THE ELLIPTIC σ-FUNCTIONS CONSIDERED AS A SPECIAL CASE OF THE HYPERELLIPTIC σ-FUNCTIONS*

BY

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The object of the following paper is two-fold:

1) To give a sketch of the theory of the elliptic σ-functions as they appear in the light of the theory of the hyperelliptic σ-functions.

2) To serve as an introduction for a subsequent paper in which an analogous presentation of the theory of the hyperelliptic σ-functions will be given. Only such methods will therefore be used which are capable of an extension to the hyperelliptic case.

§1. *Canonical systems of associated integrals of the first, second and third kind.*†

We start from the algebraic equation

$$y^2 = R(x), \tag{1}$$

where

$$\begin{aligned} R(x) &= A_0x^4 + 4A_1x^3 + 6A_2x^2 + 4A_3x + A_4 \\ &= A_0(x - a_0)(x - a_1)(x - a_2)(x - a_3) \end{aligned} \tag{2}$$

is a biquadratic whose roots are all distinct.

We construct the Riemann-surface T and the canonical cross cuts A_1, A_2 by which T is transformed into the simply connected surface T', as in the adjoined diagram, in which the full lines are drawn in the upper sheet, the dotted lines in the lower sheet.

We shall say that an integral of the first kind, w, and an integral of the

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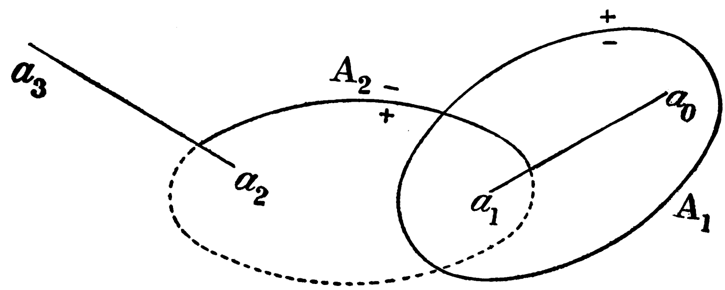
† Compare: BOLZA, *On Weierstrass' Systems of hyperelliptic Integrals of the first and second kind, Chicago Congress Papers*, 1893, p. 1, and the references there given. Also BAKER, *Abel's Theorem*, art. 138–140.

second kind, Z, form a *canonical system*, if the bilinear relation between their periods:

	A_1	A_2
w	$2\omega_1$	$2\omega_2$
Z	$2\eta_1$	$2\eta_2$

has the "canonical" form

$$\eta_1\omega_2 - \eta_2\omega_1 = \frac{\pi i}{2}. \tag{3}$$



There always exists an infinity of such canonical systems, and if w, Z is one such system, every other, $\overline{w}$, $\overline{Z}$, is contained in the form

$$\begin{aligned} \overline{w} &= cw \\ c\overline{Z} &= Z + ew + r(x, y), \end{aligned} \tag{4}$$

where c and e are arbitrary constants and $r(x, y)$ an arbitrary rational function of x and y, (x, y) being the variable upper limit of the integrals.

The most general integral of the third kind $I^{x_1x_0}_{\xi_1\xi_0}$ with the parameters ξ_0, ξ_1 and the limits x_0, x_1 contains one arbitrary constant; its periods at the cross-cuts A_λ are expressible in terms of the integrals of the first and second kind $w^{\xi_1\xi_0}$, $Z^{\xi_1\xi_0}$, taken in J' from ξ_0 to ξ_1. By a proper choice of the constant we can make the integral of the second kind disappear, which determines the constant unambiguously. The *integral of the third kind thus uniquely associated with a given canonical system* we denote by $P_{\xi_1\xi_0}$; its period P_λ at A_λ turns out to be

$$P_\lambda = -2\eta_\lambda w^{\xi_1\xi_0} \qquad (\lambda = 1, 2). \tag{5}$$

As a consequence of (5), this integral is always *commutative*:

$$P^{x_1x_0}_{\xi_1\xi_0} = P^{\xi_1\xi_0}_{x_1x_0}. \tag{6}$$

The *explicit expression* of such a system of associated integrals of the first, second and third kind can be obtained, in the most general way, as follows:

