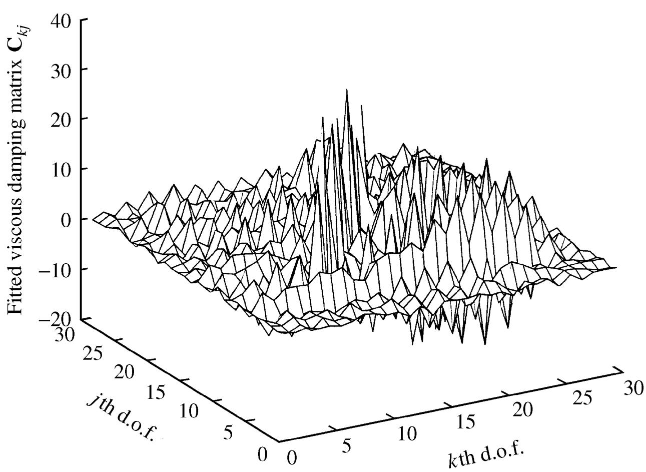

Figure 4. Fitted viscous damping matrix for $\gamma = 0{\cdot}02$, damping model 2, noise case (c).

Figures 4 and 5 respectively. For both cases the fitted viscous damping matrix is noisy and asymmetric. This, however, does not completely invalidate the fitting procedure: one can see the noisy peak along the diagonal indicating the spatial distribution of the damping. From this study it may be concluded that the presence of noise in the real and imaginary parts of the mode shapes affects the fitting procedure. Further, upon comparing Figures 2 and 3 with Figures 4 and 5 it is clear that the fitted viscous damping matrix is more sensitive to errors in the mode shapes than to errors in the complex natural frequencies.

Now consider the noise case (e), with noise levels assumed for all four quantities which approximate to the case one might expect in a real experimental situation. The fitted viscous damping matrix for damping model 2 corresponding to this noise case is shown in Figure 6. The fitted matrix is noisy, but a "peak" corresponding to the positions of the dampers can still be identified. Thus, for small values of γ, the method of viscous damping identification works reasonably well for realistic noise levels.

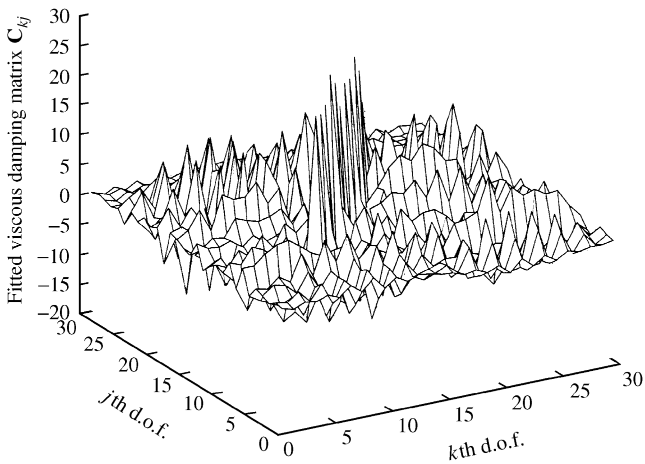

Figure 5. Fitted viscous damping matrix for $\gamma = 0{\cdot}02$, damping model 2, noise case (d).

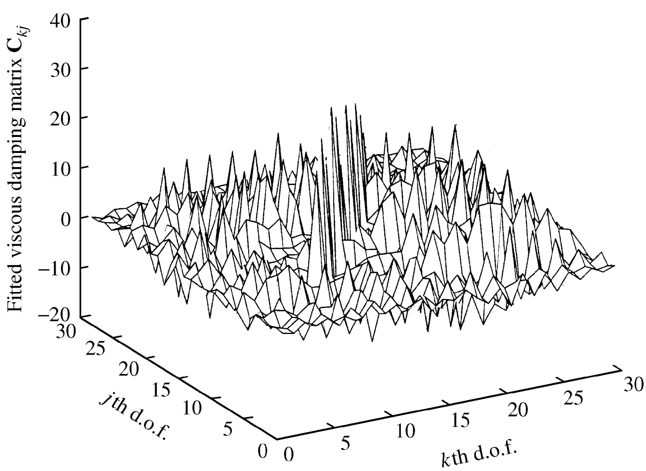

Figure 6. Fitted viscous damping matrix for $\gamma = 0{\cdot}02$, damping model 2, noise case (e).

For the results shown so far it is assumed that the complete set of modes is available. In practice it would be expected that some modes will not be available. For damping model 2 corresponding to the noise cases (e), Figures 7 and 8 show the fitted viscous damping matrix using the first 20 and first 10 modes respectively. Comparing Figures 6–8 one observes that reduction in the number of modes used for fitting the damping matrix results in a reduction of noise in the fitted matrix. The spatial resolution of the fitted matrix reduces with the reduction in the number of modes used for fitting. As a result, noise is sampled at fewer points and consequently the fitted matrix is less noisy. The same behaviour has been observed for other noise cases also.

