ON RELATIVE MOTION*

BY

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PART I.

Contents.

INTRODUCTION.

The theory of relative motion as developed in this paper originated in a memoir by E. BOUR in 1863.† It deals mainly with the so-called "second form" of differential equations of LAGRANGE and with the canonical system of differential equations of HAMILTON-JACOBI. BOUR, very modestly, claimed no dis-

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† *Journal de Liouville*, ser. 2, vol. 8.

covery. "There is no theory of relative motion, properly speaking," he remarks in the introduction to his memoir, and, in a certain sense, this is true as I have shown in the course of this paper (See chap. X, §1). Nevertheless, BOUR'S memoir was followed by several important applications, and the differential equations established by him for relative motion have been used since with great advantage. In this connection the memoir: *Sur l'application de la méthode de Lagrange à divers problèmes de mouvement relatif*,* by PH. GILBERT, and the dissertation: *Over de bettrekkelijke Beweging* † by KAMERLINGH-ONNES require special mention. The first is given over entirely to applications of LAGRANGE'S equations, as the title clearly indicates. The second is an application of the canonical system of equations of HAMILTON-JACOBI. But KAMERLINGH-ONNES' paper is not restricted to applications. The author first generalizes BOUR'S equations by assuming that the equations of constraint may involve the time explicitly. To KAMERLINGH-ONNES we also owe the first general expression of what I have called *the perturbative function of convective motion* in the subsequent pages.

The first part of my paper deals only with the *theory* of relative motion. The differential equations are derived from one fundamental principle embodied in the so-called "theorem of CORIOLIS." This enables us not only to write down the differential equations of relative motion *immediately* from the corresponding equations of absolute motion, but to obtain equations *as general as those known for absolute motion.* When the forces have a potential function my equations reduce to those of KAMERLINGH-ONNES. When, moreover, the constraints do not involve the time explicitly, they reduce to the equations of BOUR (Chapters I and II).

These equations are afterwards derived in a different way (Chapter X) and the connection between the two methods is brought forth (Chapter XI). At the same time SCHERING'S generalized notion of a force function is obtained and it is shown how this generalization may serve the theory of relative motion.

In Chapter III the notion of perturbations is introduced into the theory of relative motion, convective motion (or, what is commonly known as "space motion") being the disturbing factor. The perturbative function so obtained is the sum of three terms, only two of which had been well defined by BOUR and KAMERLINGH-ONNES, namely, the functions K and L; the form and mechanical significance of the third term (denoted in this paper by G_2) remained indefinite if not unknown. For the function G_2 several expressions are given which show explicitly that its existence is due solely to the constraints of the material

*Annales de la Société scientifique de Bruxelles; 1882, pp. 270–373; 1883, pp. 11–110. Also in book form, Paris, ed. Gauthier-Villars. The book had two editions.

†Nieuw Archief voor Wiskunde (Amsterdam); vol. 5. This paper reproduces only the mathematical part of KAMERLINGH-ONNES' dissertation which deals with physical problems.

system (Chapters V and VII) and the mechanical significance of this function is established (Chapters VIII and IX).

Chapters IV and VI contain some necessary digressions into transformations of coördinates. The formulæ obtained in these two chapters are capable of wider applications.*

I should like to say a few words in regard to chapters VIII and IX. The manner in which the kinetic energy of a material system may be decomposed seems to me worthy of notice. Indeed, while it is quite obvious that the effect of constraints involving the time explicitly could be produced by submitting the given material system to a succession of different constraints not involving the time explicitly and equivalent to the given system of constraints at each particular moment; while, moreover, it is quite as obvious that the velocity w_i of a particle at a given moment will be the resultant of two velocities w_{i1} and w_{i2}, w_{i1} being that velocity which the particle would have if the time, as far as it enters explicitly in the equations of constraint, became fixed at that moment, and w_{i2} being the component velocity of the particle arising from the variation of the time, as far as it enters explicitly in the equations of constraint; yet, the fact that the kinetic energy of the material system $\frac{1}{2}\sum m_i w_i^2$ will be equal to the sum of the kinetic energies $\frac{1}{2}\sum m_i w_{i1}^2$ and $\frac{1}{2}\sum m_i w_{i2}^2$ is by no means obvious. I venture to say that this proposition may find more than one application in physics.

The second part of my paper which I hope to bring out in the near future will contain some applications of the theory expounded in Part I. Among the problems to be treated may be mentioned the intricate problem of FOUCAULT'S pendulum when the oscillations are not infinitely small, and the problem of FOUCAULT'S top which PH. GILBERT was unable to solve (loc. cit. § XX). Both these problems can be easily solved by means of the theory and formulas given in Part I, and they yield highly interesting results.

CHAPTER I.

DIFFERENTIAL EQUATIONS OF LAGRANGE FOR RELATIVE MOTION.

§1. *Preliminaries.* Let $\Xi H Z$ be a system of coördinate axes fixed in space and XYZ another system of axes moving with regard to the first. The motion of a material system referred to the axes $\Xi H Z$ will be called *absolute*; referred to the axes XYZ it will be called *relative.* The motion of the system XYZ itself with regard to the fixed axes $\Xi H Z$ will be called *convective motion.* I introduce this term in preference to the one commonly used "space motion." It

*See in this connection a paper by D. BOBYLEF in vol. 58 of the Memoirs (Zapiski) of the Imperial Academy of Sciences of St. Petersburg, 1888.

will be observed that the term here proposed comes nearer to the corresponding terms in French and in German: *mouvement d'entrainement* and *Führungsbewegung*. Sometimes the word *convection* will be used in place of convective motion.

Having broached the question of terminology I also propose to use the term *turning acceleration* in lieu of "compound acceleration." "Turning acceleration" is a translation of the term used by J. SOMOF in his treatise of mechanics and it owes its origin to its true mechanical significance. In his German translation of Somof's classical book Professor A. Ziwet uses the term "Rückkehrbeschleunigung,"* which may well be translated "turning acceleration."

§2. *Theorem of Coriolis.* I derive the differential equations for relative motion from the following principle:

Let (E) be the differential equations of motion of a material system with regard to the invariable system XYZ on the supposition that convection did not exist, i. e., that the axes XYZ were fixed in space. To obtain the differential equations of relative motion of the material system with regard to the moving axes XYZ we only need to add in the equations (E) to the real forces acting at a particle m_i, two fictitious forces, namely $-m_i j_{ci}$ and $m_i j_{ti}$, denoting by m_i the mass of the particle, by j_{ci} its convective acceleration (i. e., acceleration due to convective motion) *and by j_{ti} its turning acceleration.*

I will not stop to demonstrate this principle which follows at once from the well-known theorem of Coriolis.

§ 3. *Differential equations of relative motion.* As a first application of the principle just enunciated, the differential equations of LAGRANGE for relative motion will be derived. Let $\mu = 3n - s$ be the degree of freedom of a given system of n particles subject to s conditions of constraint, and $q_1, q_2, \cdots, q_\mu$ the μ independent coördinate parameters which define the relative position of the system. By reason of the principle of § 2 we can write at once the differential equations of Lagrange for relative motion:

$$
(1) \qquad p_k = \frac{\partial T^{(r)}}{\partial q'_k}, \qquad \frac{\partial p_k}{dt} = \frac{\partial T^{(r)}}{\partial q_k} + Q_k, \qquad (k = 1, 2, \cdots, \mu),
$$

where $T^{(r)}$ denotes the relative kinetic energy of the system, and

$$
(2) \qquad Q_k = \sum_{i=1}^{i=n} \left(F_{ix} \frac{\partial x_i}{\partial q_k} + F_{iy} \frac{\partial y_i}{\partial q_k} + F_{iz} \frac{\partial z_i}{\partial q_k} \right),
$$

provided the forces F_i comprise the fictitious forces of Coriolis as well as the real forces, i. e.,

* See Professor ZIWET'S translation of SOMOF'S *Mechanics*: Kinematics, chap. 17, p. 394.

$$(3)\qquad \begin{aligned} F_{ix} &= X_i + m_i j_{tix} - m_i j_{cix}\,, \\ F_{iy} &= Y_i + m_i j_{tiy} - m_i j_{ciy}\,, \\ F_{iz} &= Z_i + m_i j_{tiz} - m_i j_{ciz}\,, \end{aligned}$$

X_i, Y_i, Z_i being the projections of the real force acting on the particle m_i.

§ 4. Let us now transform the expression of Q_k by introducing certain new functions K, L, G, defined as follows:

$$(4)\qquad K = -\sum_{i=1}^{i=n} m_i(x_i\dot{v}_{0x} + y_i\dot{v}_{0y} + z_i\dot{v}_{0z})\,,$$

$$(5)\qquad L = \sum_{i=1}^{i=n} m_i[\,p(y_iz_i' - z_iy_i') + q(z_ix_i' - x_iz_i') + r(x_iy_i' - y_ix_i')\,]\,,$$

$$(6)\qquad G = \tfrac{1}{2}\sum_{i=1}^{i=n} m_i[\,(qz_i - ry_i)^2 + (rx_i - pz_i)^2 + (py_i - qx_i)^2\,]\,,$$

where $\dot{v}_0$ denotes the acceleration of the origin of the axes of relative coördinates, and, p, q, r, the components of the angular velocity ω of convective rotation.

By means of the familiar formulas:

$$\begin{aligned} j_{cx} &= \dot{v}_{0x} + p(px + qy + rz) - \omega^2 x + \left(z\frac{dq}{dt} - y\frac{dr}{dt}\right), \\ j_{cy} &= \dot{v}_{0y} + q(px + qy + rz) - \omega^2 y + \left(x\frac{dr}{dt} - z\frac{dp}{dt}\right), \\ j_{cz} &= \dot{v}_{0z} + r(px + qy + rz) - \omega^2 z + \left(y\frac{dp}{dt} - x\frac{dq}{dt}\right); \end{aligned}$$

$$\begin{aligned} j_{tx} &= -\,2(\,qz' - ry')\,, \\ j_{ty} &= -\,2(\,rz' - pz')\,, \\ j_{tz} &= -\,2(py' - qx')\,; \end{aligned}$$

one can readily verify that

$$\sum_{i=1}^{i=n} m_i\left[(j_{tix} - j_{cix})\frac{\partial x_i}{\partial q_k} + (j_{tiy} - j_{ciy})\frac{\partial y_i}{\partial q_k} + (j_{tiz} - j_{ciz})\frac{\partial z_i}{\partial q_k}\right] = \frac{\partial}{\partial q_k}(K + L + G) - \frac{d}{dt}\left(\frac{\partial L}{\partial q_k'}\right),$$

and consequently,

$$(7)\qquad Q_k = Q_k^{(0)} + \frac{\partial}{\partial q_k}(K + L + G) - \frac{d}{dt}\left(\frac{\partial L}{\partial q_k'}\right),$$

where I have put

$$(8)\qquad Q_k^{(0)} = \sum_{i=1}^{i=n}\left(X_i\frac{\partial x_i}{\partial q_k} + Y_i\frac{\partial y_i}{\partial q_k} + Z_i\frac{\partial z_i}{\partial q_k}\right).$$

If now we introduce the expression (7) of Q_k into (1) and put

$$(9) \qquad \mathfrak{p}_k = p_k + \frac{\partial L}{\partial q_k'} = \frac{\partial (T^{(r)} + L)}{\partial q_k'},$$

we obtain the required differential equations :

$$(10) \qquad \frac{d\mathfrak{p}_k}{dt} = \frac{\partial}{\partial q_k}(T^{(r)} + K + L + G) + Q_k^{(0)} \qquad (k=1, 2, \cdots, \mu).$$

§ 5. *Bour's equations.* In particular, if the real forces have a potential function U and if we put for the sake of convenience,

$$(11) \qquad U_1 = U + K + G,$$

equations (10) become:

$$(10') \qquad \frac{d\mathfrak{p}_k}{dt} = \frac{\partial}{\partial q_k}(T^{(r)} + L + U_1) \qquad (k=1, 2, \cdots, \mu).$$

The last equations differ from those obtained by BOUR * only in form. Bour introduces the function T_2:

$$(12) \qquad T_2 = \frac{1}{2}\sum_{i=1}^{i=n} m_i[(x_i' + qz_i - ry_i)^2 + (y_i' + rx_i - pz_i)^2 + (z_i' + py_i - qx_i)^2].$$

It is easily seen that

$$(13) \qquad T_2 = T^{(r)} + L + G,$$

and since the function G does not contain the variables q_1'; q_2', $\cdots$, q_μ',

$$\frac{\partial T_2}{\partial q_k'} = \frac{\partial (T^{(r)} + L)}{\partial q_k'} = \mathfrak{p}_k,$$

so that equations (10′) are identical with the following :

$$(14) \qquad \frac{d}{dt}\left(\frac{\partial T_2}{\partial q_k'}\right) - \frac{\partial T_2}{\partial q_k} = \frac{\partial (U + K)}{\partial q_k}, \qquad (k=1, 2, \cdots, \mu),$$

given by BOUR.

§ 6. *The functions K, L, G.* These three functions may be thrown into more convenient forms which, moreover, have the advantage of making the results of this chapter applicable to solid bodies without further developments.

From the definitions of K, L and G we have:

$$(4') \qquad K = - M\dot{v}_0 r_c \cos(\dot{v}_0, r_c)$$

* Journal de Liouville, ser. 2, vol. 8, p. 15.

where M denotes the total mass of the material system and r_c the radius vector of its center of inertia;

$$(5') \qquad L = \omega \mathfrak{M}_0 \cos(\omega, \mathfrak{M}_0),$$

where $\mathfrak{M}_0$ denotes the principal moment of relative momentum with regard to the origin of the relative coördinates; and finally, G is the kinetic energy of the material system due to convective rotation, so that for a rigid system or solid body

$$(6') \qquad G = \tfrac{1}{2} I_\omega \omega^2,$$

where I_ω denotes the moment of inertia of the rigid system or solid body with regard to the instantaneous axis (ω).

§7. It will be observed that $K = 0$ in the following cases:

(1) If the origin of the relative coördinates is fixed or moving uniformly in a straight line;

(2) If this origin coincides with the center of inertia of the material system;

(3) If the acceleration of the origin of relative coördinates is at right angles to the radius vector of the center of inertia.

When the convective motion is *translatory*, $\omega = 0$ and consequently $L = G = 0$. The function L vanishes in two other cases:

(1) If the principal moment of relative momentum about the origin of relative coördinates vanishes;

(2) If this moment of relative momentum is at right angles to the instantaneous axis of convective rotation.

CHAPTER II.

Canonical Equations of Hamilton-Jacobi for Relative Motion.

§1. *Differential equations of relative motion.* According to the principle derived from the theorem of Coriolis we can write at once the differential equations of Hamilton-Jacobi for relative motion in the form:

$$(1) \qquad \begin{cases} \dfrac{dp_k}{dt} = -\dfrac{\partial \Theta}{\partial q_k} + Q_k, \quad \dfrac{dq_k}{dt} = \dfrac{\partial \Theta}{\partial p_k} \qquad (k = 1, 2, \cdots, \mu), \\[2ex] \Theta = \displaystyle\sum_{e=1}^{e=\mu} p_e q'_e - T^{(r)}, \end{cases}$$

where Q_k is again given by formula (7) of the preceding chapter.

§ 2. The variables $\mathfrak{p}_k$ will now be introduced instead of the p_k into formulas (1). Let Θ_1 denote the function Θ after this change of variables is effected. Then

$$\frac{\partial \Theta_1}{\partial q_k} = \frac{\partial \Theta}{\partial q_k} - \sum_{e=1}^{e=\mu} \frac{\partial \Theta}{\partial p_e} \frac{\partial}{\partial q_k}\left(\frac{\partial L}{\partial q'_e}\right) = \frac{\partial \Theta}{\partial q_k} - \sum_{e=1}^{e=\mu} q'_e \frac{\partial}{\partial q_k}\left(\frac{\partial L}{\partial q'_e}\right)$$
$$= \frac{\partial \Theta}{\partial q_k} - \frac{\partial}{\partial q_k} \sum_{e=1}^{e=\mu} q'_e \frac{\partial L}{\partial q'_e}.$$

On the other hand

$$\sum_{e=1}^{e=\mu} q'_e \frac{\partial L}{\partial q'_e} = L - \lambda, \tag{2}$$

where λ denotes the function

$$\lambda = \sum_{i=1}^{i=n} m_i \left[p\left(y_i \frac{\partial z_i}{\partial t} - z_i \frac{\partial y_i}{\partial t}\right) + q\left(z_i \frac{\partial x_i}{\partial t} - x_i \frac{\partial z_i}{\partial t}\right) + r\left(x_i \frac{\partial y_i}{\partial t} - y_i \frac{\partial x_i}{\partial t}\right)\right]. \tag{3}$$

In the partial derivatives, x_i, y_i, z_i are considered functions of $q_1, q_2, \cdots q_\mu, t$. Hence

$$\frac{\partial \Theta_1}{\partial q_k} = \frac{\partial \Theta}{\partial q_k} - \frac{\partial (L - \lambda)}{\partial q_k},$$

and therefore

$$-\frac{\partial \Theta}{\partial q_k} + Q_k = - \frac{\partial}{\partial q_k}(\Theta_1 - K - G - \lambda) - \frac{d}{dt}\left(\frac{\partial L}{\partial q'_k}\right) + Q_k^{(0)}.$$

Further, it is obvious that

$$\frac{\partial \Theta}{\partial p_k} = \frac{\partial \Theta_1}{\partial \mathfrak{p}_k}.$$

By means of the last two formulas the differential equations (1) may be given the new form :

$$\frac{d\mathfrak{p}_k}{dt} = - \frac{\partial}{\partial q_k}(\Theta_1 - K - G - \lambda) + Q_k^{(0)}, \tag{4}$$

$$\frac{dq_k}{dt} = \frac{\partial \Theta_1}{\partial \mathfrak{p}_k} = \frac{\partial}{\partial \mathfrak{p}_k}(\Theta_1 - K - G - \lambda), \tag{5}$$

This is the required system of differential equations.

§ 3. *Equations of Kamerlingh-Onnes.* In particular, if the real forces have a potential function U we arrive at the formulas given by KAMERLINGH-ONNES :*

* *Over de bettrekkelijke Beweging,* Nieuw Archief voor Wiskunde (Amsterdam), vol. 5.

$$(6)\qquad \begin{cases} \dfrac{d\mathfrak{p}_k}{dt} = -\dfrac{\partial H}{\partial q_k}, \quad \dfrac{dq_k}{dt} = \dfrac{\partial H}{\partial \mathfrak{p}_k}, \\ H = \Theta_1 - U_1 - \lambda, \end{cases}$$

U_1 having the same meaning as in the first chapter.

The mechanical significance of the function λ is similar to that of the function L, the difference being that into λ enter not the whole relative velocities x'_i, y'_i, z'_i, but only those parts of these velocities which would remain if the independent coördinate parameters became constant.

