

A PURELY GEOMETRIC REPRESENTATION OF ALL POINTS IN THE PROJECTIVE PLANE*

BY

JULIAN LOWELL COOLIDGE

Introduction.

The advantage of having a basis of Geometric reality for the complex numbers of Algebra has long been apparent to mathematicians. More than two centuries ago JOHN WALLIS † devised a method of representing imaginary numbers by points in a plane which is truly ingenious, although hopelessly inferior to the well-known method with which we habitually associate the names of ARGAND and GAUSS. ‡ From another point of view, this latter method may be looked upon as a method of representing, by means of real points in a plane, the imaginary points of a real straight line.

The first mathematician to give a satisfactory purely geometric definition of imaginary elements was VON STAUDT, and in his *Beiträge zur Geometrie der Lage* § he shows how the GAUSS construction may be reached from a totally different point of view. He does not however outline his method in its most general form, as he makes use of the circular points at infinity, and a better statement has been given by KÖTTER. || The same construction was used by HENRY J. S. SMITH ¶, but the end in view was different and he did not employ VON STAUDT's distinction between conjugate imaginaries.

The problem of representing by means of real elements all points, not of a real straight line merely, but of a real plane, has not, to my knowledge, been

* Presented to the Society October 28, 1899, under a slightly different title. Received for publication March 12, 1900.

† WALLIS, *Algebra*, chapter 68.

‡ The true discoverer seems to have been WESSEL: *Om Direktionens analytiske Betegning*, Memoirs of Danish Academy, vol. 5, 1799. A French translation was laid before the same society by ZEUTHEN in 1897. A brief history of this whole subject will be found in *A Chapter in the History of Mathematics*, by W. W. BEMAN, Proceedings of the American Association, vol. 46, 1897.

§ VON STAUDT, *Beiträge zur Geometrie der Lage*, p. 264, Nuremberg 1856-1860.

|| KÖTTER, *Grundzüge einer rein geometrischen Theorie der algebraischen ebenen Curven*, p. 21, Berlin 1887.

¶ *Mémoire sur Quelques Problèmes Cubiques et Biquadratiques*, *Annali di Matematica pura ed applicata*, series 2, vol. 3, Milan 1869.

seriously attempted; or, at least, no purely geometric solution has been published. * One sees at once that the attempt to represent these points by the real points in space, to which one would naturally be led by the analogy of the method of GAUSS, must of necessity be fruitless, for the real points of space form but a triply infinite system, while the system of imaginary points of a real plane is quadruply infinite. We notice however that the system of real straight lines in space is quadruply infinite, and it is the object of this paper to show how they may be employed to represent the aforesaid system of imaginary points. After the necessary preliminary definitions of §1 I give in §2 a representation of all points in a real line by lines in a real plane † and then (§3) extend this representation so as to include all points in a real plane, noticing in particular those systems of lines which represent points on an imaginary line. I shall then (§§4, 5) take up the subject of chains of points, showing their application to the general theory of projectivity. After that (in §§6, 7) I shall glance briefly at the system of lines which represent points on a real conic, and conclude with a few remarks as to other possible solutions of our problem, and the extension to three dimensions.

§1. Definitions.

A linear range of points in elliptic involution, to which a particular sense of description is attached, is defined as an *imaginary point*. If the involution be looked upon as having the contrary sense, one has the *conjugate point*. The real line may be spoken of as the *base* of the point.

A linear pencil in elliptic involution is defined as the one or the other of two conjugate *imaginary lines of the first sort*, according as it is supposed to have the one or the other sense of description.

An analogous definition of an *imaginary plane* is obtained from an axial pencil in elliptic involution.

An *imaginary line of the second sort* is defined as the intersection of two imaginary planes whose real axes do not meet. This definition is from AUGUST ‡ and differs slightly in wording from that of VON STAUDT § though entirely equivalent thereunto in content. The imaginary line of the second sort differs from that of the first sort in that it contains no real point, and lies in no

*[Since this article went to press I have had brought to my notice that DUPORT in the *Annales scientifiques de l'École Normale*, ser. 2, vol. 9, p. 301, has given a very neat analytic solution of the problem of representing all points in a plane by real lines. (See the addition to §7.)]

