DEFINITION OF THE ABELIAN, THE TWO HYPOABELIAN, AND RELATED LINEAR GROUPS AS QUOTIENT-GROUPS OF THE GROUPS OF ISOMORPHISMS OF CERTAIN ELEMENTARY GROUPS*

BY

LEONARD EUGENE DICKSON

1. The present paper aims to give a natural definition of the Abelian and two hypoabelian groups, which moreover preserves the essence of JORDAN'S definition based upon his important, but artificial, conception of "exposants d'échange" (*Traité*, pp. 179, 195). A second formal definition, by means of the invariants (11), (14) and (21) below, may be obtained from JORDAN (l. c. p. 217 and pp. 438–440).

Following in the main the developments of JORDAN (l. c. pp. 420–447, in particular) on the construction of solvable groups, we may obtain the above groups as quotient-groups of the groups of isomorphisms of certain elementary groups.† In May, 1898, I communicated such a treatment to Professor MOORE, who emphasized the desirability of presenting the definitions thus obtained for the Abelian and hypoabelian groups in two distinct ways: viz., from the standpoint‡ of JORDAN'S linear groups and from the standpoint§ of abstract groups.

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For cross-reference and abstract, cf. Bulletin, April, 1899, pp. 331-2 (where p. 331, line 2 from bottom, for *holoedrically* one should read *meriedrically*).

† For an outline, see §§ 8–10 below. A different aim being in view, the details of the calculation become much simpler than those of JORDAN.

‡ For example, JORDAN (l. c. §§ 118–119) obtains the *non-homogeneous* linear group as the largest group of literal substitutions transforming into itself the commutative literal group readily defined by means of the linear substitutions modulo p,

$$x_i' \equiv x_i + a_i \qquad (i = 1, \cdots, n).$$

§ The idea of the group of isomorphisms of an *abstract* group seems to have been arrived at independently by MOORE and HÖLDER, the former applying it to define the linear *homogeneous* group as the group of transformation into itself of the Abelian group of type $\{1, 1, \cdots, 1\}_n$. See his article in the Bulletin of the American Mathematical Society, Nov., 1895.

I must however refer to the paper by ENRICO BETTI, *Sopra la teorica delle sostituzioni*, Annali di Scienze Matematiche e Fisiche, pp. 5–34, 1855. He obtains a group equivalent to the *homogeneous* linear group as the "massimo Moltiplicatore" of a cyclic group of order

Quite recently I have made the investigation from the latter point of view and have found the method so much simpler and more desirable that I have abandoned my earlier work, which was not fully complete, and give in § § 8–10 a mere outline of it.

2. Consider the group F generated by the operators of period p (p being prime),

$$\Theta,\ A_i,\ B_i \qquad (i=1,\cdots,m)$$

with the generational relations,*

$$(1)\qquad \Theta^p=1,\ A_i^p=1,\ B_i^p=1 \qquad (i=1,\cdots,m)$$

$$(2)\qquad \begin{cases} \Theta A_i = A_i\Theta, & \Theta B_i = B_i\Theta \qquad (i=1,\cdots,m) \\ A_iA_j = A_jA_i, & B_iB_j = B_jB_i \qquad (i,j=1,\cdots,m) \\ A_iB_j = B_jA_i, & A_iB_i = \Theta B_iA_i \qquad \left(\begin{matrix} i,j=1,\cdots,m \\ i\neq j\end{matrix}\right). \end{cases}$$

It follows at once that

$$(3)\qquad A_i^xB_i^y = \Theta^{xy}B_i^yA_i^x,$$

and hence, in general, if we set

$$S\equiv \Theta^t A_1^{x_1}B_1^{y_1}\cdots A_m^{x_m}B_m^{y_m},$$

$$\Sigma\equiv \Theta^\tau A_1^{\xi_1}B_1^{\eta_1}\cdots A_m^{\xi_m}B_m^{\eta_m},$$

$$(4)\qquad S\Sigma = \Theta^{\sum_{i=1}^{m}{}'(x_i\eta_i-\xi_iy_i)}\Sigma S,$$

a relation which includes all of the relations (2).