§ 4. *Bour's equations.* If x_i, y_i, z_i are functions of the variables $q_1, q_2, \cdots, q_\mu$ only, and if the equations of constraint do not involve the time explicitly, we shall have $\lambda = 0$ and $\Theta_1 = T^{(r)}$, so that equations (6) take the form:

$$(7)\qquad \begin{cases} \dfrac{d\mathfrak{p}_k}{dt} = -\dfrac{\partial H_1}{\partial q_k}, \quad \dfrac{dq_k}{dt} = \dfrac{\partial H_1}{\partial \mathfrak{p}_k}, \\ H_1 = T^{(r)} - U_1. \end{cases}$$

These equations were given first by BOUR (loc. cit.).

CHAPTER III.

CONVECTION AS A PERTURBATION. THE PERTURBATIVE FUNCTION OF CONVECTIVE MOTION.

§1. *Convection as a perturbation.* If convection did not exist, the absolute and the relative motions of a given material system would be identical. Convection may therefore be considered as a disturbing element of that motion which the material system would have if the axes XYZ were fixed in space. From this point of view the relative motion of the material system becomes a *disturbed motion*, the *undisturbed motion* being that which would take place if convection did not exist, *i. e.*, if the axis XYZ were fixed in space.

This theory gives rise to a *perturbative function* whose general expression I will now proceed to derive.

§2. *Differential equations of disturbed and undisturbed motion.* It is assumed that the real forces have a potential function U. Then we have to integrate the system:

$$(1)\qquad \begin{cases} \dfrac{d\mathfrak{p}_k}{dt} = -\dfrac{\partial H}{\partial q_k}, \quad \dfrac{dq_k}{dt} = \dfrac{\partial H}{\partial \mathfrak{p}_k} \qquad (k = 1, 2, \cdots, u), \\ H = \Theta_1 - U_1 - \lambda = \Theta_1 - U - K - G - \lambda. \end{cases}$$

If convection did not exist we should have

$$K = L = G = \lambda = 0, \quad \mathfrak{p}_k = p_k, \quad \Theta_1 = \Theta,$$

and the differential equations of undisturbed motion would reduce to

$$(2) \qquad \begin{cases} \dfrac{dp_k}{dt} = -\dfrac{\partial E}{\partial q_k}, \quad \dfrac{dq_k}{dt} = \dfrac{\partial E}{\partial p_k}; \\ E = \Theta - U. \end{cases}$$

§3. *The perturbative function of convective motion.* Let us denote by the symbol [ ] that in the enclosed function the letter p has been replaced by the letter $\mathfrak{p}$.

Now, put

$$(8) \qquad H = [E] + [\Omega],$$

and let us see what is the meaning of the function Ω.

First of all it is clear that $\Omega = 0$ when convection does not exist, because in that case $[E] = E = H$.

Further, it is readily seen that the canonical integrals of the differential equations (1) and those of the differential equations:

$$(4) \qquad \frac{dp_k}{dt} = -\frac{\partial(E+\Omega)}{\partial q_k}, \quad \frac{dq_k}{dt} = \frac{\partial(E+\Omega)}{\partial p_k} \qquad (k = 1, 2, \cdots, \mu),$$

are identical. In fact, the integration of the system (1) is equivalent to solving the partial differential equation:

$$(5) \qquad \frac{\partial V}{\partial t} + \{H\}_{\mathfrak{p} = \partial V/\partial q} = 0,$$

indicating by the symbol $\{\ \}_{\mathfrak{p} = \partial V/\partial q}$ that in the expression of H the letter $\mathfrak{p}$ is replaced by the partial derivative $\partial V/\partial q$; and the canonical integrals of the differential equations (1) will be:

$$(6) \qquad \frac{\partial V}{\partial \alpha_k} = -\beta_k,$$

where $\alpha_1, \alpha_2, \cdots, \alpha_\mu$ are the arbitrary constants arising from the integration of equation (5) and $\beta_1, \beta_2, \cdots, \beta_\mu$ denote new arbitrary constants. Now to integrate the system (4) we have to solve the partial differential equation:

$$\frac{\partial V}{\partial t} + \{E + \Omega\}_{p = \partial V/\partial q} = 0,$$

which is obviously identical with equation (5) and therefore leads to the same canonical integrals (6).

It follows from the above that Ω is the required perturbative function. It will be called the *perturbative function of convective motion.*

§4. *Solution of the problem when the integrals of undisturbed motion are known.* We have at the same time arrived at the following important result.

Suppose we have integrated the canonical system of differential equations (2) defining the motion of a material system on the assumption that the axes XYZ are fixed in space, and let $a_1, \beta_1; a_2, \beta_2; \cdots; a_\mu, \beta_\mu$ be the canonical system of arbitrary constants. *To obtain the integrals of relative motion of the given material system with regard to the moving axes XYZ, we only need to substitute in the integrals of the equations* (2), *instead of the constants $a_1, a_2, \cdots; \beta_1, \beta_2, \cdots$, their values as functions of t obtained through the integration of the new canonical system of differential equations*:

$$\frac{da_k}{dt} = \frac{\partial\Omega}{\partial\beta_k}, \quad \frac{d\beta_k}{dt} = -\frac{\partial\Omega}{\partial a_k}, \qquad (k=1, 2, \cdots, \mu),$$

where Ω is given by equation (3).

It remains to give the general expression of the perturbative function Ω.

§5. *Expression of the perturbative function.* The function Θ introduced in the preceding chapter must be expressed in terms of the variables $q_1, q_2, \cdots, q_\mu; p_1, p_2, \cdots, p_\mu; t$. If therefore we put

$$(7) \qquad q'_k = \xi_k + \sum_{e=1}^{e=\mu} h_{ke} p_e,$$

$$(8) \qquad T^{(r)} = \frac{1}{2}\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} h_{ke} p_k p_e + \tau,$$

where the coefficients h_{ke} and ξ_k, and the function* τ are given functions of $q_1, q_2, \cdots, q_\mu, t$, and $h_{ke} = h_{ek}$, we obtain:

$$\Theta = \frac{1}{2}\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} h_{ke} p_k p_e + \sum_{k=1}^{k=\mu} \xi_k p_k - \tau;$$

from which Θ_1 is derived by introducing the variables $\mathfrak{p}_k$ in lieu of the p_k by means of the formula:

$$p_k = \mathfrak{p}_k - \frac{\partial L}{\partial q'_k}.$$

* For the meaning of the function τ see (17) chap. IV and (21) chap. VI.

Thus we find that

$$\Theta_1 = \tfrac{1}{2}\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} h_{ke}\mathfrak{p}_k\mathfrak{p}_e - \sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} h_{ke}\mathfrak{p}_e\frac{\partial L}{\partial q_k'} + \tfrac{1}{2}\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} h_{ke}\frac{\partial L}{\partial q_k'}\frac{\partial L}{\partial q_e'}$$

$$+ \sum_{k=1}^{k=\mu}\xi_k\mathfrak{p}_k - \sum_{k=1}^{k=\mu}\xi_k\frac{\partial L}{\partial q_k'} - \tau .$$

Denote for a moment by Θ_1' the function Θ_1 when the letter $\mathfrak{p}$ is replaced by the letter p. Then

$$\Theta_1' = \Theta - \sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} h_{ke}p_e\frac{\partial L}{\partial q_k'} - \sum_{k=1}^{k=\mu}\xi_k\frac{\partial L}{\partial q_k'} + \tfrac{1}{2}\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} h_{ke}\frac{\partial L}{\partial q_k'}\frac{\partial L}{\partial q_e'}$$

$$= \Theta - \sum_{k=1}^{k=\mu} q_k'\frac{\partial L}{\partial q_k'} + \tfrac{1}{2}\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} h_{ke}\frac{\partial L}{\partial q_k'}\frac{\partial L}{\partial q_e'},$$

or, taking into account (2), chap. II,

(9) $$\Theta_1' = \Theta - L + \lambda + \tfrac{1}{2}\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} h_{ke}\frac{\partial L}{\partial q_k'}\frac{\partial L}{\partial q_e'} .$$

Now it follows from formula (3) that

$$\Omega = \Theta_1' - U_1 - \lambda - E = \Theta_1' - \Theta - K - G - \lambda ,$$

and substituting here the value of $\Theta_1' - \Theta$ from (9) we find :

(10) $$\Omega = -K - L - G + \tfrac{1}{2}\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} h_{ke}\frac{\partial L}{\partial q_k'}\frac{\partial L}{\partial q_e'},$$

the desired expression of the perturbative function.

This formula was given first by KAMERLINGH-ONNES (l. c.). Observing that

$$\sum_{e=1}^{e=\mu} h_{ke}\frac{\partial L}{\partial q_e'} = \frac{\partial L}{\partial p_k},$$

we may give to Ω the slightly different form :

(10′) $$\Omega = -K - L - G + \tfrac{1}{2}\sum_{k=1}^{k=\mu}\frac{\partial L}{\partial q_k'}\frac{\partial L}{\partial p_k} .$$

CHAPTER IV.

Digression on some Transformation Formulas in passing from Cartesian Coördinates to a system of Independent Coördinate Parameters.*

§ 1. *Preliminaries.* Let us first consider a moving system composed of n particles subject to the s conditions of constraint:

(1) $$\phi_1(u_1, u_2, \cdots, u_{3n}, t) = 0, \quad \phi_2 = 0, \cdots, \quad \phi_s = 0,$$

where

$$u_1 = x_1 \sqrt{m_1}, \quad u_4 = x_2 \sqrt{m_2}, \quad \cdots \quad u_{3n-2} = x_n \sqrt{m_n},$$
$$u_2 = y_1 \sqrt{m_1}, \quad u_5 = y_2 \sqrt{m_2}, \quad \cdots \quad u_{3n-1} = y_n \sqrt{m_n},$$
$$u_3 = z_1 \sqrt{m_1}, \quad u_6 = z_2 \sqrt{m_2}, \quad \cdots \quad u_{3n} \ = z_n \sqrt{m_n},$$

$m_1, m_2, \cdots, m_n$ denoting, as usual, the masses of the n particles.

Let, further, $q_1, q_2, \cdots, q_\mu$ be a system of independent coördinate parameters, μ being the degree of freedom of the given material system, i. e., $\mu = 3n - s$.

If we put

$$q_i = f_i\,(u_1, u_2, \cdots, u_{3n}, t) \qquad (i = 1, 2, \cdots, \mu),$$

the functions $f_1, f_2, \cdots, f_\mu$ will involve the time explicitly as well as the functions $\phi_1, \phi_2, \cdots, \phi_s$.

§ 2. *Object of this chapter.* The object of this chapter is to prove the formulas:

(2) $$X_i = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} A_{ke} X^{(k)} \frac{\partial u_i}{\partial q_e} + \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Phi_{ke} X(\phi_k) \frac{\partial \phi_e}{\partial u_i},$$

(3) $$\sum_{i=1}^{i=3n} (X_i)^2 = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} A_{ke} X^{(k)} X^{(e)} + \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Phi_{ke} X(\phi_k) X(\phi_e),$$

where the several symbols have the following meanings: $X_1, X_2, \cdots, X_{3n}$ are certain functions which will be determined later; $X^{(k)}$ and $X(\phi_k)$ are defined by the formulas:

(*a*) $$X^{(k)} = \sum_{i=1}^{i=3n} X_i \frac{\partial u_i}{\partial q_k},$$

(*b*) $$X(\phi_k) = \sum_{i=1}^{i=3n} X_i \frac{\partial \phi_k}{\partial u_i};$$

* See the note at the close of this chapter.

while to define A_{ke} and Φ_{ke} we have the formulas:

$$(c)\qquad \begin{cases} A_{ke} = A_{ek} = \dfrac{\partial \log A}{\partial a_{ke}} = \dfrac{\partial \log A}{\partial a_{ek}}, \\ A = | a_{ek} | \qquad\qquad (e, k=1, 2, \cdots, \mu), \\ a_{ek} = a_{ke} = \displaystyle\sum_{i=1}^{i=3n} \frac{\partial u_i}{\partial q_k}\frac{\partial u_i}{\partial q_e}, \end{cases}$$

$$(d)\qquad \begin{cases} \Phi_{ke} = \Phi_{ek} = \dfrac{\partial \log \Phi}{\partial \phi_{ke}} = \dfrac{\partial \log \Phi}{\partial \phi_{ek}}, \\ \Phi = | \phi_{ek} | \qquad\qquad (e, k=1, 2, \cdots, \mu), \\ \phi_{ek} = \phi_{ke} = \displaystyle\sum_{i=1}^{i=3n} \frac{\partial \phi_k}{\partial u_i}\frac{\partial \phi_e}{\partial u_i}. \end{cases}$$

§ 3. *Proof of the proposed formulas (2), (3).* Let δu_i, δq_k denote the variations of u_i and of q_k respectively. The $\delta q_1, \delta q_2, \cdots, \delta q_\mu$ will be arbitrary, while the $\delta u_1, \delta u_2, \cdots, \delta u_{3n}$ must satisfy the s conditions:

$$(4)\qquad 0 = \sum_{i=1}^{i=3n} \frac{\partial \phi_e}{\partial u_i}\delta u_i \qquad (e=1, 2, \cdots, s).$$

I shall now form the expression of the function

$$\sum_{k=1}^{k=\mu} \frac{\partial u_i}{\partial q_k}\delta q_k$$

in terms of the variations $\delta u_1, \delta u_2, \cdots, \delta u_{3n}$. Two such expressions will be obtained, and by comparing them the formulas (2) will be proved.

To begin with, it is obvious that

$$(5)\qquad \sum_{k=1}^{k=\mu} \frac{\partial u_i}{\partial q_k}\delta q_k = \delta u_i,$$

which is the first of the two desired expressions. On the other hand, the variations $\delta q_1, \delta q_2, \cdots, \delta q_\mu$ may be expressed in terms of the variations $\delta u_1, \delta u_2, \cdots, \delta u_{3n}$ as follows:

$$\delta q_k = \sum_{i=1}^{i=3n} \frac{\partial q_k}{\partial u_i}\delta u_i;$$

and thus the second of the two desired expressions is now obtained, namely,

$$(6)\qquad \sum_{k=1}^{k=\mu} \frac{\partial u_i}{\partial q_k}\delta q_k = \sum_{k=1}^{k=\mu}\sum_{j=1}^{j=3n} \frac{\partial u_i}{\partial q_k}\frac{\partial q_k}{\partial u_j}\delta u_j.$$

The right-hand sides of (5) and of (6) must be equal for all variations δu_1, δu_2, $\cdots$, δu_{3n} which satisfy the s conditions (4). Hence, introducing certain factors ρ_{ei} which will be determined later, we may write down the following equations:

$$(7) \qquad \delta u_i = \sum_{k=1}^{k=\mu} \sum_{j=1}^{j=3n} \frac{\partial u_i}{\partial q_k} \frac{\partial q_k}{\partial u_j} \delta u_j + \sum_{e=1}^{e=s} \sum_{j=1}^{j=3n} \rho_{ei} \frac{\partial \phi_e}{\partial u_j} \delta u_j \qquad (i=1, 2, \cdots, 3n),$$

from which we obtain the relations:*

$$(8) \qquad \sum_{k=1}^{k=\mu} \frac{\partial u_i}{\partial q_k} \frac{\partial q_k}{\partial u_i} = 1 - \sum_{e=1}^{e=s} \rho_{ei} \frac{\partial \phi_e}{\partial u_i},$$

$$(9) \qquad \sum_{k=1}^{k=\mu} \frac{\partial u_i}{\partial q_k} \frac{\partial q_k}{\partial u_j} = \quad - \sum_{e=1}^{e=s} \rho_{ei} \frac{\partial \phi_e}{\partial u_j} \qquad (j \neq i).$$

With the notations of §2, and with the following:

$$(e) \qquad c_{ke} = \sum_{i=1}^{i=3n} \frac{\partial q_k}{\partial u_i} \frac{\partial \phi_e}{\partial u_i},$$

the factors ρ_{ei} in equations (8), (9) are readily obtained in the form †

$$\rho_{ei} = \sum_{m=1}^{m=s} \Phi_{me} \left(\frac{\partial \phi_m}{\partial u_i} - \sum_{k=1}^{k=\mu} c_{km} \frac{\partial u_i}{\partial q_k} \right).$$

Introducing these expressions into formulas (8) and (9) we find:

$$(10) \quad \sum_{k=1}^{k=\mu} \frac{\partial u_i}{\partial q_k} \frac{\partial q_k}{\partial u_i} = 1 - \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \Phi_{me} \frac{\partial \phi_m}{\partial u_i} \frac{\partial \phi_e}{\partial u_i} + \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \sum_{k=1}^{k=\mu} \Phi_{me} c_{km} \frac{\partial u_i}{\partial q_k} \frac{\partial \phi_e}{\partial u_i},$$

$$(11) \quad \sum_{k=1}^{k=\mu} \frac{\partial u_j}{\partial q_k} \frac{\partial q_k}{\partial u_i} = \quad - \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \Phi_{me} \frac{\partial \phi_m}{\partial u_j} \frac{\partial \phi_e}{\partial u_i} + \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \sum_{k=1}^{k=\mu} \Phi_{me} c_{km} \frac{\partial u_j}{\partial q_k} \frac{\partial \phi_e}{\partial u_i} \qquad (j \neq i).$$

* Compare with the relations:

$$\sum_{i=1}^{i=3n} \frac{\partial q_k}{\partial u_i} \frac{\partial u_i}{\partial q_k} = 1, \qquad \sum_{i=1}^{i=3n} \frac{\partial q_k}{\partial u_i} \frac{\partial u_i}{\partial q_e} = 0 \qquad (k \neq e),$$

which follow from the definition of $q_1, q_2, \cdots, q_\mu$.

† Proceed as follows: multiply both sides of (8) and of (9) by $\partial \phi_m / \partial u_i$ and by $\partial \phi_m / \partial u_j$ respectively; then add together the resulting equations, thus obtaining the s equations:

$$\sum_{k=1}^{k=\mu} \sum_{j=1}^{j=3n} \frac{\partial u_i}{\partial q_k} \frac{\partial q_k}{\partial u_j} \frac{\partial \phi_m}{\partial u_j} = \sum_{k=1}^{k=\mu} c_{km} \frac{\partial u_i}{\partial q_k} = \frac{\partial \phi_m}{\partial u_i} - \sum_{e=1}^{e=s} \rho_{ei} \, \varphi_{em} \qquad (m=1, 2, \cdots, s),$$

from which the above expression of ρ_{ei} is readily derived.

