A NEW DEFINITION OF THE GENERAL ABELIAN LINEAR GROUP*

BY

LEONARD EUGENE DICKSON

1. We may give a striking definition of the general Abelian group, making use of the fruitful conception of the "compounds of a given linear homogeneous group," introduced in recent papers by the writer.† In § 3 we determine the multiplicity of the isomorphism of a given linear homogeneous group to its compound groups. This result is applied in § 4 to show the simple relation of the Abelian group to the general linear homogeneous group in the same number of variables. In § 5 it is shown that the simple groups of composite order which are derived from the decompositions of the quaternary Abelian group and the quinary orthogonal group, each in the $GF[p^n], p > 2$, are simply isomorphic. The investigation affords a proof of the simple isomorphism between the corresponding ten-parameter projective groups without the consideration of their infinitesimal transformations.

2. It will be convenient to introduce a notation more compact than that usually employed‡ for the substitutions of the general Abelian group A_{2m, p^n} on $2m$

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† *Concerning a linear homogeneous group in $C_{m,q}$ variables isomorphic to the general linear homogeneous group in m variables*, Bulletin, Dec., 1898.

The structure of certain linear groups with quadratic invariants, Proceedings of the London Mathematical Society, vol. 30, pp. 70–98.

‡ Dickson, *The Structure of the Hypoabelian Groups*, Bulletin, pp. 495–510, July, 1898; *A Triply Infinite System of Simple Groups*, The Quarterly Journal of Pure and Applied Mathematics, pp. 169–178, 1897; Jordan, *Traité des Substitutions*, pp. 171–179, for the case $n = 1$.

Instead of considering the cogredient linear substitutions leaving invariant (up to a factor a) the usual bilinear function it is convenient to consider here the substitutions A leaving invariant the function

$$\psi \equiv \sum_{j=1}^{m} \begin{vmatrix} \xi_{2j-1} & \xi_{2j} \\ \xi_{2j-1}^{p^n} & \xi_{2j}^{p^n} \end{vmatrix}$$

The conditions that A shall leave ψ invariant are seen to be (1). The *hyperabelian* group of linear homogeneous substitutions in the $GF[p^{2n}]$ on $2m$ indices which leave ψ invariant has been studied by the writer in an article presented to the London Mathematical Society.

indices in the Galois field of order p^n. The conditions that a substitution

$$A: \qquad \xi_i' = \sum_{j=1}^{2m} a_{ij}\xi_j \qquad (i=1, \cdots, 2m)$$

shall be Abelian are the following:

$$(1) \qquad \sum_{j=1}^{m} \begin{vmatrix} a_{2j-1\,i} & a_{2j-1\,k} \\ a_{2j\,i} & a_{2j\,k} \end{vmatrix} = \alpha\epsilon_{ik} \qquad (i, k=1, \cdots, 2m;\ i<k)$$

where α is a parameter $\neq 0$ depending upon the particular substitution A and where every $\epsilon_{ik} = 0$ unless $k = i + 1 =$ even, when

$$\epsilon_{2l-1\,2l} = 1 \qquad (l=1, \cdots, m).$$

The second compound $C_{2m,2}$ of the $2m$-ary group A_{2m,p^n} is formed by the substitutions

$$(2) \qquad Y'_{i_1 i_2} = \sum_{j_1<j_2}^{j_1,j_2=1\ldots 2m} \begin{vmatrix} a_{i_1 j_1} & a_{i_1 j_2} \\ a_{i_2 j_1} & a_{i_2 j_2} \end{vmatrix} Y_{j_1 j_2} \qquad \left(\begin{matrix} i_1, i_2 = 1, \cdots, 2m \\ i_1 < i_2 \end{matrix}\right).$$

We readily verify that the group $C_{2m,2}$ has the relative invariant

$$Z \equiv \sum_{l=1}^{m} Y_{2l-1\,2l}.$$

Indeed, in virtue of the relations (1), we have, on applying to Z the substitution (2),

$$\sum_{l=1}^{m} Y'_{2l-1\,2l} = \sum_{j_1,j_2} \left\{ \sum_{l=1}^{m} \begin{vmatrix} a_{2l-1\,j_1} & a_{2l-1\,j_2} \\ a_{2l\,j_1} & a_{2l\,j_2} \end{vmatrix} \right\} Y_{j_1 j_2}$$

$$= \alpha \sum_{j_1,j_2} \epsilon_{j_1 j_2} Y_{j_1 j_2} = \alpha \sum_{l=1}^{m} Y_{2l-1\,2l}.$$

Inversely, if the substitution (2) multiply the function Z by a constant α, the relations (1) hold true. We have proved the result:

THEOREM.—*The general Abelian group A_{2m,p^n} is the largest $2m$-ary linear homogeneous group whose second compound has as a relative invariant the linear function Z.*

3. To establish the more important theorem of § 4, we determine the multiplicity of the isomorphism of a given m-ary linear homogeneous group G_m to its q^{th} compound $C_{m,q}$, supposing that $q<m$. To the substitution

$$(a_{ij}) \qquad (i, j=1, \cdots, m)$$

of G_m there corresponds the following substitution of $C_{m\,q}$: *

* We employ Sylvester's umbral notation for determinants.