† The construction appears as semi-correlative to that of KÖTTER. He establishes a correspondence of point to point by means of linear pencils; I shall establish a correspondence of point to line by means of projective ranges.

‡ *Untersuchungen über das Imaginäre in der Geometrie*; Programm der Friedrichsrealschule, Berlin 1872.

§ loc. cit., p. 77.

real plane. The system of all real lines meeting an imaginary line of the second sort (and also its conjugate, the intersection of the conjugate planes), that is, the bases of all points on such a line, form a congruence of the first order and class; one such line passes through every real point, and one lies in every real plane. All lines of the system which meet a given line, not of the system, are generators of a ruled quadric. No two lines of the system intersect.

§2. Representation of Points of a Real Line.

Let the real line be l , lying in a real plane π . Let a' be another real line in π cutting l in A and containing two other chosen real points O and Q . Finally let an imaginary line i' pass through Q . We may represent any point P of l by the real line or lines p through the intersection of OP with i' . If P be a real point distinct from A , p will coincide with OP and be unique. If P be the same as A , p will be any real line through Q . If P be imaginary, our statement of the construction has merely metaphorical significance, and the real geometric meaning must be examined more closely. Let i' be determined by the linear pencil in elliptic involution $a'a'_1 b'b'_1$, through Q , of which $a'a'_1$, $b'b'_1$ are two harmonically separated pairs, and $a'b'a'_1$ gives the sense of i' . In the same way let the imaginary point P be determined by the linear range AA_1BB_1 in elliptic involution, of which AA_1 , BB_1 are harmonically separated pairs, and ABA_1 gives the sense P . The lines $O(BA_1B_1)$ may be called ba_1b_1 respectively. Since $a'ba_1b_1 \bar{\cap} a'b'a'_1b'_1$ the lines bb' , $a_1a'_1$, and $b_1b'_1$ will meet on a line p . The pencils $a'ba_1b_1$ and $a'b'a'_1b'_1$ will cut this line in pairs of the same involution, and to the sense $a'ba_1$ will correspond the sense $a'b'a'_1$. The line p will then be the base of the intersection of OP with i' , and may be taken to represent P . Owing to the harmonic nature of the sets, $a'b'a'_1b'_1 \bar{\cap} a'b'_1a'_1b' \bar{\cap} a'b_1a'_1b$. The lines $a_1a'_1$, bb' , $b_1b'_1$ will meet on a line p_1 bearing an involution of like sort to that on p with the exception that to the sense $a'ba_1$ will correspond the sense $a'b'_1a'_1$. The line p_1 will then represent P_1 the conjugate of P . It will be seen that pp_1 meet at the intersection of $a_1a'_1$ and are harmonically divided by them.

Conversely, suppose that we have given a line p in π , and wish to find the corresponding point P on l . If p pass through O , P will be its intersection with l . If p pass through Q , P will coincide with A . If p pass through neither of these points, find its intersections with $b'a'_1b'_1$ and connect with O by lines ba_1b_1 . The involution determined by $a'a'_1$, bb_1 on l in sense $a'ba_1$ will be the point P required.

§3. Representation of Points of a Real Plane.

In representing all points of a real plane, we shall follow a method entirely analogous to that just explained for points on a line. Let the plane be called λ .

Let O be a real point and q a real line both in a plane a' , neither however lying in the plane λ . Let an imaginary plane i' pass through q . Any point P may be represented by the real line or lines p through the intersection of OP with i' . If P be a real point not on the line $(a'\lambda)$, p will be the line OP . If P be a real point on $(a'\lambda)$, the lines p will be the bundle of lines through the intersection of OP with q .