§ 4. Now, let $X_1, X_2, \cdots, X_{3n}$ be certain functions to be fixed later. Multiply both sides of (10) and of (11) by X_i and by X_j respectively, then give j all the values $1, 2, \cdots, 3n$ except i and add together the resulting equations. The relations so obtained will be:

$$(12)\quad \sum_{k=1}^{k=\mu} X^{(k)} \frac{\partial q_k}{\partial u_i} = X_i - \sum_{m=1}^{m=s}\sum_{e=1}^{e=s} \Phi_{me} X(\phi_m) \frac{\partial \phi_e}{\partial u_i} + \sum_{m=1}^{m=s}\sum_{e=1}^{e=s}\sum_{k=1}^{k=\mu} \Phi_{me} c_{km} X^{(k)} \frac{\partial \phi_e}{\partial u_i} \qquad (i=1, 2, \cdots, 3n).$$

§ 4 bis. A first selection of the functions $X_1, X_2, \cdots, X_{3n}$ will now be made. Let $X_i = \partial u_i/\partial q_r$. Then

$$X^{(k)} = \sum_{i=1}^{i=3n} \frac{\partial u_i}{\partial q_r}\frac{\partial u_i}{\partial q_k} = a_{kr},$$

$$X(\phi_m) = \sum_{i=1}^{i=3n} \frac{\partial \phi_m}{\partial u_i}\frac{\partial u_i}{\partial q_r} = 0 \qquad (m=1, 2, \cdots, s);$$

and, therefore, by (12),

$$(13)\qquad \sum_{k=1}^{k=\mu} a_{kr} \frac{\partial q_k}{\partial u_i} = \frac{\partial u_i}{\partial q_r} + \sum_{k=1}^{k=\mu}\sum_{m=1}^{m=s}\sum_{e=1}^{e=s} \Phi_{me} c_{km} a_{kr} \frac{\partial \phi_e}{\partial u_i} \qquad (r=1, 2, \cdots, \mu).$$

From the μ equations (13) we obtain

$$(14)\qquad \frac{\partial q_k}{\partial u_i} = \sum_{e=1}^{e=\mu} A_{ke} \frac{\partial u_i}{\partial q_e} + \sum_{m=1}^{m=s}\sum_{e=1}^{e=s} \Phi_{me} c_{km} \frac{\partial \phi_e}{\partial u_i} \qquad (k=1, 2, \cdots, \mu).$$

Equations (13), (14) may be found in a paper by D. Bobylef.*

§ 5. Substituting the expressions (14) of $\partial q_k/\partial u_i$ into formula (12) the latter becomes:

$$(2)\qquad X_i = \sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} A_{ke} X^{(k)} \frac{\partial u_i}{\partial q_e} + \sum_{k=1}^{k=s}\sum_{e=1}^{e=s} \Phi_{ke} X(\phi_k) \frac{\partial \phi_e}{\partial u_i} \qquad (i=1, 2, \cdots, 3n),$$

which is the first of the two formulas we proposed to prove.

Let now $Y_1, Y_2, \cdots, Y_{3n}$ be a new set of functions left for the present indeterminate. If we multiply both sides of (2) by Y_i and add all the resulting equations ($i = 1, 2, \cdots, 3n$) we shall have the formula:

$$(15)\qquad \sum_{i=1}^{i=3n} X_i Y_i = \sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} A_{ke} X^{(k)} Y^{(e)} + \sum_{k=1}^{k=s}\sum_{e=1}^{e=s} \phi_{ke} X(\phi_k) Y(\phi_e),$$

where

$$(f)\qquad Y^{(k)} = \sum_{i=1}^{i=3n} Y_i \frac{\partial u_i}{\partial q_k}, \quad Y(\phi_k) = \sum_{i=1}^{i=3n} Y_i \frac{\partial \phi_k}{\partial u_i}.$$

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* *On the change of coördinates in the differential equations of dynamics.* Supplement to the 58th vol. of the Memoirs (Zapiski) of the Imperial Academy of Sciences of St. Petersburg; 1888 (Russian).

In particular, if $X_i = Y_i$ $(i = 1, 2, \cdots, 3n)$, (15) reduces to (3) which is thus proved.

§ 6. *Kinetic energy of a material system. Bobylef's formula.* To conclude, I will make an application of formulas (2) and (3).

Let $A_i = du_i/dt$. Then we shall have:

$$X(\phi_k) = \sum_{i=1}^{i=3n} \frac{\partial \phi_k}{\partial u_i} \frac{du_i}{dt} = -\frac{\partial \phi_k}{\partial t},$$

$\partial\phi_k/\partial t$ denoting the partial derivative of the function ϕ_k with respect to the time; and, further,

$$X^{(k)} = \sum_{i=1}^{i=3n} \frac{\partial u_i}{\partial q_k} \frac{du_i}{dt} = \sum_{i=1}^{i=3n} \frac{\partial u_i'}{\partial q_k'} u_i' = \frac{\partial T}{\partial q_k'} = p_k,$$

where $T = \frac{1}{2} \sum_{i=1}^{i=3n} u_i'^2$. Hence, by formula (2),

$$\frac{du_i}{dt} = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} A_{ke} \frac{\partial u_i}{\partial q_k} p_e - \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Phi_{ke} \frac{\partial \phi_k}{\partial t} \frac{\partial \phi_e}{\partial u_i}, \tag{16}$$

and, by formula (3),

$$T = \tfrac{1}{2} \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} A_{ke} p_k p_e + \tfrac{1}{2} \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Phi_{ke} \frac{\partial \phi_k}{\partial t} \frac{\partial \phi_e}{\partial t}. \tag{17}$$

Formula (17) was given for the first time by D. BOBYLEF (loc. cit.).

Note. In deriving the formulas of this chapter no mention was made as to whether the axes XYZ were fixed in space or moving. Indeed, the results here obtained are entirely independent of such considerations: x, y, z may be either *absolute* or *relative* coördinates. A similar remark must be made with regard to the result of chapter VI.

CHAPTER V.

APPLICATION OF THE FORMULAS OF THE PRECEDING CHAPTER TO THE PERTURBATIVE FUNCTION OF CONVECTIVE MOTION.

§1. *The function* G_2. If we put

$$G_2 = G - \tfrac{1}{2} \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} h_{ke} \frac{\partial L}{\partial q_k'} \frac{\partial L}{\partial q_e'}, \tag{1}$$

the perturbative function of convective motion takes the form:

$$\Omega = -K - L - G_2. \tag{2}$$

Systems consisting of n particles will be first considered and the function G_2 for such systems will be transformed by means of formula (3) of the preceding chapter.

To begin with I remark that for a system of n particles the coefficients h_{ke} are identical with the coefficients denoted by A_{ke} in the preceding chapter, so that

$$(1') \qquad G_2 = G - \tfrac{1}{2}\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} A_{ke}\frac{\partial L}{\partial q'_k}\frac{\partial L}{\partial q'_e}.$$

Let us now put $X_i = \partial L/\partial u'_i$ in formula (3), chapter IV. Then we shall have:

$$X^{(k)} = \sum_{i=1}^{i=3n}\frac{\partial L}{\partial u'_i}\frac{\partial u_i}{\partial q_k} = \sum_{i=1}^{i=3n}\frac{\partial L}{\partial u'_i}\frac{\partial u'_i}{\partial q'_k} = \frac{\partial L}{\partial q'_k},$$

$$\sum_{i=1}^{i=3n}(X_i)^2 = \sum_{i=1}^{i=3n}\left(\frac{\partial L}{\partial u'_i}\right)^2 = 2G;$$

and therefore, by formula (3),

$$(3) \qquad G = \tfrac{1}{2}\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} A_{ke}\frac{\partial L}{\partial q'_k}\frac{\partial L}{\partial q'_e} + \tfrac{1}{2}\sum_{k=1}^{k=s}\sum_{e=1}^{e=s}\Phi_{ke}L(\phi_k)L(\phi_e),$$

where I have put

$$(4) \qquad L(\phi_k) = \sum_{i=1}^{i=3n}\frac{\partial L}{\partial u'_i}\frac{\partial \phi_k}{\partial u_i}.$$

Substituting the expression (3) of G into formula (1′) we find that

$$(5) \qquad G_2 = {}^1\sum_{k=1}^{k=s}\sum_{e=1}^{e=s}\Phi_{ke}L(\phi_k)L(\phi_e).$$

§2. *Perturbative function for a free system of particles.* It follows from this expression of the function G_2 that $G_2 = 0$ when $s = 0$. Hence this theorem due to KAMERLINGH-ONNES but not rigorously proved by him:

For a free system of particles the perturbative function of convective motion reduces to $-(K + L)$.

§ 3. *Perturbative function for a freely moving rigid system.* Another interesting case when $G_2 = 0$ is that of a freely moving rigid system, as will presently be shown. First it may be remarked that

$$\begin{aligned}\sum_{i=1}^{i=3n}\frac{\partial L}{\partial u'_i}\frac{\partial \phi_k}{\partial u_i} &= \sum_{i=1}^{i=3n}\frac{1}{m_i}\left(\frac{\partial L}{\partial x'_i}\frac{\partial \phi_k}{\partial x_i} + \frac{\partial L}{\partial y'_i}\frac{\partial \phi_k}{\partial y_i} + \frac{\partial L}{\partial z'_i}\frac{\partial \phi_k}{\partial z_i}\right)\\ &= \sum_{i=1}^{i=3n}\left((qz_i - ry_i)\frac{\partial \phi_k}{\partial x_i} + (rx_i - pz_i)\frac{\partial \phi_k}{\partial y_i} + (py_i - qx_i)\frac{\partial \phi_k}{\partial z_i}\right)\\ &= p\sum_{i=1}^{i=3n}\left(y_i\frac{\partial \phi_k}{\partial z_i} - z_i\frac{\partial \phi_k}{\partial y_i}\right) + q\sum_{i=1}^{i=n}\left(z_i\frac{\partial \phi_k}{\partial x_i} - x_i\frac{\partial \phi_k}{\partial z_i}\right)\\ &\qquad + r\sum_{i=1}^{i=3n}\left(x_i\frac{\partial \phi_k}{\partial y_i} - y_i\frac{\partial \phi_k}{\partial x_i}\right),\end{aligned}$$

and therefore, if we put for the sake of convenience,

$$(6) \qquad \begin{cases} R_x^{(k)} = \sum_{i=1}^{i=3n} \left(y_i \frac{\partial \phi_k}{\partial z_i} - z_i \frac{\partial \phi_k}{\partial y_i} \right), \\ R_y^{(k)} = \sum_{i=1}^{i=3n} \left(z_i \frac{\partial \phi_k}{\partial x_i} - x_i \frac{\partial \phi_k}{\partial z_i} \right), \\ R_z^{(k)} = \sum_{i=1}^{i=3n} \left(x_i \frac{\partial \phi_k}{\partial y_i} - y_i \frac{\partial \phi_k}{\partial x_i} \right), \end{cases}$$

we shall have:

$$(7) \qquad L(\phi_k) = pR_x^{(k)} + qR_y^{(k)} + rR_z^{(k)};$$

or, denoting by $R^{(k)}$ a vector whose projections on the axes XYZ are $R_x^{(k)}$, $R_y^{(k)}$, $R_z^{(k)}$ respectively,

$$(8) \qquad L(\phi_k) = \omega R^{(k)} \cos(\omega, R^{(k)}).$$

This expression of $L(\phi_k)$ shows that $L(\phi_k)$ vanishes with $R^{(k)}$, i. e., when

$$(9) \qquad R_x^{(k)} = R_y^{(k)} = R_z^{(k)} = 0.$$

Conditions (9) will be satisfied if the constraint ϕ_k is given by an equation of the form:

$$\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2} = \text{constant}.$$

If *all* the constraints are of this form, then we have a freely moving rigid system of n particles ($\mu = 6$, $s = 3n - 6$); and since in this case $L(\phi_1) = L(\phi_2) = \cdots = L(\phi_s) = 0$, the function G_2 will vanish. Thus I have proved the following theorem:

For a freely moving rigid system of particles the perturbative function of convective motion reduces to $-(K + L)$.

§ 4. *The perturbative function for very small values of* ω. A glance at formulas (4), (5) of chap. I and (2), (5), (7) of this chapter shows that

The perturbative function Ω *is the sum of three functions* $-K$, $-L$, $-G_2$, *all three homogeneous in* p, q, r. *The degree of homogeneity is* 0 *for* K, 1 *for* L, 2 *for* G_2.

Sometimes it is convenient to take ω as a small quantity of the first order (as in the case of motion on the surface of the earth). The function L will then be of the first order in ω, the function G_2 of the second. Looking upon Ω as a function of ω we can say that

The perturbative function of the second order in ω *is due solely to the existence of constraints. It vanishes when the system is free or when the particles form a freely moving rigid system.*

§ 5. *Cases when* $G_2 = 0$. More generally, if we observe that *the vector* $R^{(k)}$ *is parallel and proportional to the principal moment of the reactions due to the constraint* ϕ_k, it will be clear that $L(\phi_k) = 0$ when this moment vanishes or when its direction forms a right angle with the direction of the instantaneous axis (ω). Hence,

$G_2 = 0$ *when the principal moments of the reactions due to the constraints* $\phi_1, \phi_2, \cdots, \phi_s$, *either vanish or are at right angles to the direction of the instantaneous axis* (ω).

This theorem includes the case considered in § 3.

§ 6. *Case of a solid body. Expression of the function* G. By a process commonly used in mechanics we at once pass from rigid systems to solid bodies.

Let $\eta_1, \eta_2, \cdots, \eta_6$ be any system of six coördinate parameters which define the position of a free solid body with regard to the invariable * system XYZ and such that the coördinates x, y, z of any point of the body be functions of $\eta_1, \eta_2, \cdots, \eta_6$ only, i. e., these functions *do not involve the time explicitly.* Then, as is well known,

$$T = \tfrac{1}{2} \int (x'^2 + y'^2 + z'^2) dm \tag{10}$$

will be a homogeneous function of the second degree in $\eta_1', \eta_2', \cdots, \eta_6'$, and therefore,

$$2T = \sum_{i=1}^{i=6} \frac{\partial T}{\partial \eta_i'} \eta_i' . \tag{11}$$

Now, let us put

$$\mu_{ij} = \mu_{ji} = \int \left(\frac{\partial x}{\partial \eta_i} \frac{\partial x}{\partial \eta_j} + \frac{\partial y}{\partial \eta_i} \frac{\partial y}{\partial \eta_j} + \frac{\partial z}{\partial \eta_i} \frac{\partial z}{\partial \eta_j} \right) dm. \tag{12}$$

Then, as is readily seen from formulas (10) and (12),

$$\frac{\partial T}{\partial \mu_i'} = \int \left(x' \frac{\partial x}{\partial \eta_i} + y' \frac{\partial y}{\partial \eta_i} + z' \frac{\partial z}{\partial \eta_i} \right) dm = \sum_{j=1}^{j=6} \mu_{ij} \eta_j' ;$$

and, therefore, by formula (11),

$$T = \tfrac{1}{2} \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} \eta_i' \eta_j' . \tag{13}$$

The coefficients μ_{ij} given by formula (12) take the place, in the case of solid bodies, of the coefficients a_{ek} introduced in chapter IV for systems of particles. Likewise to the coefficients A_{ke} will now correspond certain coefficients M_{ij} defined as follows:

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* Not to be confused with the term *fixed*. An invariable system is *not deformable* but it may be moving.

$$(14)\qquad \begin{cases} M_{ij} = M_{ji} = \dfrac{\partial \log M}{\partial \mu_{ij}} = \dfrac{\partial \log M}{\partial \mu_{ji}}, \\ M = |\mu_{ij}| \qquad (i, j = 1, 2, \cdots 6). \end{cases}$$

§ 7. We have seen that for a freely moving rigid system of n particles $G_2 = 0$, and therefore, by (3), for such a system,

$$(15)\qquad G = \tfrac{1}{2} \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} A_{ke} \frac{\partial L}{\partial q'_k} \frac{\partial L}{\partial q'_e},$$

where now $\mu = 6$. Passing in the familiar way from rigid systems to solid bodies we only have to replace in formula (15) the coefficients A_{ke} by the coefficients M_{ke} and the letter q by the letter η; so that for a free solid body,

$$(16)\qquad G = \tfrac{1}{2} \sum_{k=1}^{k=6} \sum_{e=1}^{e=6} M_{ke} \frac{\partial L}{\partial \eta'_k} \frac{\partial L}{\partial \eta'_e}.$$

Thus we have the following theorem:

If $\eta_1, \eta_2, \cdots, \eta_6$ be any system of coördinate parameters which define the position of a free solid body with regard to the invariable system XYZ; and if the kinetic energy of the body be given by an expression of the form (13); then the function G will be given by formula (16) where the coefficients M_{ke} are derived from the coefficients μ_{ke}, functions of $\eta_1, \eta_2, \cdots, \eta_6$, by means of the relations (14).

§ 8. *Perturbative function for a free solid body.* We may now extend the results of § 3 to solid bodies, namely:

For a free solid body the perturbative function of convective motion reduces to $-(K + L)$.

CHAPTER VI.

Another digression on Transformations of Coördinates in the Equations of Dynamics.*

§1. *Preliminaries.*—We will now pass on to the motion of a solid body subject to s conditions of constraint:

$$(1)\qquad \psi_1(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6, t) = 0, \ \psi_2 = 0, \ \cdots, \ \psi_s = 0,$$

where $\eta_1, \cdots, \eta_6$ form a system of coördinates defined as in the preceding chapter. Let again,

* See the note at the end of chapter IV.

$$(2)\qquad T=\tfrac{1}{2}\sum_{i=1}^{i=6}\sum_{j=1}^{j=6}\mu_{ij}\eta'_i\eta'_j$$

be the kinetic energy of the body, the coefficients μ_{ij} being given functions of $\eta_1, \cdots, \eta_6$. Let, further, $q_1, \cdots, q_\mu$ be a system of independent coördinate parameters, μ being the degree of freedom of the body, i. e., $\mu = 6 - s$. If we put

$$q_1 = f_1(\eta_1, \cdots, \eta_6, t),$$
$$\cdots\cdots\cdots\cdots$$
$$q_\mu = f_\mu(\eta_1, \cdots, \eta_6, t),$$

the functions $f_1, f_2, \cdots, f_\mu$, as well as the functions $\psi_1, \cdots, \psi_s$, will involve the time explicitly.