Numerical experiments have been carried out using a wide range of damping models and parameter values. For small values of γ the results are, in general, quite similar to what has been shown already. For large values of γ, the results are somewhat different. For large values of γ with noise cases (a) and (b), the fitted damping matrix is not very different from the noise-free case reported in reference [1]. However, for noise cases (c)–(e), the damping

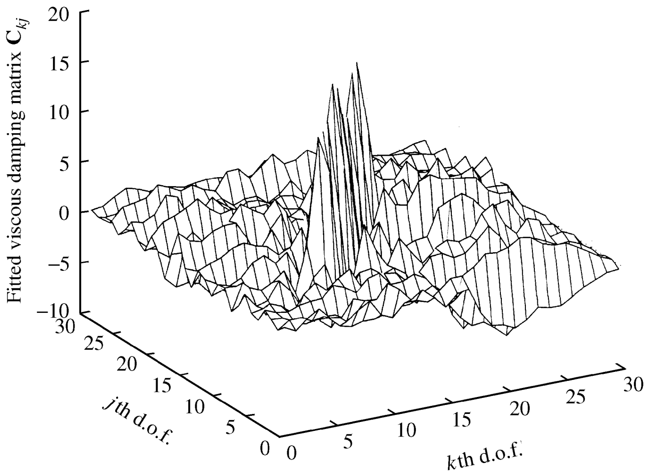

Figure 7. Fitted viscous damping matrix using first 20 modes for $\gamma = 0{\cdot}02$, damping model 2, noise case (e).

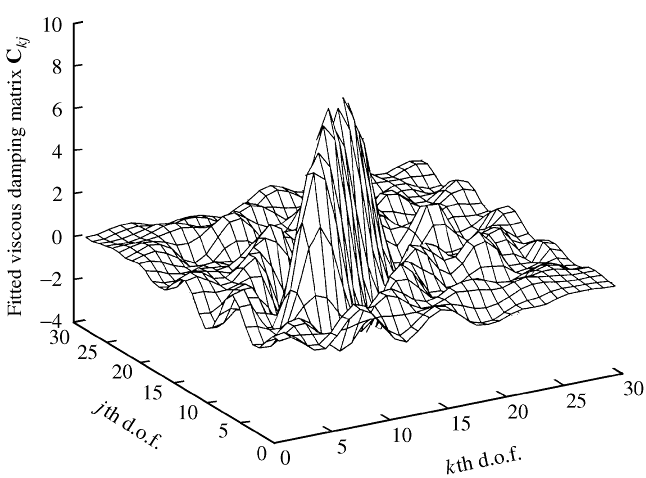

Figure 8. Fitted viscous damping matrix using first 10 modes for $\gamma = 0{\cdot}02$, damping model 2, noise case (e).

identification procedure yields unacceptable results for large values of γ. An illustration of this fact is shown here. Figure 9 shows the fitted viscous damping matrix for damping model 1 with $\gamma = 0{\cdot}5$ corresponding to the noise case (e). The fitted matrix is asymmetric and also it is no longer possible to see the spatial distribution of the damping. This result is obviously unacceptable. Similar results were obtained for noise cases (c) and (d) also.

From this numerical study, a general picture has emerged of the sensitivity of the fitted viscous damping to errors in the measured modal data. No matter what the value of γ, relative errors in the real and imaginary parts of the complex natural frequencies do not affect the fitted damping matrix very badly. However, errors in the real and imaginary parts of the complex modes affect the fitted damping matrix significantly. When γ is small, that is when the fitted damping model is not very different from the "true" damping model, the effect of errors in the mode shapes on the identified damping matrix is acceptable. But when γ is higher, that is when the fitted damping model is significantly different from the "true"

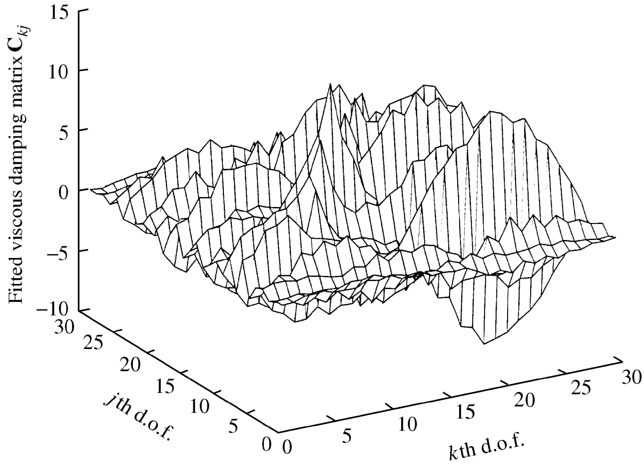

Figure 9. Fitted viscous damping matrix for $\gamma = 0{\cdot}5$, damping model 1, noise case (e).

damping model, error in the identified damping matrix due to errors in the mode shapes becomes unacceptable.