Let $F(x, \xi)$ be an integral function of x and ξ of degree two in each variable, satisfying the following conditions:

$$(7)\qquad \begin{aligned} &1)\quad F(\xi, x) = F(x, \xi), \\ &2)\quad F(\xi, \xi) = R(\xi), \\ &3)\quad \left(\frac{\partial F(x, \xi)}{\partial x}\right)_{x=\xi} = \tfrac{1}{2}R'(\xi), \end{aligned}$$

$R'(x)$ being the derivative of $R(x)$; then the integrals

$$(8)\qquad \begin{aligned} w^{xx_0} &= \int_{x_0}^{x}\frac{G\,dx}{y} \\ Z^{xx_0} &= -\frac{1}{G}\int_{x_0}^{x}\frac{y\eta + F(x, \xi)}{2(x-\xi)^2 y}dx + r(x, y) \\ P^{xx_0}_{\xi_1\xi_0} &= \int_{x_0}^{x}\int_{\xi_0}^{\xi_1}\frac{yy' + F(x, x')}{2(x-x')^2 yy'}dx\,dx', \end{aligned}$$

where G is an arbitrary constant, $r(x, y)$ an arbitrary rational function of x, y, $\eta^2 = R(\xi)$, and $y'^2 = R(x')$, form a system of associated integrals, and every such system can be obtained in this manner.

If we pass from the canonical system w, Z to another $\overline{w}$, $\overline{Z}$ by the transformation (4), the integral $\overline{P}_{\xi_1\xi_0}$ and the function $\overline{F}(x, \xi)$ connected with $\overline{w}$, $\overline{Z}$ are:

$$(9)\qquad \begin{aligned} \overline{P}^{xx_0}_{\xi_1\xi_0} &= P^{xx_0}_{\xi_1\xi_0} - e w^{xx_0} w^{\xi_1\xi_0}, \\ \overline{F}(x, \xi) &= F(x, \xi) - 2eG^2(x-\xi)^2. \end{aligned}$$

The following *two special values of* $F(x, \xi)$ are of particular importance for the sequel:

$$(10)\qquad F(x, \xi) = A_x^2 A_\xi^2,$$

where symbolically

$$R(x) = A_x^4;$$

$$(11)\qquad F(x, \xi) = \tfrac{1}{2}[\phi(x)\psi(\xi) + \phi(\xi)\psi(x)],$$

where

$$R(x) = \phi(x)\psi(x)$$

is a decomposition of $R(x)$ into two quadratic factors.

§2. *Weierstrass' Θ-functions.**

(*a*) *As functions of* u.

With every canonical system of integrals w, Z *and every system of cross-cuts* A_1, A_2, *there is further uniquely associated a function* $\Theta\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right](u)$ as follows:

Three constants a, β, τ being defined by

$$(12) \qquad 2\omega_1 a = \eta_1, \quad 2\beta\omega_1 = 1, \quad \omega_1\tau = \omega_2,$$

$$(13) \qquad \Theta\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right](u) = \sum_{\nu=-\infty}^{+\infty} e^{au^2 + 2\pi i\beta u\left(\nu+\frac{g}{2}\right) + \pi i\tau\left(\nu+\frac{g}{2}\right)^2 + \pi i h\left(\nu+\frac{g}{2}\right)},$$

g and h being integers. The series is convergent since in consequence of our agreements concerning the cross-cuts

$$\Re\left(\frac{\omega_2}{i\omega_1}\right) > 0.$$

The principal properties of these functions are:

$$(14) \qquad \begin{aligned} \Theta\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right](u + 2\omega_1) &= (-1)^g e^{2\eta_1(u+\omega_1)}\,\Theta\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right](u), \\ \Theta\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right](u + 2\omega_2) &= (-1)^h e^{2\eta_2(u+\omega_2)}\,\Theta\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right](u). \end{aligned}$$

Every integral transcendental function of u which satisfies the two relations (14) differs from $\Theta\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right](u)$ only by a constant factor.