By application of (2) any operator of F may be put into the form S above. Since A_i is not expressible in terms of the remaining generators, and likewise for B_i, it follows that $S=1$ if and only if

$$x_i\equiv y_i\equiv t\equiv 0 \pmod p \qquad (i=1,\cdots,m)$$

Further, $S=\Sigma$ requires that

$$x_i\equiv\xi_i,\quad y_i\equiv\eta_i,\quad t\equiv\tau \pmod p.$$

Indeed, $S\Sigma^{-1}=1$ may be given the form

$$\Theta^{t-\tau+\sum_{i=1}^{m}\xi_i(y_i-\eta_i)}A_1^{x_1-\xi_1}B_1^{y_1-\eta_1}\cdots A_m^{x_m-\xi_m}B_m^{y_m-\eta_m}=1.$$

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p^ν. BETTI observes (p. 34) that this result was given by GALOIS without proof. Although I have not found the passage in the papers of GALOIS referred to by BETTI, nevertheless I am confident that GALOIS had the conception of the group of isomorphisms of the Abelian group of type $\{1,1,\cdots 1\}_n$, at least for $n=2$. See the passage at the bottom of p. 58 of the fragment preserved to us of the posthumous paper, *Des équations primitives qui sont soluble par radicaux*, *Oeuvres Mathématiques D' Évariste Galois*, Paris, 1897.

* The relation $\Theta^p=1$ follows from (3); thus $A_iB_i^p=\Theta^pB_i^pA_i$. We may evidently drop Θ from the list of generators.

It follows from (4) that the powers of Θ are the only operators of F which are commutative with every operator of F.

We may prove by induction the formula

$$S^k = \Theta^{kt - \frac{1}{2}k(k-1)\sum_{i=1}^{m} x_i y_i} A_1^{kx_1} B_1^{ky_1} \cdots A_m^{kx_m} B_m^{ky_m}. \tag{5}$$

Hence, for $p > 2$, every operator of F has the period p and, for $p = 2$, the period 2 or 4, viz.,

$$S^2 = \Theta^{\Sigma x_i y_i}.$$

3. Every isomorphism I of the group F into itself is obtained by introducing a new set of generators

$$\Theta',\ A'_i,\ B'_i \qquad (i = 1, \cdots, m)$$

satisfying the relations (1) and (2) and capable of generating the entire group F. We may set

$$\begin{cases} \Theta' = \Theta^s \\ A'_i = \Theta^{l_i} A_1^{a_{1i}} B_1^{b_{1i}} \cdots A_m^{a_{mi}} B_m^{b_{mi}} \\ B'_i = \Theta^{r_i} A_1^{c_{1i}} B_1^{d_{1i}} \cdots A_m^{c_{mi}} B_m^{d_{mi}} \end{cases} \tag{6}$$

where $s \not\equiv 0 \pmod{p}$.

Since Θ', A'_i, B'_i shall generate F, we must be able, for arbitrarily given values of t', x'_i, y'_i, to determine t, x_i, y_i so that

$$\Theta^{t'} A_1^{x'_1} B_i^{y'_1} \cdots \equiv \Theta'^{t} A_1'^{x_1} B_1'^{y_1} \cdots \tag{7}$$