If P be imaginary, we shall, as before, need a more elaborate construction. Let i' be determined by an axial pencil in elliptic involution through q of which $a'a'_1$, $\beta'\beta'_1$ are two harmonic pairs and $a'\beta'a'_1$ gives the sense of i' . Let P be an imaginary point of λ not on the line $(a'\lambda)$ but on some real line l . Let it be determined by an elliptic involution, whereof AA_1 , BB_1 are harmonically separated pairs, A lying in $(a'\lambda)$, and ABA_1 giving the sense P . The plane Ol will cut i' in an imaginary line i , and we shall represent P by the base of the intersection of OP and i , exactly as in two dimensions. If P be an imaginary point of the line $(a'\lambda)$ it will, in every case, be represented by the line q .

Conversely, for a real line p we find the corresponding point P . If p pass through O , the point sought is the intersection with λ . If p do not pass through O , nor yet meet the line q , we may confine ourselves to the plane Op and find P by the construction already explained for two dimensions. If p meet the line q , P will be the projection of this intersection from O on $(a'\lambda)$. If p coincides with q , P will be any point real or imaginary on $(a'\lambda)$.

Our scheme of correspondence is now complete, and, if we leave out the exceptional points on the line $(a'\lambda)$ which all correspond to the line q (and those which are real, to whole bundles concurrent on q besides), we see that to a single point corresponds a single real line, while to the points of a real line correspond the lines of a real plane through O , and conversely.

Let us now see how those lines lie which correspond to the points on an imaginary line (naturally, of the first sort) lying in our plane λ . Two important cases must be distinguished: (1) M the real point of the imaginary line i lies on $(a'\lambda)$; (2) M does not lie on $(a'\lambda)$. In the first case i will be determined in the sense $(a'\lambda)ba_1$ by an elliptic involution of lines, of which $(a'\lambda)a_1$, bb_1 are harmonically separated pairs. Let the planes Ob , Oa_1 , Ob_1 be called $\beta a_1 \beta_1$ respectively. Since $a'\beta'a'_1\beta'_1 \bar{\cap} a'\beta a_1 \beta_1$, while a' is self-corresponding, the intersections of $\beta'\beta$, a'_1a_1 , $\beta'_1\beta_1$ will be three rays of a linear pencil in elliptic involution having its center at the point (OM, q) and lying in some plane μ . This pencil will be perspective both with $a'\beta'a'_1\beta'_1$ and with $a'\beta a_1 \beta_1$, and to the sense $a'\beta'a'_1$ will correspond the sense $a'\beta a_1$. It will therefore determine an imaginary line k which is the intersection of planes Oi and i' . Every point in i will thus be represented by the base p of a point in k , that is, by a real line in μ . Conversely, suppose that we have a real plane μ intersecting q . Connect the lines $(a'\mu)(\beta'\mu)(a'_1\mu)(\beta'_1\mu)$ with O by means of planes $a'\beta a_1 \beta_1$. These planes will

cut λ in lines $(a'\lambda)ba_1b_1$ determining in sense $(a'\lambda)ba_1$ an imaginary line i whose real point M clearly lies on $(a'\lambda)$. To every line p in μ will correspond the intersection of the plane Op with the line i . It will be noticed that if μ pass through q the corresponding points will all line on $(a'\lambda)$.

Let us now take up the second case, in which M , the real point of i , does not lie on $(a'\lambda)$. Using the same notation as before and calling the plane Oi ι we see that the axes of the pencils $a'\beta'a'_1\beta'_1$ and $\beta a_1\beta_1$ do not meet and the intersections of the planes ι' is an imaginary line of the second sort. The points of i are thus represented by the lines of a congruence. Any point of i , say that which lies on the real line l , is represented by that line of the congruence which lies in the plane Ol . The point M will be represented by the line OM , which is indeed the line of the congruence which passes through O . Conversely, an imaginary line k of the second sort lying in ι' will, with its conjugate, determine such a congruence of the first order and class. The congruence will determine about that ray which passes through O an axial pencil in elliptic involution, which will intersect λ in a linear pencil of the same sort. This latter, when taken in the sense corresponding to k , will give an imaginary line i , the locus of points represented by the lines of the congruence, since the plane Ok passes through ι' .