If we pass from one canonical system to another by means of the transformation (4), the Θ-functions corresponding to the two systems are connected by the relation

$$(15) \qquad \bar{\Theta}\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right](cu) = e^{\frac{1}{2}eu^2}\,\Theta\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right](u).$$

On the other hand, if we pass to another system of cross-cuts by the linear transformation of periods

$$(16) \qquad \begin{aligned} \tilde{\omega}_1 &= p\omega_1 + q\omega_2, \\ \tilde{\omega}_2 &= p'\omega_1 + q'\omega_2, \end{aligned}$$

the old and the new Θ-functions are connected by the relation

$$(17) \qquad \Theta\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right](u) = C\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right] \cdot \tilde{\Theta}\left[\begin{smallmatrix} \tilde{g} \\ \tilde{h} \end{smallmatrix}\right](u),$$

where $C\left[\begin{smallmatrix} g \\ h \end{smallmatrix}\right]$ is independent of u and

$$\begin{aligned} \tilde{g} &= pq + pg + qh, \\ \tilde{h} &= p'q' + p'g + q'h. \end{aligned}$$

* Compare: WEIERSTRASS, Lectures on elliptic and hyperelliptic functions; SCHOTTKY, *Abel'sche Functionen von drei Variabeln*, §1; BAKER, *Abel's Theorem*, art. 189.

(*b*) *In the Riemann-surface.*

If we substitute in the odd Θ-function for u an integral of the first kind: $u = w^{x\xi}$, the function $\Theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](w^{x\xi})$ vanishes in the point $x = \xi$ and in no other point of the Riemann-surface.

More generally, for every characteristic $\left[\begin{smallmatrix}g\\h\end{smallmatrix}\right]$ there exists a half-period $k\left[\begin{smallmatrix}g\\h\end{smallmatrix}\right]$ such that $\Theta\left[\begin{smallmatrix}g\\h\end{smallmatrix}\right](w^{x\xi} + k\left[\begin{smallmatrix}g\\h\end{smallmatrix}\right])$ vanishes for $x = \xi$ and in no other point of T'. Now $k\left[\begin{smallmatrix}g\\h\end{smallmatrix}\right]$, as every half-period, can be expressed in two different ways in the form

$$k\left[\begin{smallmatrix}g\\h\end{smallmatrix}\right] \equiv \epsilon_0 w^{a_0} + \epsilon_1 w^{a_1} + \epsilon_2 w^{a_2} + \epsilon_3 w^{a_3} \pmod{2\omega_1,\ 2\omega_2}$$

where ϵ_0, ϵ_1, ϵ_2, ϵ_3 are each 0 or 1 and w^x is w^{xa_0}.

For the dissection of § 1 the result is

$$\begin{aligned} k\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] &\equiv w^{a_0} + w^{a_1} \equiv w^{a_2} + w^{a_3} \\ k\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right] &\equiv w^{a_0} + w^{a_2} \equiv w^{a_3} + w^{a_1} \\ k\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] &\equiv w^{a_0} + w^{a_3} \equiv w^{a_1} + w^{a_2} \end{aligned}$$

With every even characteristic $\left[\begin{smallmatrix}g\\h\end{smallmatrix}\right]$ there is therefore associated a decomposition of the biquadratic R into two quadratic factors: $R = \phi\psi$, called the *algebraic characteristic of the function* $\Theta\left[\begin{smallmatrix}g\\h\end{smallmatrix}\right](u)$.

Setting

$$\phi_\lambda = \text{const. } (x - a_0)(x - a_\lambda), \qquad \psi_\lambda = R/\phi_\lambda,$$

we shall denote by $\Theta_{\phi_\lambda\psi_\lambda}(u)$ or shorter $\Theta(u)_\lambda$ the Θ-function with the algebraic characteristic $\phi_\lambda\psi_\lambda$, by k_λ the corresponding constant $k\left[\begin{smallmatrix}g\\h\end{smallmatrix}\right]$, and by $\Theta(u)_0$ or simply $\Theta(u)$ the odd Θ-function, the elements of all the characteristics being reduced to 0 or 1.

If we substitute in (17):

$$u = w^{x\xi} + k\left[\begin{smallmatrix}g\\h\end{smallmatrix}\right],$$

the left-hand side vanishes for $x = \xi$, hence also the right-hand side, therefore

$$\tilde{k}\left[\begin{smallmatrix}\tilde{g}\\\tilde{h}\end{smallmatrix}\right] \equiv k\left[\begin{smallmatrix}g\\h\end{smallmatrix}\right], \tag{18}$$

i. e., the algebraic characteristic remains unaltered under a linear transformation of the periods.