Replacing Θ', A'_i, B'_i by their values (6) and applying (2) to bring to the left of the product the operators A_1, and next the B_1, etc., we obtain the condition

$$\Theta^{t'} A_1^{x'_1} B_1^{y'_1} \cdots \equiv \Theta^{ts + \sigma} A_1^{\sum_{j=1}^{m}(a_{1j}x_j + c_{1j}y_j)} B_1^{\sum_{j=1}^{m}(b_{1j}x_j + d_{1j}y_j)} \cdots \tag{8}$$

where the exact expression for σ is immaterial. It follows from §2 that the corresponding exponents must be congruent modulo p. Thus t is determined uniquely from

$$t' \equiv ts + \sigma \pmod{p}.$$

We have further the following $2m$ conditions

$$\begin{cases} x'_i \equiv \sum_{j=1}^{m}(a_{ij}x_j + c_{ij}y_j) \\ y'_i \equiv \sum_{j=1}^{m}(b_{ij}x_j + d_{ij}y_j) \end{cases} \pmod{p} \qquad (i = 1, \cdots, m) \tag{9}$$

For x_i', y_i' arbitrary, the congruences (9) can be solved for x_i, y_i (and then uniquely) if and only if the determinant

(10) $$D \equiv \begin{vmatrix} a_{11} & c_{11} & \cdot & \cdot \\ b_{11} & d_{11} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} \not\equiv 0 \pmod{p}.$$

We next seek the conditions under which the new generators (6) satisfy the generational relations (2) or the equivalent single relation (4). S denoting the operator (7) and Σ the analogous operator

$$\Sigma \equiv \Theta^{\tau'} A_1^{\xi_1'} B_1^{\eta_1'} \cdots \equiv \Theta'^{\tau} A_1'^{\xi_1} B_1'^{\eta_1} \cdots,$$

the relation (4) requires

$$\Theta^{\sum_{i=1}^{m}(x_i'\eta_i' - \xi_i'y_i')} = \Theta'^{\sum_{i=1}^{m}(x_i\eta_i - \xi_i y_i)}.$$

Since $\Theta' = \Theta^s$, the substitution (9) and the cogredient substitution on the ξ_i, η_i must have the simultaneous relative invariant

$$\sum_{i=1}^{m}(x_i\eta_i - \xi_i y_i),$$

that is, must satisfy the relation

(11) $$\sum_{i=1}^{m}(x_i'\eta_i' - \xi_i'y_i') \equiv s\sum_{i=1}^{m}(x_i\eta_i - \xi_i y_i).$$

This relation imposes upon the coefficients the following relations modulo p:

(12) $$\begin{cases} \sum_{i=1}^{m}\begin{vmatrix} a_{ij} & c_{ij} \\ b_{ij} & d_{ij} \end{vmatrix} \equiv s, & \sum_{i=1}^{m}\begin{vmatrix} a_{ij} & c_{ik} \\ b_{ij} & d_{ik} \end{vmatrix} \equiv 0, \\ \sum_{i=1}^{m}\begin{vmatrix} a_{ij} & a_{ik} \\ b_{ij} & b_{ik} \end{vmatrix} \equiv 0, & \sum_{i=1}^{m}\begin{vmatrix} c_{ij} & c_{ik} \\ d_{ij} & d_{ik} \end{vmatrix} \equiv 0, \end{cases} \qquad (j, k = 1, \cdots, m;\ j \neq k).$$

From the manner of our derivation of (12) or by direct verification, we observe that (12) form the sufficient (as well as necessary) conditions in order that Θ', A_i', B_i' shall satisfy the relations (2).

It remains to require that A_i' and B_i' have the period p. By the theorem at the end of §2, no new condition is imposed if $p > 2$; while for $p = 2$, we have the conditions

(13) $$\sum_{j=1}^{m} a_{ji}b_{ji} \equiv \sum_{j=1}^{m} c_{ji}d_{ji} \equiv 0 \pmod{2} \qquad (i = 1, \cdots, m).$$

By the same theorem, we find by squaring (7) that the substitution (9) must, for $p = 2$, satisfy the relation

(14) $$\sum_{i=1}^{m} x_i'y_i' \equiv s \cdot \sum_{i=1}^{m} x_i y_i \pmod{2}.$$

Since $s = 1$ for $p = 2$, we find that (14) imposes upon the coefficients of (9) precisely the conditions (12) and (13).