Before proceeding further let us recapitulate our results:

To each point in the plane and not on $(a'\lambda)$ will correspond a real line in space not meeting q , and conversely. To each real point on $(a'\lambda)$ will correspond the bundle of lines through its projection from O on q . To each imaginary point on $(a'\lambda)$ will correspond the single line q .

To all points on a real line will correspond all lines in the plane connecting it with O .

To all points on an imaginary line having its real point in $(a'\lambda)$ will correspond all lines in a plane, not through either O or q .

To all points on an imaginary line whose real point is not on $(a'\lambda)$ will correspond all lines of a congruence of first order and class determined by a certain imaginary line of the second sort in ι' .

It might seem as though we should also consider the problem of representing all points in an imaginary plane, but that need not detain us, for every point on the real axis may be represented by that axis, and every other point by the single real line on which it lies.

§4. Chains of Points.

It is now necessary to go into the difficult subject of chains of points. A few preliminary definitions will not be out of place. Let pqr be real bases of four points of an imaginary line of the second sort k . Let aba_1 be three lines meeting pqr , while aba_1 gives the sense of the points of k on the lines pqr . Three

possible cases may arise. In the first, s is a generator of the system pqr in the ruled quadric determined by the other six lines; in the second, the pencils $p(aba_1)$ and $a(pqr)$ cut s in ranges having the same sense, while, in the third, the ranges so cut have opposite senses. In the first case, the point (sk) is defined as having a sense neutral to that determined by the points (pk) (qk) (rk) , in the second, as having the same or like sense, and in the third, as having the opposite sense. If s and t , the bases of two points of k , are separated by the quadric, the one will bear a point having the same sense as (pk) (qk) (rk) and the other a point having the opposite sense, * and conversely.

All those points of an imaginary line of the second sort whose sense is neutral to that determined by three given points of the line are said to belong to the *chain of points* determined by those three. The bases of all points of a chain are generators of a ruled quadric.

An exactly analogous treatment may be accorded to planes through an imaginary line of the second sort.

If $ABCD$ and $A_1B_1C_1D_1$ stand for two sets of points on, or planes through, an imaginary line of the second sort, the two groups of four points are said to be of the same sort, with regard to sense, if, according as D has a neutral, like, or opposite sense to that determined by ABC , D_1 has a neutral, like, or opposite sense to that determined by $A_1B_1C_1$, and in the first case if further to separated pairs separated pairs correspond.†

It may be shown by an elaborate investigation, which certainly need not be repeated here, that four points on a real line, or an imaginary one of the first sort, may be defined as having the same relation with regard to sense as the planes connecting them with an imaginary line of the second sort, and similarly for linear or axial pencils. Further, the relation of four elements with regard to sense is unchanged by an indefinite number of projections and intersections, real or imaginary.‡ If now we remember that an imaginary line in λ is represented, in the general case, by the real bases of points of an imaginary line of the second sort, we shall have:

The lines representing chains of points on an imaginary line in λ not having its real point on $(a'\lambda)$ will be generators of ruled quadrics. The lines representing points having the sense determined by three points will be separated by the quadric of the chain from the lines representing points having the opposite sense.

To investigate the system of lines representing chains of points on real lines in λ or on imaginary lines whose real points are on $(a'\lambda)$ it is necessary to look at

* VON STAUDT, loc. cit., p. 43.

† This restriction is introduced in order to establish the continuous character of projective correspondence, and may be omitted where that is reached from other considerations. Compare REYE, *Geometrie der Lage*, vol. 1, p. 12 of Introduction, third edition, Leipzig, 1886.