From the properties of the function $\Theta(w^{x\xi} + k_\lambda)_\lambda$ follows the theorem well known in the theory of abelian functions:

$$\frac{\Theta(w^{x\xi} + k_\lambda)_\lambda \Theta(w^{x_0\xi_0} + k_\lambda)_\lambda}{\Theta(w^{x\xi_0} + k_\lambda)_\lambda \Theta(w^{x_0\xi} + k_\lambda)_\lambda} = e^{P^{xx_0}_{\xi\xi_0}}, \tag{19}$$

for $\lambda = 0, 1, 2, 3$ with $k_0 = 0$, $P_{\xi\xi_0}^{xx_0}$ being the integral of the third kind associated with the canonical system w, Z from which the Θ-functions are derived.

Now determine with KLEIN* ξ, ξ_0 by means of the addition-theorem so that

$$\left.\begin{aligned} w^{x\xi_0} + k_\lambda &\equiv 0 \\ w^{x_0\xi} + k_\lambda &\equiv 0 \end{aligned}\right\} \bmod (2\omega_1, 2\omega_2),$$

λ having one of the values $1, 2, 3$, and substitute the values of ξ, ξ_0 in (19). The result is *Klein's Theorem*

$$\frac{\Theta^2(w^{xx_0})_\lambda}{\Theta^2(0)\lambda} = \frac{\left(\sqrt{\phi_\lambda(x)}\sqrt{\psi_\lambda(x_0)} + \sqrt{\phi_\lambda(x_0)}\sqrt{\psi_\lambda(x)}\right)^2}{4yy_0} e^{P_{xx_0}^{\bar{x}\bar{x}_0}} \tag{20}$$

where $\bar{x}$, $\bar{x}_0$ are the conjugate places to x, x_0 in T and also the path $\bar{x}_0\bar{x}$ is conjugate to the path x_0x.

The consideration of the zeros and poles of the quotient

$$\frac{\Theta^2(w^{xx_0})}{\Theta^2(w^{xx_0})_\lambda}$$

leads to an analogous expression for the odd function

$$\Theta^2(w^{xx_0}) = \gamma \frac{(x - x_0)^2}{yy_0} e^{P_{xx_0}^{\bar{x}\bar{x}_0}} \tag{21}$$

γ being a constant whose value is found by expansion according to powers of $x - x_0$:

$$\gamma \doteq G^2\Theta'^2(0) \tag{22}$$

$\Theta'(u)$ denoting the derivative of $\Theta(u)$.

§ 3. *The Al-functions and the functions $p(u)_\lambda$.*†

We now *define* in accordance with a notation used by WEIERSTRASS:

$$Al(u)_\lambda = \frac{\Theta(u)_\lambda}{\Theta(0)_\lambda}, \qquad Al(u) = \frac{\Theta(u)}{G\Theta'(0)} \qquad (\lambda = 1, 2, 3) \tag{23}$$

From the properties of the Θ-functions flow at once the following theorems concerning the *Al*-functions:

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*Mathematische Annalen, vol. 27, p. 447.

†Compare BOLZA, American Journal of Mathematics, vol. 17, p. 26.

1) *If we pass from one canonical system of integrals to another* by the transformation (4), the Al-functions belonging to the two systems are connected by the relation

$$\overline{Al}(cu)_\lambda = e^{\frac{1}{2}eu^2} Al(u)_\lambda \qquad (\lambda=0,\ 1,\ 2,\ 3). \tag{24}$$

2) If we pass to another system of cross-cuts *by a linear transformation of the periods, each Al-function remains unchanged.*

3) If we replace u by the integral of the first kind w^{xx_0}, we obtain:

$$Al^2(w^{xx_0})_\lambda = \frac{[\sqrt{\phi_\lambda(x)}\ \sqrt{\psi_\lambda(x_0)} + \sqrt{\phi_\lambda(x_0)}\ \sqrt{\psi_\lambda(x)}]^2}{4yy_0} e^{P^{\bar{x}\bar{x}_0}_{xx_0}} \quad (\lambda=1,\ 2,\ 3) \tag{25}$$

$$Al^2(w^{xx_0}) = \frac{(x-x_0)^2}{yy_0} e^{P^{\bar{x}\bar{x}_0}_{xx_0}} \tag{26}$$

4) The formulæ for the addition of periods to the argument have exactly the same form as for the Θ-functions.