We have now proved the following:

THEOREM.—*The set of generators* (6) *will define an isomorphism I of the group F into itself if, and only if, the exponents satisfy the relations* (10), (12) *and, for $p = 2$, also* (13). *To every isomorphism I corresponds a $2m$-ary linear homogeneous substitution* (9) *belonging to the general Abelian group if $p > 2$ and to the first hypoabelian group if $p = 2$; and inversely, every such substitution* (9) *is so obtainable.*

4. No condition is imposed upon the exponents l_i, r_i in order that (6) define an isomorphism I. Moreover, it is evident that the set of generators

$$(6_1) \qquad \Theta' = \Theta, \quad A_i' = \Theta^{l_i} A_i, \quad B_i' = \Theta^{r_i} B^i \qquad (i = 1, \cdots, m)$$

defines an isomorphism I_1 for which the corresponding substitution (9) is the identity. Inversely, when (9) is the identity, the isomorphism is one of the p^{2m} isomorphisms I_1. Since the group of the I's is isomorphic to the group of the corresponding substitutions (9), the group of the I_1's is self-conjugate under the group of the I's. We have thus the following:

THEOREM.—*The group of isomorphism I of the group F has an invariant sub-group of order p^{2m} formed by the isomorphisms* (6_1). *The quotient-group is simply isomorphic to the general Abelian group, if $p > 2$, and to the first hypoabelian group, if $p = 2$, each on $2m$ indices taken modulo p.*

5. Consider the group F_1 obtained by extending the group F by an operator J commutative with every operator of F and such that $J^p = \Theta$. J is thus of period p^2. Every operator of F_1 may be given the form

$$S \equiv J^t A_1^{x_1} B_1^{y_1} \cdots A_m^{x_m} B_m^{y_m},$$

which reduces to the identity if and only if

$$x_i \equiv y_i \equiv 0 \pmod{p}, \quad t \equiv 0 \pmod{p^2}.$$

On setting

$$\Sigma \equiv J^\tau A_1^{\xi_1} B_1^{\eta_1} \cdots A_m^{\xi_m} B_m^{\eta_m},$$

the relation (4) holds true. Further, $S = \Sigma$ requires that

$$x_1 \equiv \xi_i, \quad y_i \equiv \eta_i \pmod{p}, \quad t \equiv \tau \pmod{p^2}.$$

We readily verify the formula

$$(15) \qquad S^k = J^{kt} \Theta^{-\frac{1}{2}k(k-1)\Sigma x_i y_i} A_1^{kx_1} B_1^{ky_1} \cdots A_m^{kx_m} B_m^{ky_m}.$$

Hence, according as $p > 2$ or $p = 2$, we have

$$(16) \qquad S^p = \Theta^t \text{ or } S^p = \Theta^{t - \Sigma x_i y_i}.$$

By (4) the powers of J are the only operators of F_1 commutative with every operator of F_1. Every isomorphism of F_1 into itself is therefore obtained by introducing new generators of the form

$$(17)\quad \begin{cases} J' = J^s & (s \text{ not divisible by } p) \\ A'_i = J^{l_i} A_1^{a_{1i}} B_1^{b_{1i}} \cdots A_m^{a_{mi}} B_m^{b_{mi}}, \\ B'_i = J^{r_i} A_1^{c_{1i}} B_1^{d_{1i}} \cdots A_m^{c_{mi}} B_m^{d_{mi}}. \end{cases}$$

As in §3, J', A'_i, B'_i will generate the whole group F_1 if, and only if, (10) holds and will satisfy the generational relation (4) if, and only if, the conditions (12) be satisfied. Finally, by (16), A'_i and B'_i will be of period p if, and only if,

$$(18)\quad \begin{cases} \text{for } p > 2, \quad l_i \equiv r_i \equiv 0 \pmod{p} & (i = 1, \cdots, m), \\ \text{for } p = 2, \quad l_i - \sum_{j=1}^{m} a_{ji} b_{ji} \equiv r_i - \sum_{j=1}^{m} c_{ji} d_{ji} \equiv 0 \pmod{2} & (i = 1, \cdots, m). \end{cases}$$