2.2. PERTURBATION ANALYSIS

These findings are further explored by a simple perturbation analysis. Suppose that the real and imaginary parts of the complex modal matrix can be expressed as

$$\mathbf{U} = \mathbf{U}_0 + \mathbf{\Delta U} \quad \text{and} \quad \mathbf{V} = \mathbf{V}_0 + \mathbf{\Delta V}, \tag{1}$$

where $\mathbf{\Delta}(\cdot)$ denotes the "error part" and $(\cdot)_0$ denotes the "error-free part". Note that, in general, $\mathbf{U}$ and $\mathbf{V}$ are *not* square matrices. In reference [1] it was shown that the viscous damping matrix in the physical co-ordinates can be obtained from the following relationships:

$$\mathbf{C} = [(\mathbf{U}^{\mathrm{T}}\mathbf{U})^{-1}\,\mathbf{U}^{\mathrm{T}}]^{\mathrm{T}}\,\mathbf{C}'[(\mathbf{U}^{\mathrm{T}}\mathbf{U})^{-1}\,\mathbf{U}^{\mathrm{T}}], \tag{2}$$

where

$$\mathbf{C}' = \mathbf{B\Omega} - \mathbf{\Omega}^2\mathbf{B\Omega}^{-1}, \quad \mathbf{C}'_{jj} = 2\Im(\hat{\lambda}_j) \tag{3}$$

and

$$\mathbf{B} = [\mathbf{U}^{\mathrm{T}}\mathbf{U}]^{-1}\,\mathbf{U}^{\mathrm{T}}\mathbf{V}. \tag{4}$$

In view of equation (1), from the preceding equation the matrix constants $\mathbf{B}$ can be expressed as

$$\begin{aligned}\mathbf{B} &= [(\mathbf{U}_0 + \mathbf{\Delta U})^{\mathrm{T}}\,(\mathbf{U}_0 + \mathbf{\Delta U})]^{-1}\,(\mathbf{U}_0 + \mathbf{\Delta U})^{\mathrm{T}}\,(\mathbf{V}_0 + \mathbf{\Delta V})\\ &= [\mathbf{I} - (\mathbf{U}_0^{\mathrm{T}}\mathbf{U}_0)^{-1}\,(\mathbf{\Delta U}^{\mathrm{T}}\mathbf{U}_0 + \mathbf{U}_0^{\mathrm{T}}\mathbf{\Delta U} + \mathbf{\Delta U}^{\mathrm{T}}\mathbf{\Delta U})]^{-1}\\ &\quad\times(\mathbf{U}_0^{\mathrm{T}}\mathbf{U}_0)^{-1}\,(\mathbf{U}_0 + \mathbf{\Delta U})^{\mathrm{T}}(\mathbf{V}_0 + \mathbf{\Delta V}).\end{aligned} \tag{5}$$

Neglecting second or higher order terms involving $\mathbf{\Delta}$, one may approximate the above relationship by

$$\mathbf{B} = \mathbf{B}_0 + \Delta\mathbf{B}, \tag{6}$$

where

$$\mathbf{B}_0 = (\mathbf{U}_0^{\mathrm{T}}\mathbf{U}_0)^{-1}\mathbf{U}_0^{\mathrm{T}}\mathbf{V}_0 \tag{7}$$

and

$$\Delta\mathbf{B} \approx [(\mathbf{U}_0^{\mathrm{T}}\mathbf{U}_0)^{-1}\,\Delta\mathbf{U}^{\mathrm{T}}\mathbf{V}_0 - (\mathbf{U}_0^{\mathrm{T}}\mathbf{U}_0)^{-1}\,(\Delta\mathbf{U}^{\mathrm{T}}\mathbf{U}_0 + \mathbf{U}_0^{\mathrm{T}}\Delta\mathbf{U})(\mathbf{U}_0^{\mathrm{T}}\mathbf{U}_0)^{-1}\,\mathbf{U}_0^{\mathrm{T}}\mathbf{V}_0] + [(\mathbf{U}_0^{\mathrm{T}}\mathbf{U}_0)^{-1}\mathbf{U}^{\mathrm{T}}\Delta\mathbf{V}]. \tag{8}$$

Now, express the errors in the real and imaginary parts of the complex natural frequencies as

$$\mathbf{\Omega} = \mathbf{\Omega}_0 + \Delta\mathbf{\Omega} \quad \text{and} \quad \boldsymbol{\zeta} = \boldsymbol{\zeta}_0 + \Delta\boldsymbol{\zeta}. \tag{9}$$