‡ Compare LÜROTH, *Das Imaginäre in der Geometrie und das Rechnen mit Wurfen*, *Mathematische Annalen*, vol. 8, pp. 170-172, 1875.

bases of chains of points on imaginary lines of the first sort. I am not aware that this has ever been done; it is quite easy after what we have already accomplished. For we see that an imaginary line of the first sort may easily be projected into one of the second, the bases of the points in the former becoming those of points in the latter, while all relations of sense remain unchanged. The generators of a ruled quadric will thus appear as the projection of the tangents to a real conic. As the imaginary line of the second sort lies in the quadric, so the line of the first sort will touch the conic, as will of course its conjugate. Two lines separated by the quadric will evidently appear as two lines, one of which cuts the conic in real points, while the other does not. This shows at once the arrangement of bases of points on an imaginary line of the first sort, and if we look upon the latter as arising from the intersection of the plane ι' with a real plane through O or with an imaginary one whose real axis meets q we shall have:

The lines representing chains of points on a real line in λ , or on an imaginary one whose real point is in $(a'\lambda)$ are tangents to a conic which touches ι' , and its conjugate ι'_1 . The systems of points whose sense is like to that determined by three chain points, and those whose sense is opposite, are represented by two systems of lines, one cutting the conic in real points, and the other in imaginary points.

§5. *General Theory of Projectivity.*

The question of chains is of particular importance because it leads to the general question of the projectivity of one-dimensional geometric forms, real or imaginary. The ordinary conception of projective figures is that of two figures which may be made the first and last of a sequence of perspective figures. VON STAUDT however defined* them as figures having a one-to-one correspondence and so related that four elements of one have the same relation with regard to sense as the four corresponding elements of the other and established* the important conclusion † that these two definitions amount to exactly the same. A proof of this theorem based on what we have here worked out may not be without interest.

As the relation of four elements with regard to sense is unaltered by projection, if two figures are the first and last of a sequence of perspective figures

* VON STAUDT, loc. cit., §§ 215-218, with the reference from § 218 to § 112 of the *Geometrie der Lage*.

† In the ordinary representation of a line on the Gauss plane a chain of points appears as a circle of points and the sense of D with respect to ABC depends upon the relation of D to the circle ABC with the sense ABC of description. VON STAUDT's theorem implies that the one to one point transformations of the Gauss plane, which preserve circles and senses of point quadruples and the "projective" transformations:

$$w = \frac{az + \beta}{\gamma z + \delta},$$

of the complex variable are coextensive.

they will fulfil VON STAUDT's definition. To show the converse, that two figures which answer to this definition may be made the first and last of a sequence of perspective figures, requires a rapid glance at harmonic forms.

The simplest linear constructions, such as connecting two points by a line, or passing a plane through a line and a point, may be performed unequivocally with imaginary elements as well as with real ones. If then we define four harmonic points by the familiar relation to a complete quadrangle, and four elements of a linear or axial pencil as harmonic if they project four harmonic points, the harmonic relation will be seen to be invariant in projection, whether the elements involved be real or imaginary. It may be shown further that four harmonic elements always belong to the same chain and that their bases appear in such form as four harmonic generators of a ruled quadric or four harmonic tangents to a conic.* In particular, four harmonic imaginary points on a real line in λ will be represented by four harmonic tangents to a conic in a plane through O , touching ι' and ι'_1 , and conversely.

Suppose now that we have two figures projective according to VON STAUDT's definition: i. e., having a one-to-one correspondence, and the same relations of sense, among corresponding members. We may suppose them to be two real lines, as other cases may be reduced to this one by means of projections and sections. Consider the representing lines. To each chain-conic a^2 in one plane, will correspond a chain-conic a_1^2 in the other plane. To two chain-conics b^2 and c^2 tangent to a^2 will correspond two b_1^2 and c_1^2 touching a_1^2 . To d^2 touching the real common tangents to b^2 and c^2 will correspond d_1^2 touching the tangents common to b_1^2 and c_1^2 . Now it may easily be shown that the tangents common to $a^2 d^2$ form a harmonic set with two tangents at points of contact of $a^2 b^2$ and $a^2 c^2$; similarly, in the other plane we shall have four harmonic tangents to a_1^2 .† Hence, if our original figures are projective according to VON STAUDT's definition, four harmonic elements of one correspond to four harmonic elements of the other. But the correspondence of harmonic elements, coupled with the correspondence of continuity and of sense gives, as is well known, the necessary and sufficient condition that two figures may be made the first and last of a sequence of projections and sections. Hence VON STAUDT's definition is coextensive with the usual one.‡