For $p > 2$, we set

$$l_i = p\lambda_i, \quad r_i = p\rho_i \qquad (i = 1, \cdots, m).$$

The most general suitable set of the new generators is then

$$\begin{cases} J' = J^s, \; \Theta' = \Theta^s, \\ A'_i = \Theta^{\lambda_i} A_1^{a_{1i}} B_1^{b_{1i}} \cdots, \; B'_i = \Theta^{\rho_i} A_1^{c_{1i}} B_1^{d_{1i}} \cdots. \end{cases}$$

where a_{ji}, b_{ji}, c_{ji}, d_{ji} satisfy (10), (12), while λ_i and ρ_i are arbitrary integers.

For $p = 2$, we set

$$l_i = 2\lambda_i + \sum_{j=1}^{m} a_{ji} b_{ji}, \quad r_i = 2\rho_i + \sum_{j=1}^{m} c_{ji} d_{ji}.$$

The most general set of new generators is then

$$\begin{cases} J' = J^s, \; \Theta' = \Theta^s, \\ A'_i = \Theta^{\lambda_i} J^{\Sigma a_{ji} b_{ji}} A_1^{a_{1i}} B_1^{b_{1i}} \cdots \\ B'_i = \Theta^{\rho_i} J^{\Sigma c_{ji} d_{ji}} A_1^{c_{1i}} B_1^{d_{1i}} \cdots. \end{cases}$$

For $p \geqq 2$, the new set of generators

$$(a)\quad J' = \Theta^{\tau} J, \; \Theta' = \Theta, \; A'_i = \Theta^{\lambda_i} A_i, \; B'_i = \Theta^{\rho_i} B_i \qquad (i = 1, \cdots, m)$$

defines an isomorphism for which the corresponding substitution (9) is the identity. Inversely, if the substitution be the identity, the set of generators must be of the above kind. We may state the following:

THEOREM.—*The group of isomorphisms of the group F_1 has an invariant subgroup of order p^{2m+1} formed by the isomorphisms (a). The quotient-group is simply isomorphic with the general Abelian group on $2m$ indices taken modulo p.*

6. Consider the group F' generated by

$$\Theta, A_i, B_i \qquad (i=1, \cdots, m),$$

whose periods are defined by the relations

(19) $$A_1^p = B_1^p = \Theta, \qquad \Theta^p = A_i^p = B_i^p = 1 \qquad (i=2, 3, \cdots, m),$$

and whose further generational relations are given by (2). The general operator of F' may be given the form

$$S \equiv \Theta^t A_1^{x_1} B_1^{y_1} \cdots A_m^{y_m} B_m^{y_m},$$

where each exponent may be supposed to be less than p. In this reduced form,* S is the identity if and only if the exponents are all zero.

Two general operators S and Σ of the group F' satisfy the generational relation (4). Formula (5) evidently holds for the group F'. Applying (19) to (5) we obtain for the period of an operator S of F' the formulæ

(20) $$\begin{cases} S^p = \Theta^{x_1+y_1} & (\text{if } p>2), \\ S^p = \Theta^{x_1+y_1-\sum_{i=1}^{m} x_i y_i} & (\text{if } p=2). \end{cases}$$

To obtain the group of isomorphisms of F', we proceed as in § 3, the only variation being in regard to the periods of A_i', B_i'. Raising the identity (7) to the p^{th} power and applying (20), we obtain the periodicity conditions,

$$\Theta^{x_1'+y_1'} = \Theta'^{x_1+y_1} \qquad (\text{if } p>2).$$

$$\Theta^{x_1'+y_1'-\Sigma x_i' y_i'} = \Theta'^{x_1+y_1-\Sigma x_i y_i} \qquad (\text{if } p=2),$$