§2. *Object of this chapter. Proof of the proposed formulas.*—The object of this chapter is to prove the three formulas (13), (14) and (15) given below. To this end let $\delta\eta_i$, δq_k denote respectively the variations of η_i and of q_k $(i = 1, 2, \cdots, 6; k = 1, 2, \cdots, \mu)$. The variations $\delta q_1, \cdots, \delta q_\mu$ will be arbitrary while the variations $\delta\eta_1, \cdots, \delta\eta_6$ must satisfy the s conditions:

$$(3)\qquad 0=\sum_{i=1}^{i=6}\frac{\partial\psi_e}{\partial\eta_i}\delta\eta_i \qquad (e=1, 2, \cdots, s).$$

If we consider the expression:

$$\sum_{k=1}^{k=\mu}\frac{\partial\eta_i}{\partial q_k}\delta q_k,$$

after the variations $\delta\eta_1, \delta\eta_2, \cdots, \delta\eta_6$ are introduced instead of the variations $\delta q_1, \cdots, \delta q_\mu$, then, by a process similar to that employed in chapter IV, we arrive at the formulas:

$$(4)\qquad \sum_{k=1}^{k=\mu}\frac{\partial\eta_i}{\partial q_k}\frac{\partial q_k}{\partial\eta_i}=1-\sum_{e=1}^{e=s}\rho_{ei}\frac{\partial\psi_e}{\partial\eta_i},$$

$$(5)\qquad \sum_{k=1}^{k=\mu}\frac{\partial\eta_i}{\partial q_k}\frac{\partial q_k}{\partial\eta_j}=-\sum_{e=1}^{e=s}\rho_{ei}\frac{\partial\psi_e}{\partial\eta_j}, \qquad (j \neq i),$$

in which the ρ_{ei} denote coefficients to be determined presently.

§3. I shall use, besides the notations (14) of the preceding chapter, also the following:

$$(a)\qquad \left\{\begin{aligned} b_{ke} &= b_{ek}=\sum_{i=1}^{i=6}\sum_{j=1}^{j=6}\mu_{ij}\frac{\partial\eta_i}{\partial q_k}\frac{\partial\eta_j}{\partial q_e},\\ B &= |b_{ke}|\\ B_{ke} &= B_{ek}=\frac{\partial\log B}{\partial b_{ke}}=\frac{\partial\log B}{\partial b_{ek}}; \end{aligned}\right. \qquad (k, e=1, 2, \cdots, \mu),$$

$$(b) \qquad d_{me} = \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} M_{ij} \frac{\partial \psi_m}{\partial \eta_i} \frac{\partial q_e}{\partial \eta_j};$$

$$(c) \qquad \begin{cases} \psi_{ke} = \psi_{ek} = \sum\limits_{i=1}^{i=6} \sum\limits_{j=1}^{j=6} M_{ij} \dfrac{\partial \psi_k}{\partial \eta_i} \dfrac{\partial \psi_e}{\partial \eta_j}, \\ \Psi = |\psi_{ke}| \qquad\qquad (k, e = 1, 2, \cdots, s), \\ \Psi_{ke} = \Psi_{ek} = \dfrac{\partial \log \Psi}{\partial \psi_{ke}} = \dfrac{\partial \log \Psi}{\partial \psi_{ek}}. \end{cases}$$

With these notations we readily find the following expressions for the coefficients ρ_{ei}: *

$$\rho_{ei} = \sum_{m=1}^{m=s} \Psi_{me} \left[\sum_{n=1}^{n=6} M_{in} \frac{\partial \psi_m}{\partial \eta_n} - \sum_{k=1}^{k=\mu} d_{mk} \frac{\partial \eta_i}{\partial q_k} \right];$$

substituting these expressions, the relations (4) and (5) become:

$$(6) \quad \sum_{k=1}^{k=\mu} \frac{\partial \eta_i}{\partial q_k} \frac{\partial q_k}{\partial \eta_i} = 1 - \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \sum_{n=1}^{n=6} \Psi_{me} M_{in} \frac{\partial \psi_m}{\partial \eta_n} \frac{\partial \psi_e}{\partial \eta_i} + \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \sum_{k=1}^{k=\mu} \Psi_{me} d_{mk} \frac{\partial \eta_i}{\partial q_k} \frac{\partial \psi_e}{\partial \eta_i},$$

$$(7) \quad \sum_{k=1}^{k=\mu} \frac{\partial \eta_i}{\partial q_k} \frac{\partial q_k}{\partial \eta_j} = - \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \sum_{n=1}^{n=6} \Psi_{me} M_{jn} \frac{\partial \psi_m}{\partial \eta_n} \frac{\partial \psi_e}{\partial \eta_i} + \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \sum_{k=1}^{k=\mu} \Psi_{me} d_{mk} \frac{\partial \eta_j}{\partial q_k} \frac{\partial \psi_e}{\partial \eta_i} \qquad (j \neq i).$$

§ 4. Let again $X_1, X_2, \cdots, X_6$ denote certain functions to be fixed later. For the sake of convenience put

$$(d) \qquad X^{(k)} = \sum_{i=1}^{i=6} X_i \frac{\partial \eta_i}{\partial q_k}, \qquad \mathfrak{X}^{(k)} = \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} \cdot X_j \frac{\partial \eta_i}{\partial q_k};$$

$$(e) \qquad X(\psi_k) = \sum_{i=1}^{i=6} X_i \frac{\partial \psi_k}{\partial \eta_i}, \qquad \mathfrak{X}(\psi_k) = \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} M_{ij} \cdot X_j \frac{\partial \psi_k}{\partial \eta_i}.$$

If we multiply both sides of (6) by X_i and both sides of (7) by X_j, then give j all the values $1, \cdots, 6$, except i, and add together all the resulting equations, we shall obtain:

$$(8) \quad \left\{ X_i = \sum_{k=1}^{k=\mu} X^{(k)} \frac{\partial q_k}{\partial \eta_i} + \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \psi_{me} \mathfrak{X}(\psi_m) \frac{\partial \psi_e}{\partial \eta_i} - \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \sum_{k=1}^{k=\mu} \psi_{me} d_{mk} X^{(k)} \frac{\partial \psi_e}{\partial \eta_i} \right. \qquad (i = 1, 2, \cdots, 6).$$

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* Proceed as follows:—multiply both sides of equations (4) and (5) by $M_{in}\, \partial\psi_m/\partial\eta_n$ and $M_{jn}\, \partial\psi_m/\partial\eta_n$ respectively; let j take all the values $1, 2, \cdots, 6$ except i; add together all the resulting equations and sum with regard to the index n; then the equations:

$$\sum_{k=1}^{k=\mu} d_{mk} \frac{\partial \eta_i}{\partial q_k} = \sum_{n=1}^{n=6} M_{in} \frac{\partial \psi_m}{\partial \eta_n} - \sum_{e=1}^{e=s} \rho_{ei} \psi_{me} \qquad (i = 1, 2, \cdots, 6):$$

will be obtained, from which at once follow the above expressions for the coefficients ρ_{ei}.

Likewise, if after interchanging the letters i and j in (7) we multiply both sides of (6) by X_i and both sides of (7) by X_j, give j all the values $1, 2, \cdots, 6$ except i, and add together all the resulting equations; then we obtain:

$$(9) \quad \begin{cases} X_i = \sum_{j=1}^{j=6} \sum_{k=1}^{k=\mu} X_j \frac{\partial \eta_i}{\partial q_k} \frac{\partial q_k}{\partial \eta_j} + \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \sum_{n=1}^{n=6} \psi_{me} M_{in} X(\psi_e) \frac{\partial \psi_m}{\partial \eta_n} \\ \qquad - \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \sum_{k=1}^{k=\mu} \psi_{me} d_{mk} X(\psi_e) \frac{\partial \eta_i}{\partial q_k} \qquad (i = 1, 2, \cdots, 6). \end{cases}$$

Finally, substitute the function $\sum_{j=1}^{j=6} X_j \mu_{ij}$ for the function X_i in (8); then $X^{(k)}$ will change into $\mathfrak{X}^{(k)}$, $\mathfrak{X}(\psi_m)$ into $X(\psi_m)$, and formula (8) will become:

$$(10) \quad \begin{cases} \sum_{j=1}^{j=6} \mu_{ij} X_j = \sum_{k=1}^{k=\mu} \mathfrak{X}^{(k)} \frac{\partial q_k}{\partial \eta_i} + \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \Psi_{me} X(\psi_m) \frac{\partial \psi_e}{\partial \eta_i} \\ \qquad - \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \sum_{k=1}^{k=\mu} \Psi_{me} d_{mk} \mathfrak{X}^{(k)} \frac{\partial \psi_e}{\partial \eta_i} \qquad (i = 1, 2, \cdots, 6). \end{cases}$$

§ 4 bis. I will now make a first selection of the functions $X_1, X_2, \cdots, X_6$. Let $X_i = \partial \eta_i / \partial q_n$. Then

$$X(\psi_m) = \sum_{j=1}^{j=6} \frac{\partial \psi_m}{\partial \eta_j} \frac{\partial \eta_j}{\partial q_n} = 0 \qquad (m = 1, 2, \cdots, s),$$

$$\mathfrak{X}^{(k)} = \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} \frac{\partial \eta_i}{\partial q_k} \frac{\partial \eta_j}{\partial q_n} = b_{kn};$$

and therefore, by formula (10),

$$(11) \qquad \sum_{j=1}^{j=6} \mu_{ij} \frac{\partial \eta_j}{\partial q_n} = \sum_{k=1}^{k=\mu} b_{kn} \frac{\partial q_k}{\partial \eta_i} - \sum_{k=1}^{k=\mu} \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \Psi_{me} d_{mk} b_{kn} \frac{\partial \psi_e}{\partial \eta_i}.$$

Solving the μ equations obtained by making $n = 1, 2, \cdots, \mu$ in formula (11), with regard to the $\partial q_1 / \partial \eta_i, \cdots, \partial q_\mu / \partial \eta_i$, we find that

$$(12) \qquad \frac{\partial q_k}{\partial \eta_i} = \sum_{j=1}^{j=6} \sum_{e=1}^{e=\mu} \mu_{ij} B_{ke} \frac{\partial \eta_j}{\partial q_e} + \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \Psi_{me} d_{mk} \frac{\partial \psi_e}{\partial \eta_i} \quad \begin{pmatrix} k = 1, 2, \cdots, \mu \\ i = 1, 2, \cdots, 6 \end{pmatrix}.$$

§5. If we substitute the expressions (12) of the $\partial q_k / \partial \eta_i$ in formula (9) the latter becomes:

$$(13) \quad X_i = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} B_{ke} \mathfrak{X}^{(e)} \frac{\partial \eta_i}{\partial q_k} + \sum_{m=1}^{m=s} \sum_{e=1}^{e=s} \sum_{n=1}^{n=6} \Psi_{me} M_{in} X(\psi_e) \frac{\partial \psi_m}{\partial \eta_n} \quad (i = 1, 2, \cdots, 6).$$

The same substitution made in formula (10) yields the following result :

$$(14)\qquad \begin{cases} \sum_{j=1}^{j=6} X_j\mu_{ij} = \sum_{j=1}^{j=6}\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} B_{ke}\mathfrak{X}^{(k)}\mu_{ij}\frac{\partial\eta_j}{\partial q_e} + \sum_{m=1}^{m=s}\sum_{e=1}^{e=s}\Psi_{me}X(\psi_m)\frac{\partial\psi_e}{\partial\eta_i} \\ \qquad\qquad (i=1,\ 2,\ \cdots,\ 6). \end{cases}$$

Finally, multiplying both sides of (14) by X_i and summing the result with regard to the index i, we obtain a third formula:

$$(15)\qquad \sum_{i=1}^{i=6}\sum_{j=1}^{j=6}\mu_{ij}X_iX_j = \sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} B_{ke}\mathfrak{X}^{(k)}\mathfrak{X}^{(e)} + \sum_{m=1}^{m=s}\sum_{e=1}^{e=s}\Psi_{me}X(\psi_m)X(\psi_e).$$

Formulas (13), (14) and (15) are the three which we proposed to derive.

§6. *Kinetic energy of a solid body.* As an application of these formulas let us put $X_i = d\eta_i/dt$. Then

$$X(\psi_m) = \sum_{i=1}^{i=6}\frac{\partial\psi_m}{\partial\eta_i}\frac{d\eta_i}{dt} = -\frac{\partial\psi_m}{\partial t},$$

where $\partial\psi_m/\partial t$ is the partial derivative of the function ψ_m with regard to the time ; further,

$$\mathfrak{X}^{(k)} = \sum_{i=1}^{i=6}\sum_{j=1}^{j=6}\mu_{ij}\eta'_i\frac{\partial\eta_j}{\partial q_k} = \frac{\partial T}{\partial q'_k} = p_k;$$

and, therefore, by (13), (14), (15),

$$(16)\qquad \eta'_i = \sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} B_{ke}\frac{\partial\eta_i}{\partial q_k}p_e - \sum_{m=1}^{m=s}\sum_{e=1}^{e=s}\sum_{n=1}^{n=6}\Psi_{me}M_{in}\frac{\partial\psi_e}{\partial t}\frac{\partial\psi_m}{\partial\eta_n},$$

$$(17)\qquad \frac{\partial T}{\partial\eta'_i} = \sum_{j=1}^{j=6}\mu_{ij}\eta'_j = \sum_{e=1}^{e=\mu}\sum_{k=1}^{k=\mu}\sum_{j=1}^{j=6} B_{ke}\mu_{ij}\frac{\partial\eta_j}{\partial q_e}p_k - \sum_{k=1}^{k=s}\sum_{e=1}^{e=s}\Psi_{ke}\frac{\partial\psi_k}{\partial t}\frac{\partial\psi_e}{\partial\eta_i},$$

$$(18)\qquad T = \tfrac{1}{2}\sum_{i=1}^{i=6}\sum_{j=1}^{j=6}\mu_{ij}\eta'_i\eta'_j = \tfrac{1}{2}\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu}B_{ke}p_kp_e + \tfrac{1}{2}\sum_{k=1}^{k=s}\sum_{e=1}^{e=s}\Psi_{ke}\frac{\partial\psi_k}{\partial t}\frac{\partial\psi_e}{\partial t}.$$

§7. *A modification of the three proposed formulas.* If we substitute the function

$$\sum_{j=1}^{j=6} M_{ij}X_j$$

for X_i in formulas (13)–(15) and observe that this will change $\mathfrak{X}^{(k)}$ into $X^{(k)}$, $X(\psi_k)$ into $\mathfrak{X}(\psi_k)$, we shall obtain :

$$(19)\qquad \begin{cases} \sum_{j=1}^{j=6} M_{ij}X_j = \sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} B_{ke}X^{(e)}\frac{\partial\eta_j}{\partial q_k} + \sum_{m=1}^{m=s}\sum_{e=1}^{e=s}\sum_{n=1}^{n=6}\Psi_{me}M_{in}\mathfrak{X}(\psi_e)\frac{\partial\psi_m}{\partial\eta_n} \\ \qquad\qquad (i=1,\ 2,\ \cdots,\ 6), \end{cases}$$

$$(20)\qquad X_i=\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu}\sum_{j=1}^{j=6}B_{ke}X^{(k)}\mu_{ij}\frac{\partial\eta_j}{\partial q_e}+\sum_{m=1}^{m=s}\sum_{e=1}^{e=s}\Psi_{me}\mathfrak{X}(\psi_m)\frac{\partial\psi_e}{\partial\eta_i},$$

$$(21)\qquad \sum_{i=1}^{i=6}\sum_{j=1}^{j=6}M_{ij}X_iX_j=\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu}B_{ke}X^{(k)}X^{(e)}+\sum_{k=1}^{k=s}\sum_{e=1}^{e=s}\Psi_{ke}\mathfrak{X}(\psi_k)\mathfrak{X}(\psi_e).$$

The last formula will be required in the next chapter.

CHAPTER VII.

Return to the Perturbative Function of Convective Motion.

§1. *The function* G.—Let us again take up the expression:

$$(1)\qquad G_2=G-\tfrac{1}{2}\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu}h_{ke}\frac{\partial L}{\partial q'_k}\frac{\partial L}{\partial q'_e},$$

and let us now consider the case of motion of a solid body subject to s conditions of constraint as in the preceding chapter. It will be observed that in this case the coefficients h_{ke} which figure in (1) are identical with the coefficients B_{ke} introduced in chapter VI, so that now

$$(1')\qquad G_2=G-\tfrac{1}{2}\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu}B_{ke}\frac{\partial L}{\partial q'_k}\frac{\partial L}{\partial q'_e}.$$

This expression will be transformed by means of the formulas of chapter VI. For this purpose put $X_i=\partial L/\partial\eta'_i$. Then

$$X^{(k)}=\sum_{i=1}^{i=6}\frac{\partial L}{\partial\eta'_i}\frac{\partial\eta_i}{\partial q_k}=\frac{\partial L}{\partial q'_k},$$

and therefore, by formula (21) of the preceding chapter,

$$\sum_{i=1}^{i=6}\sum_{j=1}^{j=6}M_{ij}\frac{\partial L}{\partial\eta'_i}\frac{\partial L}{\partial\eta'_j}=\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu}B_{ke}\frac{\partial L}{\partial q'_k}\frac{\partial L}{\partial q'_e}+\sum_{k=1}^{k=s}\sum_{e=1}^{e=s}\Psi_{ke}L(\psi_k)L(\psi_e),$$

where

$$(2)\qquad L(\psi_k)=\sum_{i=1}^{i=6}\sum_{j=1}^{j=6}M_{ij}\frac{\partial L}{\partial\eta'_i}\frac{\partial\psi_k}{\partial\eta_j}.$$

But we have found in §6, chapter V, that

$$\sum_{i=1}^{i=6}\sum_{j=1}^{j=6}M_{ij}\frac{\partial L}{\partial\eta'_i}\frac{\partial L}{\partial\eta'_j}=2G;$$

hence,

$$(3)\qquad G=\tfrac{1}{2}\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu}B_{ke}\frac{\partial L}{\partial q'_k}\frac{\partial L}{\partial q'_e}+\tfrac{1}{2}\sum_{k=1}^{k=s}\sum_{e=1}^{e=s}\Psi_{ke}L(\psi_k)L(\psi_e).$$

§2. *Perturbative function for a free solid body.*—Substituting the expression (3 of G into formula (1′) we find that, for a solid body,

$$(4) \qquad G_2 = \tfrac{1}{2} \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Psi_{ke} L(\psi_k) L(\psi_e) ;$$

i. e., the function G_2 is due solely to the existence of constraints.

If $s = 0$, i. e., if the body is free, then $G_2 = 0$. We thus find again the theorem of §7, chapter V, namely, that *for a free solid body the perturbative function of convective motion reduces to* $-(K + L)$.

§3. *Cases when $G_2 = 0$.*—We have already seen in chapter V that by extending in a familiar way the results obtained for rigid systems of particles to solid bodies we shall have:

$$(5) \qquad L(\psi_k) = pR_x^{(k)} + qR_y^{(k)} + rR_z^{(k)} = \omega R^{(k)} \cos(\omega , R^{(k)}),$$

where $R^{(k)}$ is a vector parallel and proportional to the moment of reaction relatively to the origin of the axes XYZ due to the constraint ψ_k. Formula (5) shows that $L(\psi_k)$ *vanishes whenever this moment vanishes or when its direction forms a right angle with the direction of the instantaneous axis* (ω).

Two interesting cases when all the $L(\psi_k)$ vanish may be mentioned.