$$[a]_q: \qquad Y'_{i_1\cdots i_q} = \sum_{\substack{j_1\cdots j_q\\ j_1<j_2\cdots<j_q}}^{1\ldots m} \begin{vmatrix} i_1 \cdots i_q \\ j_1 \cdots j_q \end{vmatrix} Y_{j_1\cdots j_q} \qquad \begin{pmatrix} i_1, \cdots, i_q = 1, \cdots, m \\ i_1 < i_2 < \cdots < i_q \end{pmatrix}.$$

Let j be an integer such that $q < j \leqq m$. Consider the matrix J of certain coefficients of the substitution $[a]_q$, viz.,

$$\begin{bmatrix} \begin{vmatrix} 1\,2\cdots q-1\,q \\ 1\,2\cdots q-1\,q \end{vmatrix} & -\begin{vmatrix} 1\,2\cdots q-1\,q \\ 1\,2\cdots q-1\,j \end{vmatrix} & \cdots (-1)^q & \begin{vmatrix} 1\,2\cdots q-1\,q \\ 2\,3\cdots q\quad j \end{vmatrix} \\ -\begin{vmatrix} 1\,2\cdots q-1\,j \\ 1\,2\cdots q-1\,q \end{vmatrix} & \begin{vmatrix} 1\,2\cdots q-1\,j \\ 1\,2\cdots q-1\,j \end{vmatrix} & \cdots (-1)^{q+1} & \begin{vmatrix} 1\,2\cdots q-1\,j \\ 2\,3\cdots q\quad j \end{vmatrix} \\ \cdots\cdots\cdots & \cdots\cdots\cdots & & \cdots\cdots\cdots \\ (-1)^q \begin{vmatrix} 2\,3\cdots q\quad j \\ 1\,2\cdots q-1\,q \end{vmatrix} & (-1)^{q+1} \begin{vmatrix} 2\,3\cdots q\quad j \\ 1\,2\cdots q-1\,j \end{vmatrix} & \cdots & \begin{vmatrix} 2\,3\cdots q\quad j \\ 2\,3\cdots q\quad j \end{vmatrix} \end{bmatrix}.$$

Consider also the matrix A of determinant Δ,

$$A \equiv \begin{bmatrix} a_{jj}\, a_{qj} \cdots a_{1j} \\ a_{jq}\, a_{qq} \cdots a_{1q} \\ \cdots\cdots\cdots \\ a_{j1}\, a_{q1} \cdots a_{11} \end{bmatrix}.$$

The composition of the matrices J and A gives the result

$$JA \equiv \begin{bmatrix} \Delta\, 0 \cdots 0 \\ 0\, \Delta \cdots 0 \\ \cdots\cdots \\ 0\, 0 \cdots \Delta \end{bmatrix}.$$

We seek those substitutions of G_m which correspond to the identity in $C_{m,q}$. Suppose, therefore, that $[a]_q$ reduces to the identical substitution, so that the matrix J is the identity.

In this case we have

$$\Delta^{q+1} = \Delta,\ a_{ik} = 0,\ a_{ii} = \Delta \qquad (i, k = 1, 2 \cdots, q, j;\ i \neq k).$$

Taking in turn $j = q+1,\ q+2, \cdots, m$, we have the result

$$(a_{ij}) \equiv \begin{bmatrix} \Delta\, 0 \cdots 0 \\ 0\, \Delta \cdots 0 \\ \cdots\cdots \\ 0\, 0 \cdots \Delta \end{bmatrix}.$$

Hence $\Delta \neq 0$ and therefore $\Delta^q = 1$. Inversely, every such substitution of G_m corresponds to the identity in $C_{m,q}$.

THEOREM.—*The continuous group of all m-ary linear homogeneous substitutions is* $(q, 1)$ *fold isomorphic to its* q^{th} *compound* $(q < m)$.

For linear substitutions in the $GF[p^n]$, we have

$$\Delta^q = 1, \quad \Delta^{p^n-1} = 1.$$

Thus we have the analogous.

THEOREM.—*The group of all m-ary linear homogeneous substitutions in the* $GF[p^n]$ *is* $(g, 1)$ *fold isomorphic to its* q^{th} *compound, g being the greatest common divisor of q and* $p^n - 1$.