For $p=2$, $\Theta' \equiv \Theta^s = \Theta$, so that the condition is

(21) $$x_1' + y_1' + \sum_{i=1}^{m} x_i' y_i' \equiv x_1 + y_1 + \sum_{i=1}^{m} x_i y_i \pmod 2,$$

which imposes upon the coefficients of the substitution (9) the conditions (12) together with the following conditions modulo 2 (where $\delta_{11}=1$, $\delta_{12}=\cdots=\delta_{1m}=0$),†

(22) $$a_{1j} + b_{1j} + \sum_{i=1}^{m} a_{ij} b_{ij} \equiv c_{1j} + d_{1j} + \sum_{i=1}^{m} c_{ij} d_{ij} \equiv \delta_{1j} \qquad (j=1, \cdots, m).$$

As one sees directly from (20), the conditions (22) suffice when $p=2$ to give A_i' and B_i' the periods (19). For $p=2$, the group of substitutions (9) having the invariant (21) is the second hypoabelian group. Proceeding as in § 4, we have the following result:

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* Unless so reduced, $S \equiv 1$ if and only if

$$x_i \equiv y_i \equiv t \equiv 0 \pmod p, \qquad x_1 + y_1 + pt \equiv 0 \pmod{p^2})$$

† The relation (21) is not a formal, but a numerical, identity. Indeed, we must note that $x^2 \equiv x \pmod 2$.

THEOREM.—*The group of isomorphisms of the group F'' has an invariant subgroup of order p^{2m} formed by the isomorphisms* (6_1). *Their quotient-group is simply isomorphic, for $p = 2$, to the second hypoabelian group, and, for $p > 2$, to that subgroup of the general Abelian group satisfying the relation*

$$x_1' + y_1' \equiv s(x_1 + y_1), \tag{23}$$

where s is any integer not divisible by p, but the same as in (12).

Since the Abelian group contains the substitution L_1, which alters only x_1, replacing it by $x_1 + y_1$, the subgroup defined by (23) is conjugate within the general Abelian group to that subgroup which multiplies x_1 by the parameter s.

7. Consider, finally, the group F_1' obtained by extending F' by an operator J commutative with every operator of F' and such that $J^p = \Theta$. To obtain the group of isomorphisms of F_1', we proceed as in § 5. The conditions that the new generators (17) shall have the periods (19) are seen, on applying (15), to be as follows:

(for $p > 2$) $\qquad l_i + a_{1i} + b_{1i} \equiv r_i + c_{1i} + d_{1i} \equiv \delta_{1i}$

(for $p = 2$) $\qquad l_i + a_{1i} + b_{1i} - \sum_{j=1}^{m} a_{ji} b_{ji} \equiv r_i + c_{1i} + d_{1i} - \sum_{j=1}^{m} c_{ji} d_{ji} \equiv \delta_{1i}$

$$(i = 1, \cdots, m).$$

The exponents l_i, r_i are thus determined modulo p. Proceeding as in § 5, we reach the following result:

THEOREM.—*The group of isomorphisms of F_1' has an invariant subgroup of order p^{2m+1}, the quotient-group being simply isomorphic with the general Abelian group on $2m$ indices modulo p.*

8. We may give a concrete representation * of the operators of the groups F, F_1, F', F_1', as linear homogeneous substitutions on p^m variables $Z_{\xi_1, \ldots, \xi_m}$ the indices ξ_i being taken modulo p.