We further *define* $p(u)_\lambda$ by the equation

$$p(u)_\lambda = -\frac{d^2 \log Al(u)_\lambda}{du^2}, \qquad (\lambda=0,\ 1,\ 2,\ 3) \tag{27}$$

and write $p(u) = p(u)_0$.

Now take the second logarithmic derivative of (19) with respect to x and ξ, we obtain

$$G^2 p(w^{x\xi}) = \frac{y\eta + F(x,\xi)}{2(x-\xi)^2}. \tag{28}$$

From this fundamental formula follow a number of consequences by giving ξ; or x and ξ special values:

$$p(w^{a_\lambda a_0}) = \frac{F(a_\lambda, a_0)}{2(a_\lambda - a_0)^2};$$

we denote this constant by

$$p(w^{a_\lambda a_0}) = \epsilon_\lambda \qquad (\lambda=1,\ 2,\ 3).$$

Then further

$$G^2[p(w^{xa_0}) - \epsilon_\lambda] = \frac{1}{4}\frac{R'(a_0)}{a_0 - a_\lambda}\frac{x-a_\lambda}{x-a_0} = \frac{Al^2(w^{xa_0})_\lambda}{Al^2(w^{xa_0})} \tag{29}$$

and after differentiation

$$(30) \qquad G^3 p'(w^{xa_0}) = -\tfrac{1}{4}\frac{R'(a_0)y}{(x-a_0)^2}.$$

Hence follow the equations

$$(31) \qquad \begin{aligned} G^2[p(u) - \epsilon_\lambda] &= \frac{Al^2(u)_\lambda}{Al^2(u)}, \\ p'^2(u) &= 4\big(p(u)-\epsilon_1\big)\big(p(u)-\epsilon_2\big)\big(p(u)-\epsilon_3\big), \\ &= 4p^3(u) - \gamma_1 p^2(u) - \gamma_2 p(u) - \gamma_3 \qquad \text{say}, \\ G^3 p'(u) &= -\frac{2Al(u)_1\, Al(u)_2\, Al(u)_3}{Al^3(u)}. \end{aligned}$$

We can use the differential equation (31) to determine the first terms in the expansion of $p(u)$ and obtain

$$p(u) = u^{-2} + \tfrac{1}{12}\gamma_1 + \cdots.$$

Hence

$$(32) \qquad \begin{aligned} Al(u) &= G^{-1}[u - \tfrac{1}{24}\gamma_1 u^3 + \cdots], \\ Al(u)_1 \cdot Al(u)_2 \cdot Al(u)_3 &= 1 - \tfrac{1}{8}\gamma_1 u^2 + \cdots. \end{aligned}$$

The last expansion can also be obtained independently of the differential equation from the result that

$$p(0)_\lambda = -Al''(0)_\lambda = \epsilon_\lambda;$$

hence

$$(33) \qquad Al(u)_\lambda = 1 - \tfrac{1}{2}\epsilon_\lambda u^2 + \cdots.$$

§ 4. *The σ-functions.**

So far no special assumption has been made concerning the canonical system of integrals of the first and second kind with which the Al-functions are associated.

We now propose so to determine the canonical system that in the expansions (31) the second term disappears.

According to (9) and (10) we can always write $F(x, \xi)$ in the form

$$F(x, \xi) = A_x^2 A_\xi^2 - 2eG^2(x-\xi)^2;$$

* Compare KLEIN, *Ueber hyperelliptische Sigmafunctionen*, Mathematische Annalen, vol. 27.

then ϵ_λ takes the form

$$G^2\epsilon_\lambda = \tfrac{1}{4}\frac{R'(a_0)}{a_\lambda - a_0} + \tfrac{1}{24}R''(a_0) - e = -\tfrac{1}{6}(\phi_\lambda,\ \psi_\lambda)_2 - e$$

and therefore

$$\gamma_1 = 4(\epsilon_1 + \epsilon_2 + \epsilon_3) = 12eG^{-2}.$$

Hence in order to make $\gamma_1 = 0$, we must choose $e = 0$; we further choose $G = 1$ and thus obtain the result:

For the canonical system characterized by

$$w = \int \frac{dx}{y},\quad F(x,\xi) = A_x^2 A_\xi^2 \tag{34}$$

the following simplifications take place in the expansions of the Al- and p-functions:

1) In the expansion of pu the constant term disappears.