1. The body can freely rotate about the origin of the axes XYZ.
2. The body can freely rotate about a point which constantly remains on the instantaneous axis (ω).

In fact, in the first case the moments of the constraints vanish; in the second, their directions form right angles with the direction of the instantaneous axis (ω). Hence this theorem:

When a solid body can rotate freely about the origin of the axes XYZ, or about a point constantly remaining on the instantaneous axis (ω); then the perturbative function of convective motion reduces to $-(K + L)$.

§4. *The perturbative function for very small values of ω.* Again, it will be observed as in chapter V that the perturbative function is the sum of the three functions $-K$, $-L$, $-G_2$, all three homogeneous in p, q, r; the degree of homogeneity being 0 for K, 1 for L, 2 for G_2.

If we look upon ω as a small quantity of the first order and upon Ω as a function of ω, we can say that *the perturbative function of the second order in ω is due solely to the existence of constraints and vanishes when the body is free, or when it freely rotates about the origin of the axes of relative coördinates, or finally, when it freely rotates about a point constantly remaining on the instantaneous axis* (ω).

CHAPTER VIII.

Decomposition of the Kinetic Energy of a Material System. Mechanical Significance of the Function G_2.

§1. *On the nature of constraints.*—In order that certain theorems enunciated below may be properly understood, a few words as to the manner of interpreting geometrical conditions are necessary.

Suppose that a system of n particles be subject in its motion to the condition:*

$$(1) \qquad \phi(x_1, y_1, z_1, \cdots, z_{i-1}, x_i, y_i, z_i, x_{i+1}, \cdots, x_n, y_n, z_n) = 0,$$

not involving the time explicitly. We know that, geometrically interpreted, this condition involves the following:—

1) A particle m_i (x_i, y_i, z_i) remains on the surface:

$$(2) \qquad \phi(x_1, y_1, z_1, \cdots, z_{i-1}, \mathrm{X}, \mathrm{Y}, \mathrm{Z}, x_{i+1}, \cdots, x_n, y_n, z_n) = 0,$$

whose form and position in space depend on the $3(n-1)$ parameters $x_1, y_1, \cdots, z_{i-1}, x_{i+1}, \cdots, z_n$.

2) The velocity of this particle is tangential to the surface (2).

These conditions can be expressed in a somewhat different form.

1) A particle m_i must remain on the surface (2). This surface changes its form and position in space during the motion.

2) The variation in form and position of the surface (2) imparts to each of its points a certain velocity v whose normal and tangential components will be denoted by v_N and v_T.

3) When the equations of constraint, as in (1), do not involve the time explicitly, then the normal velocity v_N at the point coinciding with the particle m_i is equal in magnitude and direction to the normal component of the velocity of this particle. In consequence of this the velocity of the particle m_i will be tangential to the surface (2).

4) The tangential velocity v_T is without effect on the particle m_i, the surface (2) being *ideal*, *i. e.*, void of friction.

Now, suppose that the given material system is subject to the condition:

$$(1') \qquad \phi(x_1, y_1, z_1, \cdots, z_{i-1}, x_i, y_i, z_i, x_{i+1}, \cdots, x_n, y_n, z_n, t) = 0,$$

involving the time explicitly. Geometrically this condition may be interpreted as follows:—

* Unless otherwise stated, x, y, z in this chapter denote coördinates with regard to any system of axes, whether fixed in space or moving.

1) A particle m_i must remain on the surface:

$$(2') \qquad \phi(x_1, y_1, z_1, \cdots, z_{i-1}, \mathrm{X}, \mathrm{Y}, \mathrm{Z}, x_{i+1}, \cdots, x_n, y_n, z_n, t) = 0,$$

whose form and position in space depend on the $3(n-1)+1$ parameters $x_1, \cdots, z_{i-1}, x_{i+1}, \cdots, z_n, t$.

2) The variation in form and position of the surface (2) imparts to each of its points a certain velocity v whose normal and tangential components are v_N and v_T.

3) When the equations of constraint, as in (1′), involve the time explicitly, then the velocity v_N, at the point coinciding with the particle m_i, is no longer equal to the normal component of the velocity of this particle. In consequence of this the surface (2′) will exercise an *impulse* on the particle m_i, normal to the surface (2), and whose components along the axes XYZ will therefore be of the form $\lambda\,\partial\phi/\partial x_i$, $\lambda\,\partial\phi/\partial y_i$, $\lambda\,\partial\phi/\partial z_i$.

4) The tangential component of the velocity of the particle m_i will satisfy the given condition (2′) if the parameter t be considered constant; or, to be more precise, if w'_i denote the tangential component of the velocity w_i of a particle m_i, we shall have:

$$0 = \sum_i \left(w'_{ix} \frac{\partial\phi}{\partial x_i} + w'_{iy} \frac{\partial\phi}{\partial y_i} + w'_{iz} \frac{\partial\phi}{\partial z_i} \right).$$

5) The tangential velocity v_T, again, is without effect on the particle m_i.

If, instead of the sole condition (1′), we had several such conditions involving the time explicitly, it is clear that the several surfaces $\phi_1 = 0$, $\phi_2 = 0$, $\cdots$, would each exercise a normal impulse on the particle; so that, $m_i(w_i)_2$ denoting the *resultant* of all these impulses, we may write

$$(2) \qquad \begin{cases} m_i(w_{ix})_2 = \sum_k \lambda_k \dfrac{\partial\phi_k}{\partial x_i}, \\ m_i(w_{iy})_2 = \sum_k \lambda_k \dfrac{\partial\phi_k}{\partial y_i}, \\ m_i(w_{iz})_2 = \sum_k \lambda_k \dfrac{\partial\phi_k}{\partial z_i}. \end{cases}$$

Now, $(w_i)_2$ being a component of the actual velocity w_i of the particle m_i, let $(w_i)_1$ be the other component. It is clear from the above that the velocities $(w_i)_1$ will comply with the given conditions of constraint if the time, so far as it enters *explicitly* into their equations, be considered constant, i. e., we shall have:

$$(4) \qquad 0 = \sum_i \left\{ (w_{ix})_1 \frac{\partial\phi_k}{\partial x_i} + (w_{iy})_1 \frac{\partial\phi_k}{\partial y_i} + (w_{iz})_1 \frac{\partial\phi_k}{\partial z_i} \right\} \qquad (k = 1, 2, \cdots).$$

§2. *Decomposition of the kinetic energy of a material system.*—It follows from formulas (3) and (4) that

$$(5)\qquad 0=\sum_{i=1}^{i=n} m_i\left\{(w_{ix})_1(w_{ix})_2+(w_{iy})_1(w_{iy})_2+(w_{iz})_1(w_{iz})_2\right\};$$

and therefore, if we put

$$(6)\qquad \mathfrak{T}=\tfrac{1}{2}\sum_{i=1}^{i=n} m_i w_i^2,\qquad \mathfrak{T}_1=\tfrac{1}{2}\sum_{i=1}^{i=n} m_i(w_i)_1^2,\qquad \mathfrak{T}_2=\tfrac{1}{2}\sum_{i=1}^{i=n} m_i(w_i)_2^2,$$

we shall have:

$$(7)\qquad \mathfrak{T}=\mathfrak{T}_1+\mathfrak{T}_2,$$

which expresses the following theorem:

The kinetic energy $\mathfrak{T}$ of a system of particles subject to conditions of constraint involving the time explicitly is the sum of the kinetic energy $\mathfrak{T}_1$ which the given material system would have if at the considered moment the time, so far as it enters explicitly in the equations of constraint, became constant; and of the kinetic energy $\mathfrak{T}_2$ which is imparted to the material system by the variation of the time, so far as it enters explicitly in these equations.

§3. *Expressions of $(w_i)_1$, $(w_i)_2$, $\mathfrak{T}_1$, $\mathfrak{T}_2$.*—To determine these put

$$(8)\qquad W^{(k)}=\sum_{i=1}^{i=n} m_i\left(w_{ix}\frac{\partial x_i}{\partial q_k}+w_{iy}\frac{\partial y_i}{\partial q_k}+w_{iz}\frac{\partial z_i}{\partial q_k}\right),$$

$q_1, q_2, \cdots, q_\mu$ being the independent Lagrangian coördinates of the system, and

$$(9)\qquad W(\phi_k)=\sum_{i=1}^{i=n}\left(w_{ix}\frac{\partial \phi_k}{\partial x_i}+w_{iy}\frac{\partial \phi_k}{\partial y_i}+w_{iz}\frac{\partial \phi_k}{\partial z_i}\right).$$

It is readily seen with the help of formulas (3) and (4) that

$$(10)\qquad W(\phi_k)=\sum_{i=1}^{i=n}\left\{(w_{ix})_2\frac{\partial \phi_k}{\partial x_i}+(w_{iy})_2\frac{\partial \phi_k}{\partial y_i}+(w_{iz})_2\frac{\partial \phi_k}{\partial z_i}\right\}=\sum_{e=1}^{e=s}\lambda_e\phi_{ke}\qquad (k=1,2,\cdots,s),$$

ϕ_{ke} being defined by (d), chap. IV. The s equations (10) being solved with regard to $\lambda_1, \cdots, \lambda_s$ give

$$(11)\qquad \lambda_k=\sum_{e=1}^{e=s}\Phi_{ke}W(\phi_e)\qquad (k=1,2,\cdots,s).$$

Substituting these expressions of the λ_k in (3) we obtain:

$$
(12)\qquad \left\{\begin{aligned}
m_i(w_{ix})_2 &= \sum_{k=1}^{k=s}\sum_{e=1}^{e=s} \Phi_{ke} W(\phi_k)\frac{\partial \phi_e}{\partial x_i},\\
m_i(w_{iy})_2 &= \sum_{k=1}^{k=s}\sum_{e=1}^{e=s} \Phi_{ke} W(\phi_k)\frac{\partial \phi_e}{\partial y_i},\\
m_i(w_{iz})_2 &= \sum_{k=1}^{k=s}\sum_{e=1}^{e=s} \Phi_{ke} W(\phi_k)\frac{\partial \phi_e}{\partial z_i}.
\end{aligned}\right.
$$

To determine the (w_i), observe that the conditions (4) will be satisfied by selecting the velocities (w_i) so that

$$
(13)\qquad \left\{\begin{aligned}
(w_{ix})_1 &= \sum_{k=1}^{k=\mu} \mu_k \frac{\partial x_i}{\partial q_k},\\
(w_{iy})_1 &= \sum_{k=1}^{k=\mu} \mu_k \frac{\partial y_i}{\partial q_k},\\
(w_{iz})_1 &= \sum_{k=1}^{k=\mu} \mu_k \frac{\partial z_i}{\partial q_k}.
\end{aligned}\right.
$$

It remains therefore to determine the μ_k. To this end I remark that

$$
\sum_{i=1}^{i=n} m_i \left\{(w_{ix})_2 \frac{\partial x_i}{\partial q_k} + (w_{iy})_2 \frac{\partial y_i}{\partial q_k} + (w_{iz})_2 \frac{\partial z_i}{\partial q_k}\right\} = \sum_{e=1}^{e=s} \lambda_e \frac{\partial \phi_e}{\partial q_k} \equiv 0
$$

$(k=1, 2, \cdots, \mu)$,

and consequently,

$$
(14)\quad W^{(k)} = \sum_{i=1}^{i=n} m_i \left\{(W_{ix})_1 \frac{\partial x_i}{\partial q_k} + (w_{iy})_1 \frac{\partial y_i}{\partial q_k} + (w_{iz})_1 \frac{\partial z_i}{\partial q_k}\right\} = \sum_{e=1}^{e=\mu} \mu_e a_{ke}
$$

$(k=1, 2, \cdots, \mu)$,

a_{ke} being defined as in §3, chap. IV. From the μ equations (14) are readily derived the following values of $\mu_1, \cdots, \mu_\mu$:

$$
(15)\qquad \mu_k = \sum_{e=1}^{e=\mu} A_{ke} W^{(e)} \qquad (k=1, 2, \cdots, \mu).
$$

Substituting these expressions of the μ_k in (13) we obtain:

$$
(16)\qquad \left\{\begin{aligned}
(w_{ix})_1 &= \sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} A_{ke} W^{(k)} \frac{\partial x_i}{\partial q_e},\\
(w_{iy})_1 &= \sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} A_{ke} W^{(k)} \frac{\partial y_i}{\partial q_e},\\
(w_{iz})_1 &= \sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} A_{ke} W^{(k)} \frac{\partial z_i}{\partial q_e}.
\end{aligned}\right.
$$

By means of (16) and (12) we find without difficulty that

$$(17) \qquad \mathfrak{T}_1 = \tfrac{1}{2} \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} A_{ke} W^{(k)} W^{(e)},$$

$$(18) \qquad \mathfrak{T}_2 = \tfrac{1}{2} \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Phi_{ke} W(\phi_k) W(\phi_e).$$

§4. *A special case.*—In particular let

$$w_{ix} = \frac{dx_i}{dt} = x_i', \qquad w_{iy} = y_i', \qquad w_{iz} = z_i'.$$

Then

$$\mathfrak{T} = T, \qquad W^{(k)} = \frac{\partial T}{\partial q_k'} = p_k, \qquad W(\phi_k) = -\frac{\partial \phi_k}{\partial t};$$

and the results of §6, chapter IV are obtained in a new form which lends itself more readily to a mechanical interpretation. Put

$$(19) \qquad u_i' = (u_i')_1 + (u_i')_2,$$

the u_i being defined as in chapter IV, and we shall then have:

$$(20) \qquad (u_i')_1 = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} A_{ke} \frac{\partial u_i}{\partial q_e} p_k,$$

$$(21) \qquad (u_i')_2 = -\sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Phi_{ke} \frac{\partial \phi_e}{\partial t} \frac{\partial \phi_k}{\partial u_i},$$

simultaneously with the formulas:

$$(22) \qquad T = T_1 + T_2,$$

$$(23) \qquad T_1 = \tfrac{1}{2} \sum_{i=1}^{i=3n} (u_i')_1^2 = \tfrac{1}{2} \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} A_{ke} p_k p_e,$$

$$(24) \qquad T_2 = \tfrac{1}{2} \sum_{i=1}^{i=3n} (u_i')_2^2 = \tfrac{1}{2} \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Phi_{ke} \frac{\partial \phi_k}{\partial t} \frac{\partial \phi_e}{\partial t}.$$

The mechanical interpretation of these formulas follows at once from the theorem enunciated in §2. Indeed, nothing need be changed in the enunciation of that theorem in order to apply it in the present case, except the letter $\mathfrak{T}$ into the letter T.

§5. *New expressions for formulas* (2) *and* (3) *of chapter IV.*—Another demonstration of the theorem of §2, based on the formulas of chapter IV, will now be given. It was shown in §5 of that chapter that we may write

$$(25) \qquad X_i = (X_i)_1 + (X_i)_2,$$

where

$$(26) \qquad (X_i)_1 = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} A_{ke} X^{(k)} \frac{\partial u_i}{\partial q_e},$$

(27) $$(X_i)_2 = \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Phi_{ke}(\phi_k) \frac{\partial \phi_e}{\partial u_i}.$$

These formulas show that

(28) $$\sum_{i=1}^{i=3n} (X_i)_1 \frac{\partial \phi_k}{\partial u_i} = 0 \qquad (k = 1, 2, \cdots, s),$$

and subsequently, that

(29) $$\sum_{i=1}^{i=3n} (X_i)_1 (X_i)_2 = 0.$$

Therefore

(30) $$\sum_{i=1}^{i=3n} (X_i)^2 = \sum_{i=1}^{i=3n} (X_i)_1^2 + \sum_{i=1}^{i=3n} (X_i)_2^2,$$

with

(31) $$\sum_{i=1}^{i=3n} (X_i)_1^2 = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} A_{ke} X^{(k)} X^{(e)},$$

(32) $$\sum_{i=1}^{i=3n} (X_i)_2^2 = \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Phi_{ke} X(\phi_k) X(\phi_e).$$

§6. *Another proof of the theorem of* §2.—Let w_i be the velocity of a particle and let $X_1, X_2, \cdots, X_{3n}$ be respectively $\sqrt{m_1}\, w_{1x}, \sqrt{m_1}\, w_{1y}, \cdots, \sqrt{m_n}\, w_{nz}$; let moreover $\mathfrak{T}$, $W^{(k)}$, $W(\phi_k)$ be defined as before by (6), (8), (9). Applying, first, formula (25) of the preceding paragraph, we find:

(33) $$\begin{cases} w_{ix} = (w_{ix})_1 + (w_{ix})_2, \\ w_{iy} = (w_{iy})_1 + (w_{iy})_2, \\ w_{iz} = (w_{iz})_1 + (w_{iz})_2; \end{cases}$$

where, according to (25) and (27),

(34) $$\begin{cases} (w_{ix})_1 = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} A_{ke} W^{(k)} \frac{\partial x_i}{\partial q_e}, \\ (w_{iy})_1 = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} A_{ke} W^{(k)} \frac{\partial y_i}{\partial q_e}, \\ (w_{iz})_1 = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} A_{ke} W^{(k)} \frac{\partial z_i}{\partial q_e}; \end{cases}$$

(35) $$\begin{cases} m_i(w_{ix})_2 = \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Phi_{ke} W(\phi_k) \frac{\partial \phi_e}{\partial x_i}, \\ m_i(w_{iy})_2 = \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Phi_{ke} W(\phi_k) \frac{\partial \phi_e}{\partial y_i}, \\ m_i(w_{iz})_2 = \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Phi_{ke} W(\phi_k) \frac{\partial \phi_e}{\partial z_i}. \end{cases}$$

At the same time, by (30),

$$\sum_{i=1}^{i=n} m_i w_i^2 = \sum_{i=1}^{i=n} m_i (w_i)_1^2 + \sum_{i=1}^{i=n} m_i (w_i)_2^2,$$

or, in the notation (6),

$$\mathfrak{T} = \mathfrak{T}_1 + \mathfrak{T}_2$$

which is the formula (7) of §2 and expresses the theorem there enunciated.

Formulas (34) and (35) are identical with (16) and (12). The proof of these formulas and of the theorem of §2, as given in this paragraph, is possibly more rigorous than the demonstration previously given. The advantage of the other method lies in its immediate geometrical interpretation. For these reasons it seemed expedient to give both proofs.

§ 7. *Specific virtual velocities. Constraint velocities.*—For the convenience of formulating the results two new terms will now be introduced. Referring to the equation (4) it is readily seen that the velocities $(w_i)_1$ belong to the class of so-called *virtual velocities*, a term used in deriving D'Alembert's principle. The velocities $(w_i)_1$ however are *perfectly determinate* at each moment, while in general to each particle m_i, at a given moment, correspond an infinity of virtual velocities. Among the latter is the component $(w_i)_1$ of the actual velocity w_i. I call $(w_i)_1$ for lack of a better term the *specific virtual velocity* of the particle m_i.