For the groups F and F_1 we may take

$$(24) \qquad \begin{cases} A_i: & Z'_{\xi_1 \ldots \xi_m} = \theta^{\xi_i} Z_{\xi_1 \ldots \xi_m}, \\ B_i: & Z'_{\xi_1 \ldots \xi_i \ldots \xi_m} = Z_{\xi_1 \ldots \xi_i + 1 \ldots \xi_m}, \\ \Theta: & Z'_{\xi_1 \ldots \xi_m} = \theta Z_{\xi_1 \ldots \xi_m}, \\ J: & Z'_{\xi_1 \ldots \xi_m} = j Z_{\xi_1 \ldots \xi_m}, \end{cases}$$

where θ is a primitive p^{th} root of unity and j a primitive root (always existing) of the equation

$$j^p = \theta \quad \text{or} \quad j^{p^2} = 1.$$

It is readily seen that the substitutions (24) satisfy the relations (1) and (2).

* Taken in a simplified form from JORDAN'S *Traité*, §§ 561 and 566. For JORDAN'S π, I use p, and avoid the use of his other parameter p.

The general substitution S of F takes the form

$$(25)\qquad Z'_{\xi_1 \cdots \xi_m} = \theta^{t+\sum_{i=1}^{m} x_i(\xi_i+y_i)} Z_{\xi_1+y_1 \cdots \xi_m+y_m}.$$

It is at once evident that two substitutions S are identical if and only if their corresponding exponents x_i, y_i, t are congruent modulo p.

9. By a method similar to that of JORDAN, l. c. §§ 568–9, but simpler as to details, we can readily prove that the characteristic determinant (with the parameter K) of (25) is equal modulo p to respectively

$$(1 - K^p)^{p^{m-1}} \qquad (p > 2),$$

$$(K^2 - \theta^{\Sigma x_i y_i})^{2^{m-1}} \qquad (p = 2).$$

Since the substitution S and its transformed (7) by a linear substitution must have equal characteristic determinants, we obtain (14) as a condition *necessary* for the isomorphism when $p = 2$. For the group of transformation of F_1 into itself, the characteristic determinant of

$$J^t A_1'^{x_1} B_1'^{y_1} \cdots \equiv J^{\sum_{i=1}^{m}(l_i x_i + r_i y_i)+t} \Theta^\tau A_1^{x_1'} B_1^{y_1'} \cdots$$

is, for $p = 2$,

$$\left(K^2 - \theta^{t+\sum_{i=1}^{m}(l_i x_i + r_i y_i + x_i' y_i')}\right)^{2^{m-1}}.$$

Hence a condition necessary for the isomorphism is the following,

$$\sum_{i=1}^{m} (x_i' y_i' + l_i x_i + r_i y_i) \equiv \sum_{i=1}^{m} x_i y_i \pmod 2,$$

whence follow the relations $(18)_{p=2}$.

But by this method it remains to discuss the following inverse problem. How many isomorphisms of F (or F_1) into itself correspond to each substitution (9) satisfying the above relations? Compare JORDAN, l. c. §§ 579–580.

10. For the groups F' and F_1' of §§ 6 and 7, we may take

$$\begin{cases} A_1: & Z'_{\xi_1 \cdots \xi_m} = j\theta^{\xi_1} Z_{\xi_1 \cdots \xi_m} \\ B_1: & Z'_{\xi_1 \xi_2 \cdots \xi_m} = j Z_{\xi_1+1\, \xi_2 \cdots \xi_m} \\ \Theta, J, A_i, B_i\ (i = 2, \cdots, m) & \text{defined by (24).} \end{cases}$$

Then the general substitution $\Theta_1^{t_1} A^{x_1} B^{y_1} \cdots$ is

$$Z'_{\xi_1 \cdots \xi_m} = j^{x_1+y_1} \theta^{t+\Sigma x_i(\xi_i+y_i)} Z_{\xi_1+y_1 \cdots \xi_m+y_m},$$

whose characteristic determinant is for $p = 2$,

$$(K^2 - \theta^{x_1+y_1+\Sigma x_i y_i})^{2^{m-1}}.$$

This equals the characteristic determinant of

$$\Theta'^t A_1'^{x_1} B_1'^{y_1} \cdots A_m'^{x_m} B_m'^{y_m}$$

only if (21) be satisfied.