By using these equations and equation (6), the damping matrix in the modal co-ordinates can be obtained from equation (3) as

$$\mathbf{C}' = (\mathbf{B}_0 + \Delta\mathbf{B})(\mathbf{\Omega}_0 + \Delta\mathbf{\Omega}) - (\mathbf{\Omega}_0 + \Delta\mathbf{\Omega})^2(\mathbf{B}_0 + \Delta\mathbf{B})(\mathbf{\Omega}_0 + \Delta\mathbf{\Omega})^{-1}. \tag{10}$$

Upon neglecting second or higher order terms involving $\mathbf{\Delta}$, the above relationship can be approximated by

$$\mathbf{C}' = \mathbf{C}_0' + \Delta\mathbf{C}', \tag{11}$$

where

$$\mathbf{C}_0' = \mathbf{B}_0\mathbf{\Omega}_0 - \mathbf{\Omega}_0^2\mathbf{B}_0\mathbf{\Omega}_0^{-1} \tag{12}$$

and

$$\Delta\mathbf{C}' \approx (\mathbf{\Omega}_0\Delta\mathbf{B}) + (\mathbf{B}_0\Delta\mathbf{\Omega} - 2\mathbf{\Omega}_0\Delta\mathbf{\Omega}\mathbf{B}_0\mathbf{\Omega}_0^{-1} - \mathbf{\Omega}_0\Delta\mathbf{\Omega} + \mathbf{\Omega}_0^2\mathbf{B}_0\mathbf{\Omega}_0^{-2}\Delta\mathbf{\Omega}). \tag{13}$$

Substituting $\mathbf{B}_0$ and $\Delta\mathbf{B}$ from equations (7) and (8) into the preceding expression one may write

$$\Delta\mathbf{C}' \approx \Delta\mathbf{C}'_{\Delta U} + \Delta\mathbf{C}'_{\Delta V} + \Delta\mathbf{C}'_{\Delta\Omega}, \tag{14}$$

where $\Delta\mathbf{C}'_{(\cdot)}$, the error in $\mathbf{C}'$ due to error in $(\cdot)$, are given by

$$\Delta\mathbf{C}'_{\Delta U} = \mathbf{\Omega}_0(\mathbf{U}_0^{\mathrm{T}}\mathbf{U}_0)^{-1}\,\Delta\mathbf{U}^{\mathrm{T}}\mathbf{V}_0 - \mathbf{\Omega}_0(\mathbf{U}_0^{\mathrm{T}}\mathbf{U}_0)^{-1}(\Delta\mathbf{U}^{\mathrm{T}}\mathbf{U}_0 + \mathbf{U}_0^{\mathrm{T}}\Delta\mathbf{U})(\mathbf{U}_0^{\mathrm{T}}\mathbf{U}_0)^{-1}\mathbf{U}_0^{\mathrm{T}}\mathbf{V}_0, \tag{15}$$

$$\Delta\mathbf{C}'_{\Delta V} = \mathbf{\Omega}_0(\mathbf{U}_0^{\mathrm{T}}\mathbf{U}_0)^{-1}\mathbf{U}^{\mathrm{T}}\Delta\mathbf{V} \tag{16}$$

and

$$\Delta \mathbf{C}'_{\Delta\Omega} = (\mathbf{U}_0^{\mathrm{T}}\mathbf{U}_0)^{-1}\,\mathbf{U}_0^{\mathrm{T}}\mathbf{V}_0\Delta\boldsymbol{\Omega} - 2\boldsymbol{\Omega}_0\Delta\boldsymbol{\Omega}(\mathbf{U}_0^{\mathrm{T}}\mathbf{U}_0)^{-1}\,\mathbf{U}_0^{\mathrm{T}}\mathbf{V}_0\boldsymbol{\Omega}_0^{-1} - \boldsymbol{\Omega}_0\Delta\boldsymbol{\Omega}$$

$$+ \boldsymbol{\Omega}_0^2(\mathbf{U}_0^{\mathrm{T}}\mathbf{U}_0)^{-1}\,\mathbf{U}_0^{\mathrm{T}}\mathbf{V}_0\boldsymbol{\Omega}_0^{-2}\Delta\boldsymbol{\Omega}. \tag{17}$$

The above expressions are valid for the off-diagonal terms of $\mathbf{C}'$ only. For the diagonal terms, from equation (3) one simply obtains

$$\Delta \mathbf{C}'_{jj} = 2\Delta\zeta_j. \tag{18}$$

Equation (14) separates the contributions to the error in the off-diagonal terms of $\mathbf{C}'$ arising from $\Delta\mathbf{U}$, $\Delta\mathbf{V}$ and $\Delta\boldsymbol{\Omega}$. Using equations (15)–(18) one can find the error in the modal damping matrix due to errors in the modal parameters.