The term proposed for the other component of w_i is justified by the geometrical interpretation given in the beginning of this chapter. I call $(w_i)_2$ the *constraint velocity* of the particle m_i.

The theorem of §2 will now read as follows:

The kinetic energy of a system of particles subject to conditions of constraint involving the time explicitly is the sum of the kinetic energy of the material system arising from specific virtual velocities and of the kinetic energy due to the constraint velocities of the particles.

§ 8. *Kinetic energy imparted to a material system by convection.*—Suppose that the kinetic energy $\mathfrak{T}$ is imparted to the material system through a motion of the axes XYZ. Suppose moreover that the equations of constraint, when expressed in terms of the relative coördinates of the particles, do not involve the time explicitly. Then the velocities $(w_i)_1$ will satisfy the same conditions as the *actual relative velocities* of the particles; in other words, it would be *possible* for a particle m_i to assume the velocity $(w_i)_1$ in its *relative* motion. On the other hand, the velocities $(w_i)_2$ being resultants of velocities normal to surfaces to which the relative velocities are tangential,* it is clear that it would be *impos-*

* On account of the condition that the equations of constraint expressed in terms of the relative coördinates do not involve the time explicitly.

sible for a particle m_i to assume the velocity $(w_i)_2$ in its *relative* motion. Hence the following theorem:

The convective kinetic energy of a system of particles, when the conditions of constraint expressed in the relative coördinates do not involve the time explicitly, is the sum of the kinetic energy arising from certain velocities which it is possible for the particles to assume in their relative motion, and of the kinetic energy due to certain other velocities which it would be impossible for the particles to assume in their relative motion.*†

§9. *Mechanical interpretation of the function* G_2.—The results obtained in the preceding paragraphs will now be applied to the perturbative function of convective motion.

Let again x_i, y_i, z_i denote the relative coördinates of a particle and let

$$(36) \qquad w_{ix} = qz_i - ry_i, \quad w_{iy} = rx_i - pz_i, \quad w_{iz} = py_i - qx_i.$$

Then

$$\mathfrak{T} = G,$$

and, since now

$$m_i w_{ix} = \frac{\partial L}{\partial x_i'}, \quad m_i w_{iy} = \frac{\partial L}{\partial y_i'}, \quad m_i w_{iz} = \frac{\partial L}{\partial z_i'},$$

we shall have in the present case:

$$W^{(k)} = \sum_{i=1}^{i=3n} \frac{\partial L}{\partial u_i'} \frac{\partial u_i}{\partial q_k} = \frac{\partial L}{\partial q_k'},$$

$$W(\phi_k) = \sum_{i=1}^{i=3n} \frac{\partial L}{\partial u_i'} \frac{\partial \phi_k}{\partial u_i} = L(\phi_k);$$

and the formulas of §§2, 3 yield the following:

$$(37) \qquad G = G_1 + G_2,$$

$$(38) \qquad G_1 = \tfrac{1}{2} \sum_{i=1}^{i=n} m_i (w_i)_1^2 = \tfrac{1}{2} \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} A_{ke} \frac{\partial L}{\partial q_k'} \frac{\partial L}{\partial q_e'},$$

$$(39) \qquad G_2 = \tfrac{1}{2} \sum_{i=1}^{i=n} m_i (w_i)_2^2 = \tfrac{1}{2} \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Phi_{ke} L(\phi_k) L(\phi_e),$$

which go to show that G_2, as given by the decomposition of the kinetic energy G, is nothing else than the third term of the perturbative function of convective motion, so designated before.

---

*i. e., the kinetic energy imparted to the material system by a motion of the axes XYZ.

† While this theorem, in general, involves an element of indeterminateness, namely the manifoldness of possible and of impossible velocities, in the applications this indeterminateness often vanishes; thus when a particle can move in only one direction, or, on the contrary, when it can move in every direction but one, the problem of decomposing the kinetic energy is obviously reduced. See as an illustration the example given in §13, chap. IX.

Before interpreting these results I make the following remark:—

Let $\phi(x_1, y_1, z_1, \cdots, z_n, t) = 0$ be a condition of constraint expressed in terms of the relative coördinates of the particles and let the function ϕ become $f(\xi_1, \eta_1, \zeta_1, \cdots, \zeta_n, t)$ when the absolute coördinates are introduced instead of the relative. We consider, as in §1, the two surfaces:

$$\phi(x_1, y_1, z_1, \cdots, z_{i-1}, \quad \mathrm{X}, \mathrm{Y}, \mathrm{Z}, x_{i+1}, \cdots, t) = 0, \tag{40}$$

$$f(\xi_1, \eta_1, \zeta_1, \cdots, \zeta_{i-1}, \quad \Xi, \mathrm{H}, \mathrm{Z}, \xi_{i+1}, \cdots, t) = 0. \tag{40'}$$

and we find that the variation of the surface (40) in form and position relatively to the axes XYZ imparts to a particle m_i an impulse $m_i(v_i)_2^{(r)}$; the variation of the surface (40′) in form and position in absolute space imparts to the same particle an impulse $m_i(v_i)_2^{(a)}$. Now, at each moment t, the equations (40) and (40′) represent the same surface. Hence, the variation of this surface in absolute space is equivalent to the combined, (1) variation of the same surface relatively to the axes XYZ, and (2) its convective motion with the axes XYZ. The first imparts to the particle m_i the impulse $m_i(v_i)_2^{(r)}$; the second imparts to it an impulse $m_i(v_i)_2^{(c)}$; and it is clear that $(v_i)_2^{(a)}$ will be the resultant of velocities $(v_i)_2^{(r)}$ and $(v_i)_2^{(c)}$. In the case of more than one condition of constraint a generalization of the preceding result is readily obtained. We may say therefore that

The absolute constraint velocity of a particle is the resultant of its relative and its convective constraint velocities.

A similar proposition holds for the specific virtual velocity of a particle.

In particular, if the equations of constraint expressed in terms of the relative coördinates of the particles do not involve the time explicitly, then the relative constraint velocities vanish and therefore the constraint velocity of a particle is due only to convection.

Now, if we observe that the expressions (36) are the components along the axes XYZ of the convective rotary velocity of a particle, we may interpret formula (39) as follows:—

G_2 is the kinetic energy of the material system due to constraint velocities arising from convective rotation.

Note.—It may be well to supplement the geometrical interpretation given above by the following remark which is suggested by HANSEN'S method of computing perturbations.

The transformation of coördinates:

$$(a) \qquad \begin{cases} x_i = a_i + \alpha_{i1}\xi_i + \beta_{i1}\eta_i + \gamma_{i1}\zeta_i, \\ y_i = b_i + \alpha_{i2}\xi_i + \beta_{i2}\eta_i + \gamma_{i2}\zeta_i, \\ z_i = c_i + \alpha_{i3}\xi_i + \beta_{i3}\eta_i + \gamma_{i3}\zeta_i; \end{cases}$$

may be effected by the successive steps

$$(b)\qquad \begin{cases} x_i = a_i + \bar{x}_i\,, \\ y_i = b_i + \bar{y}_i\,, \\ z_i = c_i + \bar{z}_i\,; \end{cases}$$

$$(c)\qquad \begin{cases} \bar{x}_i = \alpha_{i1}\xi_i + \beta_{i1}\eta_i + \gamma_{i1}\zeta_i\,, \\ \bar{y}_i = \alpha_{i2}\xi_i + \beta_{i2}\eta_i + \gamma_{i2}\zeta_i\,, \\ \bar{z}_i = \alpha_{i3}\xi_i + \beta_{i3}\eta_i + \gamma_{3i}\zeta_i\,. \end{cases}$$

The several coefficients entering into these formulas are functions of the time. A distinct letter for the time in each case will be used, namely t, $\bar{\tau}$, τ.

Now, let

$$(d)\qquad \phi(x_1,\ y_1,\ z_1,\ \cdots,\ x_n,\ y_n,\ z_n,\ t) = 0$$

be the equation of a constraint expressed in terms of the relative coördinates. Introducing the absolute coördinates by means of (a), equation (d) takes the form :

$$(e)\qquad f(\xi_1,\ \eta_1,\ \zeta_1,\ \cdots,\ \xi_n,\ \eta_n,\ \xi_n,\ t) = 0\,.$$

Introducing the same coördinates into (d) by the successive transformations (b) and (c) we first obtain,

$$F(\bar{x}_1,\ \bar{y}_1,\ \bar{z}_1,\ \cdots,\ \bar{x}_n,\ \bar{y}_n,\ \bar{z}_n,\ \bar{\tau},\ t) = 0\,,$$

and next,

$$(f)\qquad \Phi(\xi_1,\ \eta_1,\ \zeta_1,\ \cdots,\ \xi_n,\ \eta_n,\ \zeta_n,\ \tau,\ \bar{\tau},\ t) = 0\,.$$

The equations (e) and (f) represent the same constraint.

Now it is clear, that by varying the time in (f) only *as far as it enters explicitly and in the form of the letter t* we shall impart to a particle the same constraint velocity as the corresponding variation of the time, so far as it enters explicitly in the equation (d), would impart to this particle. This constraint velocity was termed above the *relative constraint velocity* of the particle. Similarly, the variation of the time in (f), *as far as it appears under the letter* $\bar{\tau}$, will impart to a particle the *constraint velocity of convective translation;* and finally, the variation of the time, *as far as it enters in* (f) *in the form of the letter* τ, will impart to a particle the *constraint velocity of convective rotation.*

The resultant of these velocities is the *absolute constraint velocity* of the particle. This resultant would be imparted to the particle directly, by the corresponding variation of the time, as far as it enters explicitly in the equation (e).

On the other hand, by assuming that at a given moment the time under the respective letters t, $\bar{\tau}$, τ became constant in (f), we should arrive at the notion of the *relative specific virtual velocity* of a particle and of the *specific virtual velocities of convective translation and of rotation* respectively. We should also find that the resultant of these velocities is the *absolute specific virtual velocity* of the particle, i. e., the absolute velocity which this particle would assume if the time, as far as it enters explicitly in equation (e), became constant at the considered moment.

CHAPTER IX.

Decomposition of the Kinetic Energy of a Solid Body. Mechanical Interpretation of the Function G_2.

§1. *Specific virtual velocity and constraint velocity of a solid body.*—Let $\mathfrak{v}_i$ be the velocity of the body along the parameter η_i and let $\mathfrak{T}$ denote the kinetic energy of the body due to these velocities, i. e., let

$$(1) \qquad \mathfrak{T} = \tfrac{1}{2} \sum \sum \mu_{ij} \mathfrak{v}_i \mathfrak{v}_j .$$

I remark, as in the preceding chapter, that the velocity $\mathfrak{v}_i$ of the body along the parameter η_i may be considered as the resultant of the velocity $(\mathfrak{v}_i)_1$ which the body would have along the same parameter if at the considered moment the time, as far as it enters *explicitly* into the equations of constraint, became constant; and of the velocity $(\mathfrak{v}_i)_2$ imparted to the body along the parameter η_i by the variation of the time as far as it enters *explicitly* in these equations. I shall call $(\mathfrak{v}_i)_1$ the *specific virtual velocity* and $(\mathfrak{v}_i)_2$ the *constraint velocity* of the body along the parameter η_i.*

It may be remarked, as in the preceding chapter, that the velocities $(\mathfrak{v}_i)_1$ satisfy the conditions:

$$(2) \qquad 0 = \sum_{i=1}^{i=6} (\mathfrak{v}_i)_1 \frac{\partial \psi_k}{\partial \eta_i} \qquad (k = 1, 2, \cdots, s),$$

while the impulses $(\partial \mathfrak{T} / \partial \mathfrak{v}_i)_2$ imparted to the body by the variation of the time, as far as it enters explicitly in the equations of constraint, must have the form $\sum \lambda_k \partial \psi_k / \partial \eta_i$, so that we can write:

$$(3) \qquad \sum_{j=1}^{j=6} \mu_{ij} (\mathfrak{v}_j)_2 = \left(\frac{\partial \mathfrak{T}}{\partial \mathfrak{v}_i} \right)_2 = \sum_{k=1}^{k=s} \lambda_k \frac{\partial \psi_k}{\partial \eta_i} \qquad (i = 1, 2, \cdots, 6).$$

* Unless explicitly stated otherwise the parameters $\eta_1, \cdots, \eta_6$ define the position of the body with regard to any system of axes, fixed in space or moving.

§2. *Decomposition of the kinetic energy of a solid body.*—It follows from (2) and (3) that

$$0 = \sum_{i=1}^{i=6} \sum_{j=1}^{j=2} \mu_{ij} (\mathfrak{v}_i)_1 (\mathfrak{v}_j)_2 , \tag{4}$$

and consequently,

$$\sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} \mathfrak{v}_i \mathfrak{v}_j = \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} (\mathfrak{v}_i)_1 (\mathfrak{v}_j)_1 + \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} (\mathfrak{v}_i)_2 (\mathfrak{v}_j)_2 ;$$

or, if we put

$$\mathfrak{T}_1 = \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} (\mathfrak{v}_i)_1 (\mathfrak{v}_j)_1 , \quad \mathfrak{T}_2 = \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} (\mathfrak{v}_i)_2 (\mathfrak{v}_j)_2 , \tag{5}$$

and take into account formula (1),

$$\mathfrak{T} = \mathfrak{T}_1 + \mathfrak{T}_2 . \tag{6}$$

We have thus arrived at the following theorem:

The kinetic energy of a solid body, subject to conditions of constraint involving the time explicitly, is the sum of the kinetic energy of the body arising from the specific virtual velocities and of the kinetic energy due to the constraint velocities.

§3. *Expressions of* $(\mathfrak{v}_i)_1$, $(\mathfrak{v}_i)_2$, $\mathfrak{T}_1$, $\mathfrak{T}_2$.—To determine these, put

$$\mathfrak{B}^{(k)} = \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} \mathfrak{v}_i \frac{\partial \eta_j}{\partial q_k} , \tag{7}$$

$$\mathfrak{B}(\psi_k) = \sum_{i=1}^{i=6} \mathfrak{v}_i \frac{\partial \psi_k}{\partial \eta_i} , \tag{8}$$

q_1, q_2, $\cdots$ being the independent Lagrangian coördinates of the body.

It is readily seen, by means of (2) and (3) that *

$$\mathfrak{B}(\psi_k) = \sum_{i=1}^{i=6} (\mathfrak{v}_i)_2 \frac{\partial \psi_k}{\partial \eta_i} = \sum_{e=1}^{e=s} \lambda_e \psi_{ek} \qquad (k = 1, 2, \cdots, s).$$

From these s equations we derive:

$$\lambda_k = \sum_{e=1}^{e=s} \Psi_{ke} \mathfrak{B}(\psi_e) \qquad (k = 1, 2, \cdots, s).$$

---

* The six equations (3) solved with respect to $(\mathfrak{v}_1)_2$, $\cdots$, $(\mathfrak{v}_6)_2$ give

$$(\mathfrak{v}_i)_2 = \sum_{k=1}^{k=s} \sum_{j=1}^{j=6} \lambda_k M_{ij} \frac{\partial \psi_k}{\partial \eta_j} \qquad (i = 1, 2, \cdots, 6). \tag{3'}$$

Substituting these expressions of the λ_k in (3) and (3′) we obtain:

(9) $$\left(\frac{\partial \mathfrak{T}}{\partial \mathfrak{v}_i}\right)_2 = \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Psi_{ke} \mathfrak{V}(\psi_k) \frac{\partial \psi_e}{\partial \eta_i} \qquad (i = 1, 2, \cdots, 6),$$

(10) $$(\mathfrak{v}_i)_2 = \sum_{j=1}^{j=6} \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Psi_{ke} \mathfrak{V}(\psi_k) M_{ij} \frac{\partial \psi_e}{\partial \eta_j} \qquad (i = 1, 2, \cdots, 6).$$

To determine the velocities $(\mathfrak{v}_i)_1$ I observe that the conditions (2) will be satisfied by giving to $(\mathfrak{v}_i)_1$ the form:

(11) $$(\mathfrak{v}_i)_1 = \sum_{k=1}^{k=\mu} \mu_k \frac{\partial \eta_i}{\partial q_k}.$$

It remains therefore to determine the μ_k. For this we have:

$$\sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} (\mathfrak{v}_i)_2 \frac{\partial \eta_j}{\partial q_k} = \sum_{e=1}^{e=s} \lambda_e \frac{\partial \psi_e}{\partial q_k} \equiv 0 \qquad (k = 1, 2, \cdots, \mu),$$

and consequently,

$$\mathfrak{V}^{(k)} = \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} (\mathfrak{v}_i)_1 \frac{\partial \eta_j}{\partial q_k} = \sum_{e=1}^{e=s} \mu_e b_{ke} \qquad (k = 1, 2, \cdots, \mu),$$

b_{ke} being defined in §3, chap. VI. From the last μ equations we derive

$$\mu_k = \sum_{e=1}^{e=\mu} B_{ke} \mathfrak{V}^{(e)} \qquad (k = 1, 2, \cdots, \mu);$$

and substituting these expressions of the μ_k in (11), we obtain:

(12) $$(\mathfrak{v}_i)_1 = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} B_{ke} \mathfrak{V}^{(k)} \frac{\partial \eta_i}{\partial q_e} \qquad (i = 1, 2, \cdots, 6).$$

By means of (9) and (12) we now find without difficulty that

(13) $$\mathfrak{T}_1 = \tfrac{1}{2} \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} B_{ke} \mathfrak{V}^{(k)} \mathfrak{V}^{(e)},$$

(14) $$\mathfrak{T}_2 = \tfrac{1}{2} \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Psi_{ke} \mathfrak{V}(\psi_k) \mathfrak{V}(\psi_e).$$

§4. *A special case.*—In particular, let $\mathfrak{v}_i = d\eta_i/dt = \eta_i'$. Then

$$\mathfrak{T} = T, \quad \mathfrak{V}^{(k)} = \frac{\partial T}{\partial q_k'} = p_k, \quad \mathfrak{V}(\psi_k) = -\frac{\partial \psi_k}{\partial t};$$

and the results of §6, chap. VI, are obtained in a new form which lends itself more readily to a mechanical interpretation. Put

(15) $$\eta'_i = (\eta'_i)_1 + (\eta'_i)_2,$$

and we shall then have:

(16) $$(\eta'_i)_1 = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} B_{ke} \frac{\partial \eta_i}{\partial q_e} p_k,$$

(17) $$(\eta'_i)_2 = -\sum_{n=1}^{n=6} \sum_{e=1}^{e=s} \sum_{k=1}^{k=s} \Psi_{ke} M_{in} \frac{\partial \psi_e}{\partial t} \frac{\partial \psi_k}{\partial \eta_n},$$

simultaneously with the formulas:

(18) $$T = T_1 + T_2,$$

(19) $$T_1 = \tfrac{1}{2} \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} (\eta'_i)_1 (\eta'_j)_1 = \tfrac{1}{2} \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} B_{ke} p_k p_e,$$

(20) $$T_2 = \tfrac{1}{2} \sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} (\eta'_i)_2 (\eta'_j)_2 = \tfrac{1}{2} \sum_{k=1}^{k=s} \sum_{e=1}^{e=s} \Psi_{ke} \frac{\partial \psi_k}{\partial t} \frac{\partial \psi_e}{\partial t}.$$

The mechanical interpretation of these formulas follows immediately from the theorem enunciated in §2; in fact the only change in its formulation necessary to apply it in the present case is the substitution of T for $\mathfrak{T}$.