4. From the results of § § 2–3, we derive immediately the

THEOREM.—*According as* $p = 2$ *or* $p > 2$, *the general Abelian group* A_{2m,p^n} *is holoedrically or hemiedrically* isomorphic to that subgroup of the second compound of the general 2m-ary linear homogeneous group in the* $GF[p^n]$ *which has as a relative invariant the linear function Z.*

The writer has shown (Bulletin, l. c.) that this second compound leaves invariant the Pfaffian

$$F_{2m} \equiv [1, 2, \cdots, 2m].$$

Hence A_{2m,p^n} is isomorphic to a linear homogeneous group in $m(2m-1)$ variables Y_{ij} with coefficients in the $GF[p^n]$ and having as relative invariants the functions

$$F_{2m}, \quad Z \equiv \sum_{l=1}^{m} Y_{2l-1\,2l}.$$

To the subgroup† of the latter which leaves these functions absolutely invariant there corresponds a self-conjugate subgroup of A_{2m,p^n}, which leaves the customary bilinear function absolutely invariant. This subgroup, containing only substitutions of determinant ± 1, may be designated as the *special* Abelian group A'_{2m,p^n}. It has the self-conjugate substitution which changes the signs of all the $2m$ indices. Except for $(2m, p^n) = (2, 2)$, $(2, 3)$ and $(4, 2)$, the quotient group H_{2m,p^n} is simple.‡

5. THEOREM.—*For* $p > 2$, *the simple group* H_{4,p^n}, *having the order*

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* Since the Abelian group contains the substitution $\xi_i' = -\xi_i (i = 1, 2, \cdots, 2m)$.

† This group has been studied by the writer in the Proceedings of the London Mathematical Society, l. c., §§ 22–33.

‡ DICKSON, *A triply infinite system of simple groups*, The Quarterly Journal of Mathematics, July, 1897; *ibid.*, April, 1899, for the cases $p = 2$, $2m = 4$, $n > 1$, previously unconsidered.

$$\tfrac{1}{2}(p^{4n}-1)(p^{2n}-1)p^{4n},$$

is simply isomorphic to the simple subgroup of equal order of the quinary orthogonal group in the $GF[p^n]$.

On introducing the invariant $Z \equiv Y_{12} + Y_{34}$ as a new variable in place of Y_{34}, the general substitution [see (2)] of the second compound of A'_{4,p^n} becomes, for $p > 2$:

$$\begin{array}{r|ccccc}
 & Y_{12}-\frac{1}{2}Z & Y_{13} & Y_{14} & Y_{23} & Y_{24} \\ \hline
(Y_{12}-\frac{1}{2}Z)' = & 2\begin{vmatrix}12\\12\end{vmatrix}-1 & \begin{vmatrix}12\\13\end{vmatrix} & \begin{vmatrix}12\\14\end{vmatrix} & \begin{vmatrix}12\\23\end{vmatrix} & \begin{vmatrix}12\\24\end{vmatrix} \\
Y_{13}' = & 2\begin{vmatrix}13\\12\end{vmatrix} & \begin{vmatrix}13\\13\end{vmatrix} & \begin{vmatrix}13\\14\end{vmatrix} & \begin{vmatrix}13\\23\end{vmatrix} & \begin{vmatrix}13\\24\end{vmatrix} \\
Y_{14}' = & 2\begin{vmatrix}14\\12\end{vmatrix} & \begin{vmatrix}14\\13\end{vmatrix} & \begin{vmatrix}14\\14\end{vmatrix} & \begin{vmatrix}14\\23\end{vmatrix} & \begin{vmatrix}14\\24\end{vmatrix} \\
Y_{23}' = & 2\begin{vmatrix}23\\12\end{vmatrix} & \begin{vmatrix}23\\13\end{vmatrix} & \begin{vmatrix}23\\14\end{vmatrix} & \begin{vmatrix}23\\23\end{vmatrix} & \begin{vmatrix}23\\24\end{vmatrix} \\
Y_{24}' = & 2\begin{vmatrix}24\\12\end{vmatrix} & \begin{vmatrix}24\\13\end{vmatrix} & \begin{vmatrix}24\\14\end{vmatrix} & \begin{vmatrix}24\\23\end{vmatrix} & \begin{vmatrix}24\\24\end{vmatrix}
\end{array}$$

It is therefore a substitution on five indices leaving absolutely invariant the function

$$\varphi \equiv (\tfrac{1}{2}Z)^2 - [1234] \equiv (Y_{12}-\tfrac{1}{2}Z)^2 + Y_{13}Y_{24} - Y_{14}Y_{23}.$$

This second compound is simply isomorphic to H_{4,p^n}. Indeed, the former is hemiedrically isomorphic to A'_{4,p^n} by §3; while to the substitution changing the sign of every index there corresponds the identity in the second compound.

By a simple transformation of indices* the function φ can be given the form

$$\sum_{i=1}^{5} x_i^2.$$

Hence the second compound is simply isomorphic to a subgroup O_{5,p^n} of the total orthogonal group O of determinant unity. From the result of §16 of the paper in the Proceedings of the London Mathematical Society, above cited, it follows that O_{5,p^n} does not contain the substitution which extends the simple subgroup of O of order

* See the first pages of the article, *Determination of the structure of all linear homogeneous groups in a Galois field which are defined by a quadratic invariant,* American Journal of Mathematics, July, 1899.

$$\tfrac{1}{2}(p^{4n}-1)(p^{2n}-1)p^{4n}$$

to the total group O. Hence $O_{5,\,p^n}$ is this simple subgroup.

This investigation also proves the theorem due to Lie: *The projective group of a linear complex in space of three dimensions is isomorphic to the projective group of a non-degenerate surface of the second order in space of four dimensions, each group having ten parameters.*

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