2) In the expansion of $Al(u)$ the first term is u, the term in u^3 disappears.

3) In the expansion of the product $Al(u)_1 \cdot Al(u)_2 \cdot Al(u)_3$ the term in u^2 disappears.

4) At the same time the differential equation for pu simplifies to

$$p'^2(u) = 4p^3(u) - g_2 p(u) - g_3 \tag{35}$$

where g_2, g_3 are the quadratic and cubic invariants of $R(x)$.

The last result shows that the *Al- and p-functions corresponding to this particular canonical system are identical with Weierstrass' σ- and $\wp$-functions.*

But there is still another and even more important consequence of this special choice of the canonical system: If we introduce homogeneous variables, $F(x,\xi) = A_x^2 A_\xi^2$ becomes a covariant of $R(x)$, and *among all possible forms of $F(x,\xi)$ this is the only integral rational covariant of $R(x)$.* Hence follows further that the expressions (26) and (28) for $\sigma(w^{xx_0})$ and $\wp(w^{xx_0})$ are covariants of $R(x)$ with the two sets of variables x, x_0 of index -1 and $+2$ respectively. Substituting therefore *in the expression for σu and $\wp u$*, w^{xx_0} for u, we see that *the coefficients are rational integral invariants of $R(x)$*, and that if we write

$$\sigma u = \sum_{n=0}^{\infty} a_n \frac{u^{2n+1}}{(2n+1)!},\qquad \wp u = \frac{1}{u^2} + \sum_{n=1}^{\infty} b_n \frac{u^{2n-2}}{(2n-2)!}$$

a_n and b_n are of index $2n$. This reveals the apriori reason why in the expansions of σu and $\wp u$ the second term is missing, viz: because there exists no rational integral invariant of the biquadratic of index 2.

Similar results hold for the functions $\sigma(u)_\lambda$ with respect to the simultaneous system ϕ_λ, ψ_λ.

If we wish to make the second term in the expansion of $Al(u)_\lambda$ equal to zero, we must choose for $F(x, \xi)$ the value

$$F(x, \xi) = \tfrac{1}{2}[\phi_\lambda(x)\psi_\lambda(\xi) + \phi_\lambda(\xi)\psi_\lambda(x)].$$

The coefficients of the expansion of $Al(u)_\lambda$ will then still be integral rational invariants of the system ϕ_λ, ψ_λ, and the Al-function so determined is connected with $\sigma(u)_\lambda$ by the relation

$$Al(u)_\lambda = e^{\frac{1}{2}e_\lambda u^2}\sigma(u)_\lambda.$$

It is essentially the same as the function $Al(u)$ used by WEIERSTRASS in his first paper on elliptic functions (Werke I, p. 1).

§ 5. *The partial differential equations for the σ-functions.**

If we differentiate an integral of the first or second kind with respect to one of the roots of $R(x)$, a, the result is an integral of the second kind. Hence we may write for any canonical system w, Z:

$$\frac{\partial w}{\partial a} = kw - nZ + r_1,$$
$$\frac{\partial Z}{\partial a} = -lw - k'Z + r_2,$$

where k, k', l, n are constants, and r_1, r_2 are rational functions of x, y.

Integrating around closed paths independent of a, we obtain for the periods:

$$(36)\qquad \begin{aligned} \frac{\partial \omega_\mu}{\partial a} &= k\omega_\mu - n\eta_\mu \\ \frac{\partial \eta_\mu}{\partial a} &= -l\omega_\mu - k'\eta_\mu \end{aligned} \qquad (\mu = 1, 2)$$

and the relation (3) shows that $k' = k$.