If we define the projectivity of a system of conics tangent to four given lines by that of the range of points of contact on one of these lines, they will also be

* LÜROTH, loc. cit., p. 176.

† Of course all conics here considered are chain-conics, and have two imaginary common tangents. An easy proof may be devised by considering the following special case of the correlative: If two circles touch a given circle, a circle through their intersection cuts the first in points harmonically separated by the points of contact.

‡ It is curious that the erroneous view that a one-to-one correspondence is necessarily a projective correspondence, has been widely held among mathematicians of prominence; compare:

projective with the system of tangents common to them and a fixed conic touching but three of the given lines. An easy proof of this may be obtained by transforming all the conics into points, by means of the transformation relative to the generalized form of inversion.* This shows the correspondence of planes of lines corresponding to points of projective real linear ranges in λ , or imaginary ones with their real points in $(a'\lambda)$. To each line in one plane will in general correspond one line in the other, the lines which meet $(a'\lambda)$ being exceptions. To each set of tangents to a conic touching $\iota'\iota'_1$ in one will correspond a projective set of tangents to a conic of like sort in the other: to a set of such chain-conics tangent to two given lines in one will correspond a projective set tangent to the corresponding lines. Finally, to the set of lines cutting a chain-conic in real points in one plane will correspond either the set cutting the corresponding conic in real points, or that cutting it in imaginary ones. A criterion may be established for determining in any particular case in which of these two latter ways the correspondence will appear, but the labor involved is considerable, and seems scarcely worth while when we remember that the question settles itself if we know the positions of four lines in one plane corresponding to four lines not touching the same chain-conic in the other. The relation between these two planes is in general quite different from a real collineation, for a point appears as a degenerate chain-conic, and will correspond to a proper conic in the other.

§6. *Representation of Points on a Real Conic.*

It is now worth while briefly to consider the positions of those lines which represent points on a real conic in λ . We see at once that the lines representing real points on the conic are generators of a quadric cone with its vertex at O . As every real line in λ and every imaginary one having its real vertex in $(a'\lambda)$ cuts the conic twice, we have:

The lines representing points of a real conic in λ form a congruence of the second class. It will be seen that the same is true of the lines representing an

CHASLES, *Principe de correspondance entre deux objets variables, qui peut être d'un grand usage en Geometrie*; *Comptes Rendus*, vol. 41, Dec. 24, 1855.

CREMONA, *Introduzione ad una Teoria Geometrica delle Curve piane*, p. 7, Bologna 1862.

SALMON, *Conic Sections*, p. 304, sixth edition, London 1879.

A neat and very simple analytic proof of the absurdity of this view is given by GEISER, *Sopra un Teorema Fondamentale della Geometria*; *Annali di Matematica*, series 2, vol. 4, Milan 1870.

The necessity for insisting on the correspondence of sense, even after the correspondence of harmonic elements has been proved, may not be entirely evident. A simple example will make it more clear. Suppose that on any real line, each real point be supposed to correspond to itself, each imaginary point to its conjugate. We shall have a correspondence of chains, of harmonic points, of continuity, but not of sense; the correspondence is not projective.

* Compare HIRST, *On the quadric inversion of plane curves*, *Proceedings of the Royal Society*, vol. 14, p. 91, 1865.

imaginary conic, but we shall not take any further notice of this latter curve in the present article.