The computed results from the previous section can be used to obtain numerical values. A convenient measure is the l_2 matrix norm. It was observed that for the same level of noise in the real and imaginary parts of the complex modes, the norm of $\Delta\mathbf{C}'_{\Delta U}$ is close to the norm of $\Delta\mathbf{C}'_{\Delta V}$. For noise case (e), $\|\Delta\mathbf{C}'_{\Delta V}\|/\|\Delta\mathbf{C}'_{\Delta U}\| = 2{\cdot}66$. This implies that when the error in $\mathbf{V}$ is twice that in $\mathbf{U}$, the resulting error in the modal damping matrix due to the error in $\mathbf{V}$ is approximately 2·66 times that due to $\mathbf{U}$. From equation (2) observe that only $\mathbf{U}$ is required to obtain $\mathbf{C}$ from $\mathbf{C}'$. For this reason the effect of errors in $\mathbf{U}$ on the fitted viscous damping matrix $\mathbf{C}$ may finally be more than that due to errors in $\mathbf{V}$.

3. ERROR ANALYSIS FOR NON-VISCOUS DAMPING IDENTIFICATION

3.1. NUMERICAL STUDY

Identification of an exponential damping model requires the relaxation parameter as well as the coefficient damping matrix [2]. Numerical investigations using the five noise cases introduced earlier have been carried out in this case also. Again, the locally reacting damping model shown in Figure 1 was used. It was observed that the coefficient matrix for the five noise cases behaves similar to the viscous damping matrix discussed in the last section. It is sufficient to show representative results, for noise case (e).

3.1.1. *Results for small* γ

Figure 10(a) shows the values of fitted γ for different modes for noise case (e) with damping model 2 and $\gamma = 0{\cdot}02$. The pattern of fitted γ for different modes is quite different from the corresponding noise-free case (obtained in reference [2]). For some modes the values of γ become negative, which is a non-physical result. Unlike the noise-free case, where the value of γ corresponding to the first mode turned out to be the most accurate, here it becomes the most inaccurate. In fact, values of γ for the higher modes tend towards the "true" value, as opposed to those for the lower modes as may be observed for the noise-free case. The explanation of these observations is not clear; a perturbation analysis in the next section attempts to throw some light on these issues.

The fitted coefficient matrix corresponding to this case is shown in Figure 10(b). The nature of this fitted coefficient matrix is recognizably similar to the fitted viscous damping matrix for the corresponding case shown earlier in Figure 6. The effect of modal truncation on the fitted coefficient matrix was investigated, and the result turned out to be very similar to that shown in Figures 7 and 8, the corresponding cases for viscous damping

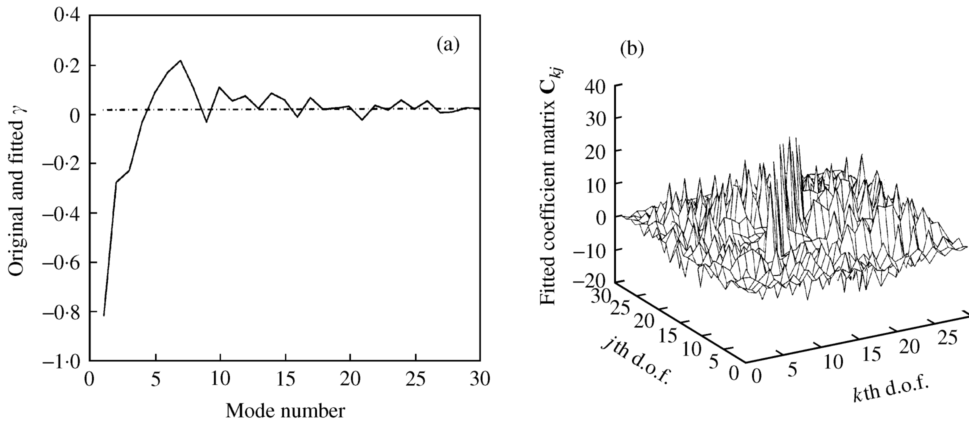

Figure 10. (a) Values of identified γ_j for damping model 2, noise case (e); ——, fitted γ_j for different modes; - · - · -, original $\gamma = 0{\cdot}02$; (b) fitted coefficient matrix of exponential model for $\gamma = 0{\cdot}02$, damping model 2, noise case (e).

identification. Recall from reference [2] that only one value of γ from those shown for different modes in Figure 10(a) should be used to obtain the coefficient matrix. It was observed that the choice of γ did not affect the fitted coefficient matrix to a great extent. For any choice of γ in the range shown in Figure 10(a), the fitted coefficient matrix in Figure 10(b) more or less remains the same. This is probably due to the fact that the variation in the values of γ is small in this case.