§5. *New expressions for formulas* (13) *and* (15) *of chap. VI.* As in the preceding chapter another proof of the theorem of §2 may be given, on the strength of the formulas of chap. VI; although the present demonstration does not yield a mechanical interpretation of the results as readily as the one previously given, it has the advantage of being more rigorous.

We have seen in chap. VI, that in the case of a solid body we may write:

(21) $$X_i = (X_i)_1 + (X_i)_2,$$

where

(22) $$(X_i)_1 = \sum_{k=1}^{k=\mu} \sum_{e=1}^{e=\mu} B_{ke} \mathfrak{X}^{(e)} \frac{\partial \eta_i}{\partial q_k},$$

(23) $$(X_i)_2 = \sum_{n=1}^{n=6} \sum_{e=1}^{e=s} \sum_{k=1}^{k=s} \Psi_{ke} M_{in} X(\psi_k) \frac{\partial \psi_i}{\partial \eta_n}.$$

These expressions show that

(24) $$\sum_{i=1}^{i=6} (X_i)_1 \frac{\partial \psi_k}{\partial \eta_i} = 0, \qquad (k = 1, 2, \cdots, s),$$

and consequently,

(25) $$\sum_{i=1}^{i=6} \sum_{j=1}^{j=6} \mu_{ij} (X_i)_1 (X_j)_2 = 0.$$

Therefore

$$(26)\qquad \sum_{i=1}^{i=6}\sum_{j=1}^{j=6}\mu_{ij}X_iX_j=\sum_{i=1}^{i=6}\sum_{j=1}^{j=6}\mu_{ij}(X_i)_1(X_j)_1+\sum_{i=1}^{i=6}\sum_{j=1}^{j=6}\mu_{ij}(X_i)_2(X_j)_2,$$

and

$$(27)\qquad \sum_{i=1}^{i=6}\sum_{j=1}^{j=6}\mu_{ij}(X_i)_1(X_j)_1=\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu}B_{ke}\mathfrak{X}^{(k)}\mathfrak{X}^{(e)},$$

$$(28)\qquad \sum_{i=1}^{i=6}\sum_{j=1}^{j=6}\mu_{ij}(X_i)_2(X_j)_2=\sum_{k=1}^{k=s}\sum_{e=1}^{e=s}\Psi_{ke}X(\psi_k)X(\psi_e).$$

§ 6. *Another proof of the theorem of* §2.—Let $\mathfrak{v}_i$ again denote the velocity of the body along the parameter η_i and let us put $X_i=\mathfrak{v}_i$; let also $\mathfrak{T}$, $\mathfrak{V}^{(k)}$, $\mathfrak{V}(\psi_k)$ be defined, as before, by (1), (7), (8). Applying formula (21) of the preceding paragraph, we shall have:

$$(29)\qquad \mathfrak{v}_i=(\mathfrak{v}_i)_1+(\mathfrak{v}_i)_2,$$

where, according to (22) and (23),

$$(30)\qquad (\mathfrak{v}_i)_1=\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu}B_{ke}\mathfrak{V}^{(e)}\frac{\partial\eta_i}{\partial q_k},$$

$$(31)\qquad (\mathfrak{v}_i)_2=\sum_{n=1}^{n=6}\sum_{k=1}^{k=s}\sum_{e=1}^{e=s}\Psi_{ke}M_{in}\mathfrak{V}(\psi_k)\frac{\partial\psi_e}{\partial\eta_n}.$$

At the same time, by (26),

$$\sum\sum\mu_{ij}\mathfrak{v}_i\mathfrak{v}_j=\sum\sum\mu_{ij}(\mathfrak{v}_i)_1(\mathfrak{v}_j)_1+\sum\sum\mu_{ij}(\mathfrak{v}_i)_2(\mathfrak{v}_j)_2,$$

or, in the notation (1) and (5),

$$\mathfrak{T}=\mathfrak{T}_1+\mathfrak{T}_2,$$

which is the formula (6) of § 2 and expresses the theorem as there enunciated.

Formulas (30) and (31) are identical with formulas (12) and (10).

§ 7. *Kinetic energy imparted to a solid body by convection.*—Suppose that the kinetic energy $\mathfrak{T}$ is imparted to the body by a motion of the invariable system XYZ. Suppose moreover that the equations of constraint, when expressed in terms of the relative coördinates of the body, do not involve the time explicitly. Then the velocities $(\mathfrak{v}_i)_1$ will satisfy the same conditions as the *actual relative velocities* of the body; in other words, it would be *possible* for this body to assume the velocity $(\mathfrak{v}_i)_1$ in its *relative* motion. On the other hand, it is clear that it would be *impossible* for the body to assume the velocity $(\mathfrak{v}_i)_2$ in its relative motion, on account of the condition that the equations of constraint expressed in terms of the relative coördinates of the body do not involve the time explicitly. Hence the following theorem:

*The convective kinetic energy $\mathfrak{T}$ of a solid body, when the conditions of constraint expressed in terms of the relative coördinates do not involve the time explicitly, is the sum of the kinetic energy $\mathfrak{T}_1$ arising from a certain motion which would be possible relatively to the axes XYZ, and of the kinetic energy $\mathfrak{T}_2$ due to a certain other motion which it would be impossible for the body to assume relatively to these axes.**

§8. *Mechanical interpretation of the function G_2.*—I will now apply the results of the preceding paragraphs to the perturbative function of convective motion.

We have found in §6, chap. IV, that $\eta_1, \cdots, \eta_6$ being the six parameters defining the relative position of a solid body we shall have:

$$G = \frac{1}{2}\sum_{i=1}^{i=6}\sum_{j=1}^{j=6} M_{ij}\frac{\partial L}{\partial \eta'_i}\frac{\partial L}{\partial \eta'_j}. \tag{32}$$

If we take

$$\mathfrak{v}_i = \sum_{j=1}^{j=6} M_{ij}\frac{\partial L}{\partial \eta'_j}, \tag{33}$$

this expression of G will become:

$$G = \frac{1}{2}\sum_{i=1}^{i=6}\sum_{j=1}^{j=6} \mu_{ij}\mathfrak{v}_i\mathfrak{v}_j, \tag{34}$$

and the formulas of §§2, 3 are immediately applicable. Thus, we shall have first

$$\mathfrak{V}^{(k)} = \frac{\partial L}{\partial q'_k}, \qquad \mathfrak{V}(\psi_k) = L(\psi_k),$$

and further,

$$G = G_1 + G_2, \tag{35}$$

$$\begin{aligned} G_1 &= \frac{1}{2}\sum_{i=1}^{i=6}\sum_{j=1}^{j=6} \mu_{ij}(\mathfrak{v}_i)_1(\mathfrak{v}_j)_1 = \frac{1}{2}\sum_{i=1}^{i=6}\sum_{j=1}^{j=6} M_{ij}\left(\frac{\partial L}{\partial \eta'_i}\right)_1\left(\frac{\partial L}{\partial \eta'_j}\right)_1 \\ &= \frac{1}{2}\sum_{k=1}^{k=\mu}\sum_{e=1}^{e=\mu} B_{ke}\frac{\partial L}{\partial q'_k}\frac{\partial L}{\partial q'_e}, \end{aligned} \tag{36}$$

$$\begin{aligned} G_2 &= \frac{1}{2}\sum_{i=1}^{i=6}\sum_{j=1}^{j=6} \mu_{ij}(\mathfrak{v}_i)_2(\mathfrak{v}_j)_2 = \frac{1}{2}\sum_{i=1}^{i=6}\sum_{j=1}^{j=6} M_{ij}\left(\frac{\partial L}{\partial \eta'_i}\right)_2\left(\frac{\partial L}{\partial \eta'_j}\right)_2 \\ &= \frac{1}{2}\sum_{k=1}^{k=s}\sum_{e=1}^{e=s} \Psi_{ke}L(\psi_k)L(\psi_e). \end{aligned} \tag{37}$$

* See the second footnote on p. 150 and *Example*, §13, chap. IX.

These formulas show that G_2 is nothing else than the third term of the perturbative function of convective motion, so designated before.

We have moreover arrived at the following mechanical interpretation * of the function G_2:

G_2 is the kinetic energy of the solid body due to constraint velocities arising from convective rotation.

§9. A slightly different form will now be given to the results of the preceding paragraph.

Let A be a point rigidly connected with the solid body. We know that the motion of the body may be decomposed into a translation of the point A and a rotation about this point. Let v_a be the velocity of A and let ω represent both the instantaneous axis and the angular velocity of rotation of the body. It is clear that we may apply to v_a and to ω the principle of decomposition into specific virtual and constraint velocities as we have applied it to the velocities $\mathfrak{v}_i$ before. In this manner ω will be now the diagonal of a parallelogram constructed on $(\omega)_1$ and $(\omega)_2$, where $(\omega)_1$ represents both the specific virtual angular velocity and the specific virtual instantaneous axis of rotation of the body, while $(\omega)_2$ stands for both the angular constraint velocity and the instantaneous constraint axis of rotation of the body. The meaning of these terms is clear from the explanation and use of similar terms on previous occasions. Namely, $(\omega)_1$ would be obtained if the time, as far as it enters *explicitly* in the equations of constraint, became constant at the moment considered, while $(\omega)_2$ arises from the variation of the time as far as it enters *explicitly* in these equations.†

§10. In particular, let us now consider the convective rotation of the solid body. Constraint velocities arise from divers sources, as was pointed out in §9, chap. VIII. We are here concerned only with those arising from convective rotation.‡ Let therefore ω, $(\omega)_1$, $(\omega)_2$ represent respectively: the angular velocity and instantaneous axis of convective rotation, the specific virtual angular velocity and specific virtual instantaneous axis of convective rotation, the angular constraint velocity and instantaneous constraint axis of convective rotation. Then

$$(38) \qquad G = I_\omega \omega^2, \quad G_1 = I_{\omega_1}(\omega)_1^2, \quad G_2 = I_{\omega_2}(\omega)_2^2,$$

where I_ω, I_{ω_1}, I_{ω_2} denote the moments of inertia of the body with regard to the instantaneous axes ω, $(\omega)_1$, $(\omega)_2$ respectively, and we shall have:

$$(39) \qquad I_\omega \omega^2 = I_{\omega_1}(\omega)_1^2 + I_{\omega_2}(\omega)_2^2.$$

* See §9 of the preceding chapter.

† See NOTE at the close of the preceding chapter.

‡ Convective translation and the explicit appearance of the time in the equations of constraint expressed in terms of *relative* coördinates also give rise to constraint velocities.

It is clear that the three axes ω, $(\omega)_1$, $(\omega)_2$ lie in the same plane and pass through the same point; but *this point does not generally coincide with the origin of the axes XYZ* about which the convective rotation of the body takes place.

When the body can rotate freely about a point on the instantaneous axis ω, or about the origin of the axes XYZ, it is obvious that the angular velocity $(\omega)_2$ vanishes and $(\omega)_1$ coincides with ω. Therefore, in these cases $G = G_1$ and $G_2 = 0$. Thus we have found again the theorem of §3, chap. VII.

§11. If we assume that the equations of constraint expressed in the relative coördinates of the body do not involve the time explicitly, it is clear that a rotation about the axis $(\omega)_1$ would be *possible* in the relative motion of the body, while it would be *impossible* for this body to rotate about the axis $(\omega)_2$ in its relative motion. Hence, we may say that

When the conditions of constraint expressed in the relative coördinates of the body do not involve the time explicitly, then G_2 is the kinetic energy of the body due to convective rotation about an axis about which relative rotation would be impossible.

§12. Let E stand for the ellipsoid of inertia of the body at the point of intersection of the three axes ω, $(\omega)_1$, $(\omega)_2$. It is obvious from (39) that *the axis $(\omega)_2$ lies in a plane conjugate to the direction* $(\omega)_1$ with regard to the ellipsoid E, and similarly that *the axis $(\omega)_1$ lies in a plane conjugate to the direction $(\omega)_2$* with regard to the same ellipsoid.

If the direction of either $(\omega)_1$ or $(\omega)_2$ be that of an axis of the ellipsoid E, the instantaneous axes $(\omega)_1$ and $(\omega)_2$ are at right angles to each other.

In this connection the following remark may be in place. Let o be the origin of a system of axes a_1, a_2, a_3 whose directions form *a conjugate system* with respect to the ellipsoid of inertia of the body at the point o. If Ω denote the angular velocity of rotation of the body in magnitude and direction, and Ω_1, Ω_2, Ω_3 the components of Ω along the axes a_1, a_2, a_3; if moreover I_Ω, I_{a_1}, I_{a_2}, I_{a_3} be the moments of inertia of the body about the axes Ω, a_1, a_2, a_3; then the kinetic energy of rotation of the body about the point o will be

$$\tag{40} \tfrac{1}{2} I_\Omega \Omega^2 = \tfrac{1}{2}(I_{a_1}\Omega_1^2 + I_{a_2}\Omega_2^2 + I_{a_3}\Omega_3^2).$$

When the directions a_1, a_2, a_3 are those of the *axes* of the ellipsoid of inertia, and λ_1, λ_2, λ_3 denote the cosines of the angles which Ω forms with these axes, the expression (40) assumes the well-known form:

$$\tag{41} \tfrac{1}{2} I_\Omega \Omega^2 = \tfrac{1}{2}(I_{a_1}\lambda_1^2 + I_{a_2}\lambda_2^2 + I_{a_3}\lambda_3^2)\Omega^2.$$

Applying this remark to the case under discussion, when the convective rotation of the body alone is considered, so that $\frac{1}{2} I_\Omega \Omega^2 = G$, we arrive at the following conclusions :

If the body in its relative motion rotates about two axes a_2 and a_3 whose directions are conjugate with respect to the ellipsoid E, then G_2 is the kinetic energy of convective rotation of the body about the axis a_1 conjugate to a_2 and a_3. Denoting by ω' the component of ω along the axes a_1 we shall have

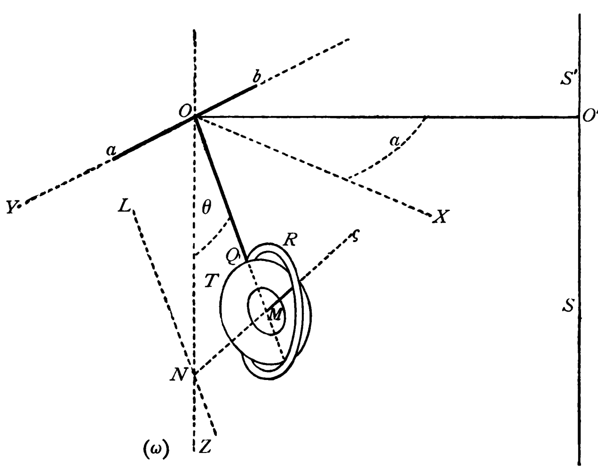
$$G_2 = \tfrac{1}{2} I_{a_1} \omega'^2 .$$

If the body in its relative motion is prevented from rotating about two axes a_2 and a_3 whose directions are conjugate with respect to the ellipsoid E, then G_1 is the kinetic energy of convective rotation of the body about the axis a_1 conjugate to a_2 and a_3 and we shall have

$$G_1 = \tfrac{1}{2} I_{a_1} \omega'^2 .$$

As an illustration let us apply these results to an example.

§13. *Example. Sire's gyroscopic pendulum.*—The pendulum consists of a shaft OQ, a ring R, and a tore T, the latter revolving about a diameter of the ring. The pendulum swings about a horizontal axis ab which forms a fixed



angle $\frac{1}{2}\pi - a$ with OO', a horizontal line of fixed length. The whole apparatus revolves about the vertical axis SS' with the angular velocity ω. The axes XYZ and their origin O will be selected as on the adjoining figure. The three axes $M\zeta$ (of the tore), OM (of the shaft) and OY (or ab) are a tright angles to each other. Denote by C and A respectively the axial and equatorial moments of inertia of the tore, by θ the angle between the axis OM and the vertical OZ, and by N the point of intersection of the axes $M\zeta$ and OZ. Draw NL parallel to the axis OM.

The tore rotates about two axes, one being its physical axis and the other perpendicular to the plane ZOM. Hence $(\omega)_1$ lies in the plane which passes through $M\zeta$. We thus obtain the point N as the intersection of ω and $(\omega)_1$. Now, the directions of $N\zeta$ and of the line perpendicular to the plane ZOM being the directions of two axes of the ellipsoid of inertia of the tore at N, and these axes being at the same time those about which the relative rotation of the tore takes place, G_2 *will be the kinetic energy of convective rotation of the body about the third axis of the ellipsoid, namely the line* NL.

The moment of inertia of the body about this line being $(md^2tg^2\theta + A)$, where m is the mass of the tore and $d = OM$, and, on the other hand, the component of ω along the line LN being $\omega \cos\theta$, we shall have:

$$G_2 = \tfrac{1}{2}(md^2 \sin^2\theta + A \cos^2\theta)\omega^2 .$$

CHAPTER X.

Other Methods of obtaining the Differential Equations of Relative Motion.

§1. *Differential equations of absolute motion.*—The position of a material system with regard to the axes $\Xi \mathrm{H} \mathrm{Z}$, fixed in space, will be given when its position relatively to the moving axes XYZ and the motion of the latter relatively to the axes $\Xi \mathrm{H} \mathrm{Z}$ are given. Hence, assuming that the motion of the axes XYZ is known, the system of coördinate parameters may be considered as defining the *absolute* as well as the *relative* motion of the material system. From this point of view relative motion and the differential equations defining it constitute in no way a theory of their own. In fact, to define the relative motion we may simply make use of the differential equations of absolute motion. Thus, if $T^{(a)}$ denote the kinetic energy of the material system due to its absolute motion, we shall have the differential equations of Lagrange:

$$(1) \qquad p_k^{(a)} = \frac{\partial T^{(a)}}{\partial q_k'}, \quad \frac{dp_k^{(a)}}{dt} = \frac{\partial (T^{(a)} + U)}{\partial q_k} \qquad (k = 1, 2, \cdots, \mu),$$

and those of Hamilton-Jacobi:

$$(2) \qquad \begin{cases} \dfrac{dp_k^{(a)}}{dt} = -\dfrac{\partial H^{(a)}}{\partial q_k}, \quad \dfrac{dq_k}{dt} = \dfrac{\partial H^{(a)}}{\partial p_k^{(a)}} \\ H^{(a)} = \sum\limits_{k=1}^{k=\mu} p_k^{(a)} q_k' - T^{(a)} - U \end{cases} \qquad (k = 1, 2, \cdots, \mu),$$

which, on integration, give the $q_1, \cdots, q_\mu$ as functions of t and 2μ arbitrary constants, thus solving the problem of relative motion.