Let us assume that the conic has at least one real point A on $(a'\lambda)$. Take an imaginary point I of the curve, and connect it with the real points of $(a'\lambda)$. We shall have one chain of a linear pencil at I , and the points where these lines meet the conic again, as they form the whole system of points collinear with I and real points of $(a'\lambda)$, will be represented by all the lines of the system which meet the line representing I . Connecting these points with A , we shall have one chain of a pencil at A projective with one chain of a pencil at I . Now, since the chain of the pencil at A will cut any real line in a chain of points, which latter will be represented by tangents to a real conic, we see that the system of all points on the chain of the pencil at A will be represented by systems of lines lying in planes which pass through a common point, the projection of A on q , and cut a real plane through O in tangents to a conic; that is to say, the planes envelop a quadric cone. The planes tangent to this cone form a system projective with the axial pencil about the line representing I , and the intersections of the corresponding planes will be generators of a ruled cubic surface.* If the conic did not meet $(a'\lambda)$ in a real point, the lines of the system which met a chosen one of their number would still generate some surface. Moreover, that surface would be of the third order, for the conic in question might be obtained from the previous one by a homographic transformation of the plane, which, while altering the positions of the representing lines, would not alter the order of ruled surfaces determined by them. The general proposition is, then, that the lines representing points on a real conic, which meet a chosen line of the system, not representing a real point, are generators of a ruled cubic surface.

§ 7. *Other Possible Constructions.*

It is scarcely necessary to say that the method here described is by no means the only one which might have been used to represent the points of a real plane by means of real lines. For instance, we might have proceeded as follows: Suppose that we have given λ and two conjugate imaginary lines of the second sort ii_1 . Each real point of λ may be represented by the real ray of the involution system ii_1 through it. Take an imaginary point ABA_1B_1 on a line l . The self-corresponding rays of the involution system which meet l will be generators of a ruled quadric forming the involution determined by ABA_1B_1 . The generators of the other system will form the involution ii_1 . As both involutions are elliptic, there will be one pair of self-corresponding lines which are axes of both.† The corresponding lines of the involution system obtained by

* Compare REYE, loc. cit., vol. 1, page 211.

† Compare VON STAUDT, loc. cit., p. 71.

joining criss-cross the intersections of these axes with λ and its mate λ_1 , may be taken to represent ABA_1B_1 and its conjugate ABA_1B_1 . It will not be difficult to find which line corresponds to which point. This method is peculiarly neat and symmetrical, for a real point is represented by a self-corresponding line while conjugate imaginaries appear as corresponding lines. It is, however, exceedingly difficult to manipulate, and for practical purposes inferior to the one which we have worked out.

[Still another method is the following. Let the plane λ be harmonically separated from a plane α by two planes $\beta\beta_1$. Let A be a chosen real point in α through which pass two conjugate imaginary lines ii_1 . To represent a point P in λ , we connect the intersection of the real axis of the plane Pi with β and the intersection of the axis of the plane Pi_1 with β_1 . This line is taken to represent the point P ; if P be a real point the line will be PA . If α is the plane at infinity, and $\beta\beta_1$ are situated at a distance 1 from λ on either side, while A is the normal point to λ and ii_1 pass through the circular points at infinity in λ , we have the construction used by DUPORT in the paper cited in the Introduction. That paper is rich in results, and the algebraic expressions involved are particularly neat and symmetrical.]

The problem of representing all points in space by real figures does not offer a very favorable field for research. For, in order to employ straight lines in an unequivocal manner we should have to make use of a four-dimensional universe, no great help towards sense-perception. Again, we might use some other sex-tuply infinite system, as that of all twisted cubic curves through three given points, or all circles in space. But there would be a decided lack of naturalness about such a proceeding, nor would the results to be obtained appear in any degree commensurate with the extraordinary amount of labor involved.

HARVARD UNIVERSITY, CAMBRIDGE, MASS., *March* 10, 1900.