3.1.2. *Results for larger γ*

Now consider a case when γ is large, so that the damping model is far from viscous. Figure 11(a) shows the values of fitted γ for different modes for the noise case (e) with damping model 1 and $\gamma = 0{\cdot}5$. Were the modal data exact, the values of γ for different modes would be constant at 0·5 since the fitted damping model is correctly “identified” as exponential (see reference [2]). However, as in the case shown in Figure 10(a), the values of fitted γ vary considerably for different modes. For the first mode the fitted γ turns out to be negative, which is non-physical.

Figure 11(b) shows the fitted coefficient matrix corresponding to this case. The fitted matrix is noisy, but it is symmetric and the nature of damping may still be understood clearly. The effect of modal truncation on the fitted coefficient matrix for this case was also investigated, and the result was found to be similar to those shown in Figures 7 and 8. The fitted viscous damping matrix corresponding to this case was shown before in Figure 9. The result shown in Figure 11(b) is clearly much better than that in Figure 9. This shows that if a correct model is fitted, then the spatial distribution of the damping can be identified even if γ is large and modal data are noisy and incomplete.

When the fitting procedure was repeated for damping model 2 with large γ it was found that the resulting fitted coefficient matrix becomes unacceptable, like that shown in Figure 9. This implies that if the fitted damping model is not close to the true model, the damping identification method proposed here produces unacceptable results in the presence of noise in the modal data. Thus, conversely, if an asymmetric matrix as shown in Figure 9 is obtained by using the damping identification procedure (whether the fitted model is viscous or non-viscous), it may be concluded that the fitted damping model is not close to the true damping model of the system. However, if a correct model of damping is fitted

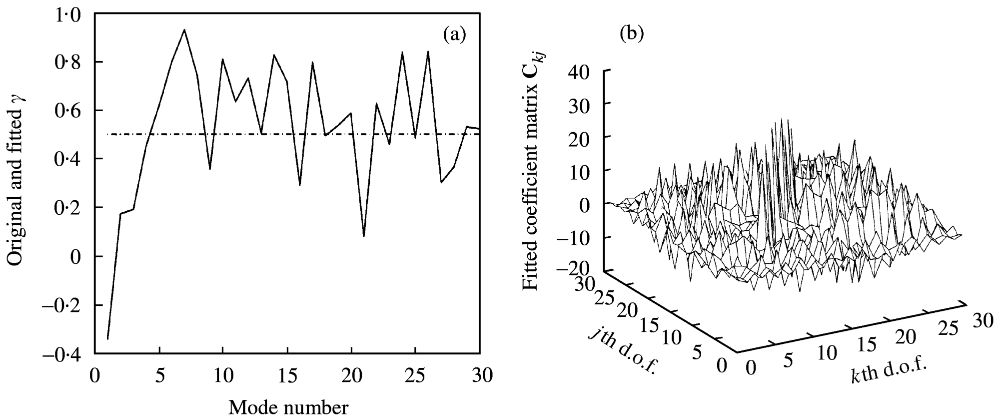

Figure 11. (a) Values of identified γ_j for damping model 1, noise case (e); ——, fitted γ_j for different modes; -·-·-, original $\gamma = 0{\cdot}5$; (b) fitted coefficient matrix of exponential model for $\gamma = 0{\cdot}5$, damping model 1, noise case (e).

(viscous or non-viscous), the spatial distribution and the nature (e.g., local or non-local) of the damping mechanism can be identified using this procedure in the presence of noise.

3.2. PERTURBATION ANALYSIS

Some explanation of the facts regarding the fitted values of γ can be given by a simple perturbation analysis. The procedure for fitting of the coefficient matrix is similar to that for the viscous damping matrix and will not be discussed again. Suppose for the jth mode

$$\mathbf{u}_j = \mathbf{u}_{0_j} + \mathbf{\Delta u}_j, \quad \mathbf{v}_j = \mathbf{v}_{0_j} + \mathbf{\Delta v}_j, \quad \text{and} \quad \omega_j = \omega_{0_j} + \Delta\omega_j. \tag{19}$$

Now, following reference [2], the expression for the normalized characteristic time constant for the jth mode, γ_j, can be written as

$$\gamma_j = 1/T_{min}\mu_j, \tag{20}$$

where T_{min} is the minimum time period of the system and the relaxation parameter for the jth mode, μ_j, is

$$\mu_j = \omega_j \frac{\mathbf{v}_j^{\mathrm{T}}\mathbf{M}\mathbf{v}_j}{\mathbf{v}_j^{\mathrm{T}}\mathbf{M}\mathbf{u}_j}. \tag{21}$$