§2. *Lagrange's equations for relative motion derived from the equations of absolute motion.*—It will now be shown how the differential equations of chaps. I, II may be obtained from the equations (1) and (2). I will introduce two new functions, namely,

$$N = \sum m_i(v_{0x}x_i + v_{0y}y_i + v_{0z}z_i) = Mv_0r_c \cos(v_0, r_c), \tag{3}$$

$$T_0 = \tfrac{1}{2} Mv_0^2, \tag{4}$$

where v_0 denotes the velocity of the origin of the axes XYZ, while r_c, M as before denote the radius vector of the center of inertia and the total mass of the material system respectively. It can be easily verified that

$$T^{(a)} = T^{(r)} + K + L + G + T_0 + \frac{dN}{dt}. \tag{5}$$

Now, since the functions K, G and T_0 do not involve the derivatives $q_1', \cdots, q_\mu'$, and moreover since

$$\frac{dN}{dt} = \frac{\partial N}{\partial t} + \sum_{k=1}^{k=\mu} q_k' \frac{\partial N}{\partial q_k}, \tag{6}$$

and therefore

$$\frac{\partial}{\partial q_k'}\left(\frac{dN}{dt}\right) = \frac{\partial N}{\partial q_k}, \tag{7}$$

it will be seen that

$$p_k^{(a)} = \frac{\partial T^{(a)}}{\partial q_k'} = \frac{\partial}{\partial q_k'}\left(T^{(r)} + L + \frac{dN}{dt}\right) = \mathfrak{p}_k + \frac{\partial N}{\partial q_k'}. \tag{8}$$

Hence,

$$\frac{d\mathfrak{p}_k}{dt} = \frac{\partial(T^{(a)} + U)}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial N}{\partial q_k}\right).$$

But it can be readily shown that

$$\frac{d}{dt}\left(\frac{\partial N}{\partial q_k}\right) = \frac{\partial}{\partial q_k}\left(\frac{dN}{dt}\right), \tag{9}$$

and therefore the preceding equation takes the form:

$$\frac{d\mathfrak{p}_k}{dt} = \frac{\partial}{\partial q_k}\left(T^{(a)} + U - \frac{dN}{dt}\right) = \frac{\partial(T^{(r)} + L + U_1)}{\partial q_k},$$

the function T_0 not involving the $q_1, \cdots, q_\mu$. The function U_1 has the same meaning here as in chap. I.

Thus equations (10′) of chap. I have been deduced from equations (1) of this chapter.

Proceeding in like manner when the real forces have no potential function, equations (10), chap. I may be obtained from the following equations of absolute motion:

$$(10) \qquad p_k^{(a)} = \frac{\partial T^{(a)}}{\partial q_k}, \quad \frac{dp_k^{(a)}}{dt} = \frac{\partial T^{(a)}}{\partial q_k} + Q_k^{(0)} \qquad (k=1, 2, \cdots, \mu).$$

§ 3. *Hamilton-Jacobi's equations derived from the equations of absolute motion.*—Passing to the canonical equations (2) I remark that

$$H^{(a)} = \sum_{k=1}^{k=\mu} p_k q_k' + \sum_{k=1}^{k=\mu} \frac{\partial L}{\partial q_k'} q_k' + \sum_{k=1}^{k=\mu} \frac{\partial N}{\partial q_k} q_k'$$
$$- T^{(r)} - K - L - G - T_0 - \frac{dN}{dt} - U,$$

or, since*

$$\sum_{k=1}^{k=\mu} \frac{\partial L}{\partial q_k'} q_k' = L - \lambda, \qquad \sum_{k=1}^{k=\mu} \frac{\partial N}{\partial q_k} q_k' = \frac{dN}{dt} - \frac{\partial N}{\partial t},$$

that

$$H^{(a)} = \sum_{k=1}^{k=\mu} p_k q_k' - T^{(r)} - U_1 - \lambda - \frac{\partial N}{\partial t} - T_0 .$$

Introducing into this expression the variables $\mathfrak{p}_1, \cdots, \mathfrak{p}_\mu$ in lieu of the $p_1, \cdots, p_\mu$, it will become:

$$(11) \qquad H^{(a)} = \Theta_1 - U_1 - \lambda - \frac{\partial N}{\partial t} - T_0,$$

Θ_1 having the same meaning as in chap. II.

Now, in the equations (2), $H^{(a)}$ is a function of the $p_1^{(a)}, \cdots, p_\mu^{(a)}$, whereas in (11) it is a function of the $\mathfrak{p}_1, \cdots, \mathfrak{p}_\mu$. Denoting by the symbol $[\;]_p$ that the function inside the brackets is expressed in terms of the $p_1, p_2, \cdots$, we shall have:

$$\frac{\partial [H^{(a)}]_{p^{(a)}}}{\partial q_k} = \frac{\partial [H^{(a)}]_{\mathfrak{p}}}{\partial q_k} - \sum_{e=1}^{e=\mu} \frac{\partial [H^{(a)}]_{p^{(a)}}}{\partial p_e^{(a)}} \frac{\partial}{\partial q_k}\left(\frac{\partial N}{\partial q_e}\right)$$
$$= \frac{\partial [H^{(a)}]_{\mathfrak{p}}}{\partial q_k} - \sum_{e=1}^{e=\mu} q_e' \frac{\partial}{\partial q_k}\left(\frac{\partial N}{\partial q_e}\right)$$
$$= \frac{\partial [H^{(a)}]_{\mathfrak{p}}}{\partial q_k} - \frac{\partial}{\partial q_k} \sum_{e=1}^{e=\mu} q_e' \frac{\partial N}{\partial q_e} = \frac{\partial (\Theta_1 - U_1 - \lambda)}{\partial q_k} - \frac{\partial}{\partial q_k}\left(\frac{dN}{dt}\right),$$

and therefore, by (2),

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* See formulas (2), chap. II and (8) of this chapter.

$$\frac{dp_k^{(a)}}{dt} = -\frac{\partial(\Theta_1 - U_1 - \lambda)}{\partial q_k} + \frac{\partial}{\partial q_k}\left(\frac{dN}{dt}\right).$$

But

$$\frac{dp_k^{(a)}}{dt} = \frac{d\mathfrak{p}_k}{dt} + \frac{d}{dt}\left(\frac{\partial N}{\partial q_k}\right) = \frac{\partial \mathfrak{p}_k}{dt} + \frac{\partial}{\partial q_k}\left(\frac{dN}{dt}\right).$$

Hence

$$\frac{d\mathfrak{p}_k}{dt} = -\frac{\partial(\Theta_1 - U_1 - \lambda)}{\partial q_k} \qquad (k = 1, 2, \cdots, \mu).$$

At the same time, as is readily seen from (2) and (11),

$$\frac{dq_k}{dt} = \frac{\partial(\Theta_1 - U_1 - \lambda)}{\partial \mathfrak{p}_k} \qquad (k = 1, 2, \cdots, \mu).$$

These 2μ equations are identical with the canonical system (6), chap. II.

When the real forces have no potential function, the system of differential equations (4), (5) may be obtained in like manner from the following equations of absolute motion:

$$(12) \qquad \left\{ \begin{aligned} &\frac{dp_k^{(a)}}{dt} = -\frac{\partial\Theta^{(a)}}{\partial q_k} + Q_k^{(0)}, \qquad \frac{dq_k}{dt} = \frac{\partial\Theta^{(a)}}{\partial p_k^{(a)}} \quad (k = 1, 2, \cdots, \mu). \\ &\Theta^{(a)} = \sum_{k=1}^{k=\mu} p^{(a)} q_k' - T^{(a)}. \end{aligned} \right.$$

§4. *Gilbert's method.*—PH. GILBERT * has indicated an exceedingly simple way of obtaining LAGRANGE'S equations of relative motion. It is based on the remark that *by adding the function K to the potential U we may assume that the origin of the axes XYZ is fixed in space.* Then $T^{(a)}$ becomes what has been denoted by T_2 in chap. I, and the differential equations of (absolute) motion will be:

$$\frac{d}{dt}\left(\frac{\partial T_2}{\partial q_k'}\right) = \frac{\partial(T_2 + U + K)}{\partial q_k} \qquad (k = 1, 2, \cdots, \mu).$$

which are equivalent to the equations (10′), chap. I, as was there explained.

§5. While GILBERT restricted himself to LAGRANGE'S equations it is easy to see how his method may be applied to the canonical system of HAMILTON-JACOBI. In fact, if we put

$$H_2 = \sum_{k=1}^{k=\mu} \mathfrak{p}_k q_k' - T_2 - (U + K),$$

where, again,

$$\mathfrak{p}_k = \frac{\partial T_2}{\partial q_k'},$$

* *Mémoire sur l'Application de la méthode de Lagrange*, etc. Part I, §1.

we shall arrive at the canonical system of differential equations:

$$(13) \qquad \frac{d\mathfrak{p}_k}{dt} = -\frac{\partial H_2}{\partial q_k}, \quad \frac{dq_k}{dt} = \frac{\partial H_2}{\partial \mathfrak{p}_k} \qquad (k = 1, 2, \cdots, \mu);$$

and it is readily seen that

$$(14) \qquad H_2 = \Theta_1 - U_1 - \lambda.$$

To show this, I remark that

$$H_2 = \sum_{k=1}^{k=\mu} p_k q_k' + \sum_{k=1}^{k=\mu} \frac{\partial L}{\partial q_k'} q_k' - T_2 - U - K$$

$$= \sum_{k=1}^{k=\mu} p_k q_k' + L - \lambda - T_2 - U - K$$

$$= \sum_{k=1}^{k=\mu} p_k q_k' - T^{(r)} - U_1 - \lambda.$$

Introducing the variables $\mathfrak{p}_1, \cdots, \mathfrak{p}_\mu$ into this expression, $\sum p_k q_k' - T^{(r)}$ will become what has been denoted by Θ_1, and H_2 will assume the form (14).

Equations (13), (14) are identical with equations (6), chap. II.

CHAPTER XI.

Some remarks concerning the Differential Equations of Lagrange and of Hamilton-Jacobi.

§1. We have seen in chapter I that the fictitious forces of Coriolis give rise to two classes of terms in the expression of Q_k. One class can be expressed as the derivatives with regard to the q_k of a function $(K + G)$ of the variables $q_1, q_2, \cdots, q_\mu, t$; while the other is capable of being thrown into the form:

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial q_k'}\right),$$

L being a function of $q_1, q_2, \cdots, q_\mu; q_1', q_2', \cdots, q_\mu'; t$. That part of Q_k, therefore, which is due to the fictitious forces of Coriolis may be omitted if, in exchange we add the function $K + G$ to the potential of the real forces and at the same time add the function L to the kinetic energy $T^{(r)}$ of relative motion. This in fact is the way in which we derived equations (9) and (10) from (1) in chap. I.

In general, let the equations:

$$(1) \qquad \frac{d}{dt}\frac{\partial T}{\partial q_k'} - \frac{\partial T}{\partial q_k} = Q_k \qquad (k = 1, 2, \cdots, \mu)$$

define the motion (absolute or relative) of a material system. It is clear that any part of Q_k such that it may be thrown into the form:

$$\frac{\partial V}{\partial q_k} - \frac{d}{dt}\frac{\partial V}{\partial q_k'},$$

V being a function of $q_1, \cdots, q_\mu;\ q_1', \cdots, q_\mu';\ t$, may be omitted if in exchange we add the function V to the function T in (1). We thus arrive at the extended notion of a force function as introduced into mechanics by SCHERING.* Namely, if the forces F_i are such that

$$(2) \qquad \sum_{i=1}^{i=n}\left(F_{ix}\frac{\partial x_i}{\partial q_k} + F_{iy}\frac{\partial y_i}{\partial q_k} + F_{iz}\frac{\partial z_i}{\partial q_k}\right) = \frac{\partial V}{\partial q_k} - \frac{d}{dt}\frac{\partial V}{\partial q_k'} \qquad (k=1, 2, \cdots, \mu),$$

V being a function of $q_1, \cdots, q_\mu;\ q_1', \cdots, q_\mu';\ t$, then LAGRANGE'S equations will take the form:

$$p_k = \frac{\partial(T+V)}{\partial q_k'}, \qquad \frac{dp_k}{dt} = \frac{\partial(T+V)}{\partial q_k} \qquad (k=1, 2, \cdots, \mu).$$

§2. From the above it follows that

If the function V satisfies the conditions:

$$(3) \qquad \frac{\partial V}{\partial q_k} - \frac{d}{dt}\frac{\partial V}{\partial q_k'} = 0, \qquad (k=1, 2, \cdots, \mu),$$

then we may add or subtract this function from T without in any way affecting the motion of a material system given by the differential equations:

$$\frac{d}{dt}\frac{\partial T}{\partial q_k'} - \frac{\partial T}{\partial q_k} = Q_k.$$

In fact such an operation is equivalent to the introduction of vanishing forces.

§3. As seen from formulas (7) and (9) of the preceding chapter, the function dN/dt satisfies the conditions (3). We may therefore, without affecting in any way the motion of the given system, add dN/dt to $T^{(r)}$ in the equations (10) or (10′) of chap. I, or subtract dN/dt from $T^{(a)}$ in the equations (1) of chap. X. The first of these operations gives:

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* *Abhandlungen der Kgl. Ges. d. Wissenschaften zu Göttingen,* vol. 18, §§16–18.

The force function of SCHERING is defined by the equation:

$$(2') \qquad \sum F_i \delta r_i = \delta V - \frac{d}{dt}\sum_{i=1}^{i=3n}\frac{\partial V}{\partial u_i}\delta u_i,$$

where u_i has the same meaning as in this paper, while r_i stands for $\sqrt{x_i^2 + y_i^2 + z_i^2}$. The μ equations (2) are equivalent to equation (2′).

It is by means of SCHERING'S extended notion of a force function that KAMERLINGH-ONNES derived the equations (10′) of chap. I and (6) of chap. II.

$$\frac{d}{dt}\frac{\partial}{\partial q'_k}\left(T^{(r)} + L + \frac{dN}{dt}\right) = \frac{\partial}{\partial q_k}\left(T^{(r)} + K + L + G + \frac{dN}{dt}\right) + Q^0_k,$$

or, as is obvious,

$$(4) \qquad \frac{dp^{(a)}_k}{dt} = \frac{\partial T^{(a)}}{\partial q_k} + Q^0_k.$$

If the real forces have a potential function U,

$$(5) \qquad \frac{dp^{(a}_k}{dt} = \frac{\partial(T^{(a)} + U)}{\partial q_k}.$$

These are exactly the equations (1) of chap. X. The second operation, as may readily be seen, will change equations (4) or (5) of this chapter into equations (10) and (10′) of chap. I.

Thus, the remark of §2 enables us to pass at once from the equations of absolute motion to those of relative motion and vice versa.

§4. In a similar way can be brought forth the connection between the canonical systems (6) of chap. II and (2) of chap. X.

In fact, in equations (2) of chap. X we may subtract dN/dt from $T^{(a)}$ without in any way affecting the motion. Thus we shall have:

$$(6) \qquad \frac{d\mathfrak{p}_k}{dt} = -\frac{\partial H'}{\partial q_k}, \qquad \frac{dq_k}{dt} = \frac{\partial H'}{\partial \mathfrak{p}_k},$$

because now

$$p^{(a)}_k = \frac{\partial}{\partial q'_k}\left(T^{(a)} - \frac{dN}{dt}\right) = \mathfrak{p}_k;$$

and as to H', it will be

$$\begin{aligned} H' &= \sum_k q'_k \frac{\partial}{\partial q'_k}\left(T^{(a)} - \frac{dN}{dt}\right) - T^{(a)} + \frac{dN}{dt} - U \\ &= \sum_k \mathfrak{p}_k q'_k - T^{(r)} - L - T_0 - U_1 \\ &= \sum_k p_k q'_k - T^{(r)} - T_0 - U_1 - \lambda \\ &= \Theta_1 - U_1 - \lambda. \end{aligned}$$

Here we may neglect the term T_0, as derivatives of H' with respect to q_k and $\mathfrak{p}_k$ only enter into (6) and H' is expressed as a function of $q_1, \cdots, q_\mu$; $\mathfrak{p}_1, \cdots, \mathfrak{p}_\mu$; t.

Hence, as anticipated, equations (6) are identical with the system (6) of chap. II.

Conversely, by adding dN/dt to $T^{(r)}$ we can obtain from equations (4), (5), (6) of chap. II the system (2) of chap. X, or a corresponding system for the case when the real forces do not have a potential function.

§5. If, considering again, more generally, the case of any motion, as expressed by the equations (1) of the present chapter, we apply the remark made in §1 to the canonical system:

$$\frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k}, \quad \frac{dq_k}{dt} = \frac{\partial H}{dp_k} \qquad (k = 1, 2, \cdots, \mu),$$

$$H = \sum_k p_k q_k' - T - U,$$

we shall obtain for the generalized force function of SCHERING the canonical system of differential equations in the following form:*

$$p_k = \frac{\partial(T + V)}{\partial q_k'}, \quad H = \sum_{k=1}^{k=\mu} p_k q_k' - T - V;$$

$$\frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k}, \quad \frac{dq_k}{dt} = \frac{\partial H}{\partial p_k} \qquad (k = 1, 2, \cdots, \mu).$$

§6. Throughout this paper, whenever Lagrangian coördinates were introduced, it was assumed that they formed an *independent* system, i. e., the number μ of such coördinates was equal to the degree of freedom of the material system. It is needless to say that to define the relative position of this system a number $\mu' > \mu$ of coördinates could be chosen, but then certain Lagrangian factors due to the reactions of some or all of the constraints would have to be introduced in the differential equations of motion. Putting $\mu' = \mu + \nu$ it would only be necessary to add the functions:

$$\sum_{k=1}^{k=\nu} \lambda_k \frac{\partial \phi_k}{\partial q_1}, \quad \cdots; \quad \sum_{k=1}^{k=\nu} \lambda_k \frac{\partial \phi_k}{\partial q_e}, \quad \cdots; \quad \sum_{k=1}^{k=\nu} \lambda_k \frac{\partial \phi_k}{\partial q_{\mu'}};$$

to $Q_1^{(0)}, \cdots, Q_e^{(0)}, \cdots, Q_{\mu'}^{(0)}$ respectively in the equations (1) chap. I and (1) chap. II.

A similar change would be necessary in the equations of motion of a solid body.

* SCHERING, loc. cit.