From (36) and the definition of the Θ-functions we easily derive the *partial differential equation for* Θ_λ

$$(37)\qquad \frac{\partial \Theta_\lambda}{\partial a} = -\tfrac{1}{2}\Theta_\lambda[lu^2 + 2na] - ku\frac{\partial \Theta_\lambda}{\partial u} + \tfrac{1}{2}n\frac{\partial^2 \Theta_\lambda}{\partial u^2} \qquad (\lambda = 0, 1, 2, 3).$$

* Compare: WEIERSTRASS, Werke II, p. 244; BOLZA, American Journal of Mathematics, vol. 21, p. 107, and the references there given; also a continuation of that paper in one of the forthcoming numbers of the American Journal.

From this partial differential equation a number of important consequences can be derived.

A. Concerning the odd σ-function.

For KLEIN'S canonical system (34), the constants k, l, n have the following values:

$$(38)\qquad \begin{aligned} k &= -\tfrac{1}{12}\frac{R''(a)}{R'(a)} = -\tfrac{1}{12}\frac{\partial}{\partial a}\log\Delta, \\ l &= -\frac{g_2}{6R'(a)}, \quad n = \frac{2}{R'(a)}, \end{aligned}$$

Δ being the discriminant of R: $\Delta = g_2^3 - 27g_3^2$.

(*a*) *Value of* $\Theta'(0)$.

Differentiating (37) and putting $u = 0$ we obtain for $\lambda = 0$:

$$\frac{\partial}{\partial a}\log\omega_1^{-\frac{1}{2}}\Theta'(0) = -\tfrac{3}{2}k.$$

Hence

$$(39)\qquad \Theta'(0) = C\omega_1^{\frac{1}{2}}\Delta^{\frac{1}{8}}$$

C being independent of the a's. The value of C is found by a limiting process:

$$C = \pi^{-\frac{1}{2}}.$$

(*b*) *Second proof of the invariantive properties of the coefficients in the expansion of σu.*

From (37) we further derive the following partial differential equation for σu:

$$(40)\qquad R'(a)\frac{\partial\sigma}{\partial a} = \tfrac{1}{12}g_2u^2\sigma + \tfrac{1}{12}R''(a)\left[u\frac{\partial\sigma}{\partial u} - \sigma\right] + \frac{\partial^2\sigma}{\partial u^2}.$$

Here we substitute for σ its expansion; then we obtain for the quotient,

$$(41)\qquad \begin{gathered} I_n = a_n\Delta^{-\frac{n}{6}}, \\ \sum_{(a)}\frac{\partial I_n}{\partial a} = 0, \quad \sum_{(a)} a\frac{\partial I_n}{\partial a} = 0, \quad \sum_{(a)} a^2\frac{\partial I_n}{\partial a} = 0, \end{gathered}$$

the summations extending over the four roots of $R(x)$. Besides I_n is a homogeneous function of dimension zero of the coefficients of $R(x)$; but these properties characterize I_n as an absolute invariant of $R(x)$; hence a_n *is an invariant of index* $2n$.

(*c*) *Recursion formula for the expansion of* σu.

Form the differential equation (40) for the four roots of $R(x)$, and introduce the coefficients of $R(x)$ instead of the roots. The final result is: *

If H *denotes the Hessian of* $R(x)$ *and* H_i *its coefficients, and if the operator* δ *is defined by*

$$\delta = \sum_{i=0}^{4} H_i \frac{\partial}{\partial A_i},$$

σu *satisfies the partial differential equation*

$$2\delta\sigma = \tfrac{1}{12} g_2 u^2 \sigma + \frac{\partial^2 \sigma}{\partial u^2}, \tag{42}$$

which furnishes for the coefficients the *recursion-formula*:

$$a_{n+1} = 2\delta a_n - \tfrac{1}{6} n(2n+1) g_2 a_{n-1}, \tag{43}$$

with $a_0 = 1$, $a_1 = 0$; for its application notice that

$$\delta g_2 = 6 g_3, \quad \delta g_3 = \tfrac{1}{3} g_2^2 .$$

B. Concerning the even σ*-functions.*

To obtain the analogous results for the even σ-functions, it is simpler to start from the canonical system characterized by (11) and $G = 1$. For this system the values of k, l, n are:

$$k = -\tfrac{1}{4} \frac{\partial \log \Delta_\phi \Delta_\psi}{\partial a}, \qquad l = -\tfrac{1}{8} \frac{R_{\phi\psi}}{R'(a)}, \qquad n = \frac{2}{R'(a)} \tag{44}$$

$R_{\phi\psi}$ being the resultant of $\phi = \phi_\lambda$ and $\psi = \psi_\lambda$, and Δ_ϕ, Δ_ψ their respective discriminants.