For the purpose of generality, assume the error in the mass matrix as

$$\mathbf{M} = \mathbf{M}_0 + \mathbf{\Delta M}. \tag{22}$$

By using equations (19) and (22), from equation (21), $1/\mu_j$ can be written as

$$\frac{1}{\mu_j} = \frac{1}{(\omega_{0_j} + \Delta\omega_j)} \frac{(\mathbf{v}_{0_j} + \mathbf{\Delta v}_j)^{\mathrm{T}}(\mathbf{M}_0 + \mathbf{\Delta M})(\mathbf{v}_{0_j} + \mathbf{\Delta v}_j)}{(\mathbf{v}_{0_j} + \mathbf{\Delta v}_j)^{\mathrm{T}}(\mathbf{M}_0 + \mathbf{\Delta M})(\mathbf{u}_{0_j} + \mathbf{\Delta u}_j)}. \tag{23}$$

Upon neglecting all terms of second or higher order involving $\mathbf{\Delta}$, equation (20) together with the above expression may be approximated as

$$\gamma_j \approx \gamma_{0_j} + \gamma_{0_j}(\Delta\gamma_j^{(1)} + \Delta\gamma_j^{(2)} + \Delta\gamma_j^{(3)} + \Delta\gamma_j^{(4)}), \tag{24}$$

where

$$\gamma_{0_j} = \frac{1}{T_{min}}\frac{1}{\omega_{0_j}}\frac{\mathbf{v}_{0_j}^{\mathrm{T}}\mathbf{M}_0\mathbf{u}_{0_j}}{\mathbf{v}_{0_j}^{\mathrm{T}}\mathbf{M}_0\mathbf{v}_{0_j}}, \tag{25}$$

$$\Delta\gamma_j^{(1)} = -\,\Delta\omega_j/\omega_{0j}, \tag{26}$$

$$\Delta\gamma_j^{(2)} = \frac{\mathbf{\Delta v}_j^{\mathrm{T}}\mathbf{M}_0\mathbf{u}_{0_j} + \mathbf{v}_{0_j}^{\mathrm{T}}\mathbf{M}_0\mathbf{\Delta u}_j}{\mathbf{v}_{0_j}^{\mathrm{T}}\mathbf{M}_0\mathbf{u}_{0_j}}, \tag{27}$$

$$\Delta\gamma_j^{(3)} = -\,2\,\mathbf{v}_{0_j}^{\mathrm{T}}\mathbf{M}_0\mathbf{\Delta v}_j/\mathbf{v}_{0_j}^{\mathrm{T}}\mathbf{M}_0\mathbf{v}_{0_j}, \tag{28}$$

$$\Delta\gamma_j^{(4)} = \frac{\mathbf{v}_{0_j}^{\mathrm{T}}\mathbf{\Delta M}\mathbf{u}_{0_j}}{\mathbf{v}_{0_j}^{\mathrm{T}}\mathbf{M}_0\mathbf{u}_{0_j}} - \frac{\mathbf{v}_{0_j}^{\mathrm{T}}\mathbf{\Delta M}\mathbf{v}_{0_j}}{\mathbf{v}_{0_j}^{\mathrm{T}}\mathbf{M}_0\mathbf{v}_{0_j}}. \tag{29}$$

Equation (24) describes how the values of γ_j are affected by the error in the modal data and mass matrix. Observe that error in the real and imaginary parts of the complex modes as well as error in the natural frequencies and the mass matrix introduce error in the estimate of the relaxation parameter. The nature of the variation of γ_j about the "true" value (γ_{0_j}), as seen in Figures 10(a) and 11(a), clearly depends on the terms $\Delta\gamma_j^{(r)}$, $r = 1, 2, 3, 4$ in equation (24).

Results from the numerical study reported in the previous section can be used to shed light on these error terms. The term $\Delta\gamma_j^{(4)}$ is zero because $\mathbf{\Delta M}$ was assumed to be zero, but the other three terms can be computed. Figure 12 shows the values of $\Delta\gamma_j^{(1)}$, $\Delta\gamma_j^{(2)}$ and $\Delta\gamma_j^{(3)}$ plotted against mode number j. It is immediately clear that $\Delta\gamma_j^{(2)}$ dominates the other two, for all values of j. This term is responsible for virtually the entire error shown in Figure 10(a). The reason this term is so big is apparent from equation (27): the expression has $\mathbf{v}_{0_j}^{\mathrm{T}}\mathbf{M}_0\mathbf{u}_{0_j}$ in the denominator. This expression is zero for a viscous damping model [2], and can be expected to be nearly zero for a near-viscous model. Of course, for such a model the numerical value of γ should be small, but the fractional error in this small value becomes very large in the presence of errors in the complex modes. The error can even become larger than the error-free value, and this accounts for the occasional appearance of non-physical negative values of γ.

Another obvious feature of Figures 10(a) and 12 is that the errors are bigger for the lower values of j. This can be explained qualitatively by the mechanism just described. So far a given damping model has been characterized as more or less close to viscous in terms of a single number, γ, which was normalized by the shortest natural period T_{min}. However, this is an over-simplification. In reality, one could define a mode-by-mode version of this parameter, normalized by the natural period $2\pi/\omega_j$. This would immediately suggest that a given damping model is likely to appear "more viscous" to the low modes of the system, and "less viscous" to the higher modes. In terms of the complex modes, this means that one might expect the low modes to satisfy the condition $\mathbf{v}_{0_j}^{\mathrm{T}}\mathbf{M}_0\mathbf{u}_{0_j} = 0$ to some accuracy, thus leading to very high values of $\Delta\gamma_j^{(2)}$. Higher modes will be constrained less by this requirement, and thus are likely to have lower errors, exactly as seen in the computed results.

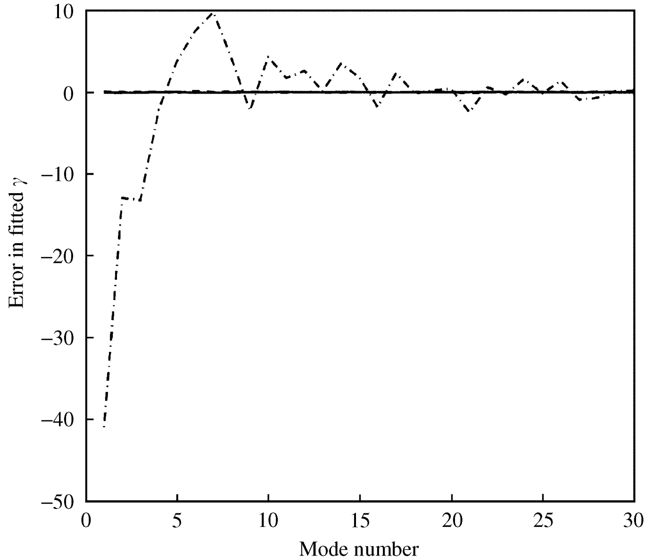

Figure 12. Errors in identified γ_j for damping model 2, noise case (e); ——, $\Delta\gamma_j^{(1)}$; –·–, $\Delta\gamma_j^{(2)}$; – – –, $\Delta\gamma_j^{(3)}$.

4. CONCLUSIONS

The effects of noise in the modal data on the identified damping properties when using the damping identification methods proposed in references [1–3] have been investigated by using numerical and analytical perturbation methods. The effect was considered of noise in four quantities, namely the real and imaginary parts of the complex natural frequencies and complex mode shapes. Both viscous and non-viscous damping models were considered for the identification. For the identification of a viscous damping matrix it was observed that the result is sensitive to errors in the real and imaginary parts of the complex modes, but very much less sensitive to errors in the complex natural frequencies. Modal truncation reduces the resolution, and actually can reduce noise in the identified damping matrix. For the identification of a non-viscous damping model, the relaxation parameter is sensitive to errors in the mode shapes and real part of the complex natural frequencies while the associated coefficient damping matrix behaves in very much the same manner as the viscous damping matrix. It was found that the value of the relaxation parameter is particularly error-prone when estimated from a system with a near-viscous damping model, and that the problem is most severe for the low-frequency modes. Fortunately, in practice this problem may be partially compensated by the fact that low-frequency modes are likely to be more accurately measured than higher modes.

It was shown that when the fitted damping model is close to the true damping model (viscous or non-viscous) of the system, it is possible to identify the correct damping distribution even if the modal data are moderately noisy. However, if the fitted damping model is far from the true damping model, a physically realistic damping matrix can only be obtained if the modal data are sufficiently accurate.

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APPENDIX A: NOMENCLATURE

$\mathbf{C}$	viscous damping matrix
$\mathbf{C}'$	viscous damping matrix in the modal co-ordinates
$\mathbf{M}$	mass matrix
$\mathbf{u}_j$	real part of the complex modes
$\mathbf{U}$	matrix containing $\mathbf{u}_j$
$\mathbf{v}_j$	imaginary part of the complex modes
$\mathbf{V}$	matrix containing $\mathbf{v}_j$
λ_j	jth complex natural frequency of the system
γ	non-dimensional characteristic time constant
ω_j	real part of the complex natural frequencies
$\boldsymbol{\Omega}$	diagonal matrix containing ω_j
ζ_j	imaginary part of the complex natural frequencies
$\boldsymbol{\varsigma}$	vector containing ζ_j
μ_j	estimated relaxation parameter for the jth mode
$r(\bullet)$	noise level in $(\bullet)$
$\boldsymbol{\Delta}(\bullet)$	error part of $(\bullet)$
$(\bullet)_0$	error-free part of $(\bullet)$
$\Im(\bullet)$	imaginary part of $(\bullet)$
$\|(\bullet)\|$	l_2 matrix norm of $(\bullet)$