(*a*) *The* Θ_λ*-zero-values.*

Put in (37) $u = 0$ and remember that for this canonical system $\Theta''(0)_\lambda = 0$. Then

$$\frac{\partial}{\partial a} \log \omega_1^{-\frac{1}{2}} \Theta(0)_\lambda = -\tfrac{1}{2} k .$$

Hence follows

$$\Theta(0)_\lambda = C_\lambda \omega_1^{\frac{1}{2}} \Delta_\phi^{\frac{1}{8}} \Delta_\psi^{\frac{1}{8}}, \tag{45}$$

where C_λ is independent of the a's and is found to be

$$C_\lambda = 2^{\frac{1}{4}} \pi^{-\frac{1}{2}} .$$

* Already given by PASCAL, Annali di Matematica (2), vol. 17, p. 278.

(*b*) *Second proof for the invariantive properties of the coefficients of the expansion of* $\sigma(u)_\lambda$.

From (37) we derive

$$(46) \qquad R'(a)\left[\frac{\partial Al_\lambda}{\partial a} - \tfrac{1}{4}u\frac{\partial Al_\lambda}{\partial u}\frac{\partial \log \Delta_\phi\Delta_\psi}{\partial a}\right] = \tfrac{1}{16}R_{\phi\psi}u^2 Al_\lambda + \frac{\partial^2 Al_\lambda}{\partial u^2}.$$

Hence follows for the coefficients c_n of the expansion

$$Al(u)_\lambda = \sum_{n=0}^{\infty} c_n \frac{u^{2n}}{(2n)!},$$

on setting

$$J_n = c_n \Delta_\phi^{-\frac{n}{2}} \Delta_\psi^{-\frac{n}{2}},$$

$$(47) \qquad \sum_{(a)} \frac{\partial J_n}{\partial a} = 0, \quad \sum_{(a)} a\frac{\partial J_n}{\partial a} = 0, \quad \sum_{(a)} a^2\frac{\partial J_n}{\partial a} = 0.$$

Besides J_n is a homogeneous function of dimension zero of the coefficients of ϕ and of the coefficients of ψ. Hence it follows that J_n is an absolute invariant of the system ϕ, ψ; consequently c_n is a simultaneous invariant of ϕ and ψ of index $2n$.

(*c*) *Recursion formula for the expansion of* $Al(u)_\lambda$.

Write (46) for the four roots of $R(x)$, add, and introduce the coefficients of ϕ and ψ instead of their roots. The final result is:

Let D denote the operator

$$D = \tfrac{1}{2}\left[\Delta_\phi \sum_{i=0}^{2} \psi_i \frac{\partial}{\partial \phi_i} + \Delta_\psi \sum_{i=0}^{2} \phi_i \frac{\partial}{\partial \psi_i}\right],$$

and $A_{\phi\psi} = (\phi, \psi)_2$; then $Al(u)_\lambda$ *satisfies the following partial differential equation*

$$(48) \qquad D(Al_\lambda) = \tfrac{1}{2}A_{\phi\psi}u\frac{\partial Al_\lambda}{\partial u} + \tfrac{1}{16}R_{\phi\psi}u^2 Al_\lambda + \frac{\partial^2 Al_\lambda}{\partial u^2}.$$

Hence follows for the coefficients c_n in the expansion

$$\sigma(u)_\lambda = e^{-\frac{1}{2}e_\lambda u^2}\sum_{n=0}^{\infty} c_n \frac{u^{2n}}{(2n)!}$$

the *recursion formula*

$$(49) \qquad c_{n+1} = D(c_n) - nA_{\phi\psi}c_n - \tfrac{1}{8}n(2n-1)R_{\phi\psi}c_{n-1}$$

with $c_0 = 1$, $c_1 = 0$.

Since

$$(50) \qquad D(R_{\phi\psi}) = 0, \quad D(A_{\phi\psi}) = A_{\phi\psi}^2 - R_{\phi\psi}$$

the coefficients are integral functions of $A_{\phi\psi}$ and $R_{\phi\psi}$.

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