# "D" LINES ON QUADRICS*

BY

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## § 1. *Introduction.*

§ 1. The "D" lines so named by E. COSSERAT † were considered originally by DARBOUX.‡ They are the lines drawn upon a surface in such a manner that the osculating sphere at every point is tangent to the surface at that point. From this definition DARBOUX deduced their differential equation :

$$(1) \qquad \frac{3d^2s}{ds} = \frac{2 \mathbf{S}\, d^2x\, d\frac{\partial F}{\partial x} + \mathbf{S}\, dx\, d^2\frac{\partial F}{\partial x}}{\mathbf{S}\, dx\, d\frac{\partial F}{\partial x}} .$$

The summations refer to the coördinates $x, y, z$; $F(x, y, z) = 0$ being the equation of the surface under consideration.

Applying this equation to the quadric surfaces we obtain

$$(2) \qquad \frac{1}{D^2} ds^2 = \mathbf{S}\, dx\, d\frac{\partial F}{\partial x} .$$

In the same memoir DARBOUX showed that by using elliptic coördinates the variables in (2) can be separated. The equation then becomes

$$(3) \qquad \frac{\sqrt{D^2 - u}\, du}{\sqrt{(a^2 - u)(b^2 - u)(c^2 - u)}} = \frac{\sqrt{D^2 - v}\, dv}{\sqrt{(a^2 - v)(b^2 - v)(c^2 - v)}} .$$

Here $u =$ constant, $v =$ constant are the equations of the lines of curvature of the quadric, and $a$, $b$, $c$ are its semi-axes.

These lines have attracted considerable attention; and ENNEPER,§ RIBAUCOUR,|| and E. COSSERAT ¶ have attempted to study them upon surfaces in general.

* Presented to the Society at the Columbus meeting, August 25, 1899, under another title and in a somewhat different form. Received for publication May 11, 1900.

† Comptes Rendus, 1895.

‡ Comptes Rendus, 1871.

§ Göttinger Nachrichten, 1871.

|| Comptes Rendus, 1875.

¶ loc. cit.

The object of this paper is to apply the theory of elliptic functions to the integration of the differential equation (3) given by DARBOUX and to obtain an idea of the appearance of these lines and some of their properties.

## §§ 2, 3. *"D" lines on central quadrics.*

§ 2. The geometrical interpretation of (2) for the case under consideration gives the following theorem:

I. *For any central quadric the semi-diameter, drawn parallel to a tangent to a "D" line on that surface, is constant in length.*

Conversely, let us find the differential equation of the curves on the surface of a central quadric such that the semi-diameter parallel to the tangent to these curves is constant in length.

We have for any central quadric,

$$\frac{1}{D^2} = \frac{\cos^2 \omega}{u} + \frac{\sin^2 \omega}{v}, \tag{4}$$

where $u$, $v$ are the parameters of the lines of curvature, $\omega$ is the angle of the curve with the lines of curvature $v =$ constant, and $D$ the semi-diameter parallel to the tangent to the curve. We suppose $D$ constant.

If $E$ and $G$ are the first fundamental coefficients of Gauss's theory and

$$f(u) = 4(a^2 - u)(b^2 - u)(c^2 - u),$$

then

$$E = \frac{u(u - v)}{f(u)}, \quad G = \frac{v(v - u)}{f(v)},$$

$$\tan \omega = \sqrt{-\frac{f(u)\, v\, dv}{f(v)\, u\, du}}.$$

Substituting into (4) we shall have

$$\frac{\sqrt{D^2 - u}\, du}{\sqrt{f(u)}} = \frac{\sqrt{D^2 - v}\, dv}{\sqrt{-f(v)}},$$

and since this equation coincides with (3) we can say:

II. *The "D" lines on central quadrics are the only lines satisfying the conditions of the theorem* I.

§ 3. The differential equation (3) when transformed into polar coördinates:

$$x = a \cos \phi \sin \psi,$$
$$y = b \sin \phi \sin \psi,$$
$$z = c \cos \psi,$$

becomes

$$d\psi^2 + \sin^2 \psi\, d\phi^2 = \frac{ds^2}{D^2}. \tag{5}$$

This equation leads easily to the theorem:

III. *The length of a portion of the central projection of the "D" line upon a concentric sphere of radius D is equal to the projected portion of the "D" line.*

## §§ 4–11. "*D*" *lines on an ellipsoid.*

§ 4. The equation (3) we write as follows:

$$\frac{(D^2-u)du}{2\sqrt{R(u)}}+\frac{(D^2-v)dv}{2\sqrt{R(v)}}=0, \tag{6}$$

where

$$R(\theta)=(a^2-\theta)(b^2-\theta)(c^2-\theta)(D^2-\theta).$$

We suppose $a>b>c$. To (6) we join the following two:

$$\left\{\begin{aligned} &\frac{du}{2\sqrt{R(u)}}+\frac{dv}{2\sqrt{R(v)}}=dU,\\ &\frac{ds}{dU}=-D\sqrt{(u-D^2)(v-D^2)}, \end{aligned}\right. \tag{7}$$

where $U$ is a new parameter, used by WEIERSTRASS; the second equation can be obtained from

$$ds=\sqrt{E}\,du\sin\omega+\sqrt{G}\,dv\cos\omega,$$

$$\sin\omega=\frac{\sqrt{(u-D^2)v}}{D\sqrt{u-v}},\qquad \cos\omega=\frac{\sqrt{(D^2-v)u}}{D\sqrt{u-v}}.$$

§ 5. Let $u=$ constant determine the confocal hyperboloid of two nappes, and $v=$ constant, that of one nappe. Then

$$a^2>u>b^2>v>c^2.$$

If $\sin\omega$ and $\cos\omega$ be real we must have

$$u>D^2>v.$$

Therefore three cases present themselves:

$$1)\ D^2=b^2,\quad 2)\ D^2>b^2,\quad 3)\ D^2<b^2.$$

The first case as indicated by DARBOUX gives us the circular sections of the ellipsoid. Theorem I gives the same result and leads to the following theorem:

IV. *Through every umbilic passes only one "D" line—the circular section parallel to the tangent plane at the opposite umbilic.*

§ 6. Passing over to the second case, in which $D^2>b^2$, we see that in (6) and (7) $u$ varies from $D^2$ to $a^2$, $v$ from $c^2$ to $b^2$.

To accomplish the inversion of the system (6) and (7) we use the substitution given by HALPHEN * for the case when the roots of the quartic are known. Put

$$\theta - a^2 = -\frac{\wp'(\frac{1}{2}\lambda)}{\wp(t) - \wp(\frac{1}{2}\lambda)}, \quad \theta - D^2 = -\frac{\wp'(\frac{1}{2}\lambda)}{\wp(\frac{1}{2}\lambda) - e_3}\,\frac{\wp(t) - e_3}{\wp(t) - \wp(\frac{1}{2}\lambda)},$$

$$\theta - b^2 = -\frac{\wp'(\frac{1}{2}\lambda)}{\wp(\frac{1}{2}\lambda) - e_2}\,\frac{\wp(t) - e_2}{\wp(t) - \wp(\frac{1}{2}\lambda)},$$

$$\theta - c^2 = -\frac{\wp'(\frac{1}{2}\lambda)}{\wp(\frac{1}{2}\lambda) - e_1}\,\frac{\wp(t) - e_1}{\wp(t) - \wp(\frac{1}{2}\lambda)},$$

When $\theta$ varies from $D^2$ to $a^2$, we must take $t$ purely imaginary and let it vary from $\omega_3$ to $0$; when $\theta$ varies from $c^2$ to $b^2$, $t - \omega_1$ must be purely imaginary and must vary from 0 to $\omega_3$; $\omega_1$ and $\omega_3$ are the half-periods of the elliptic functions introduced, the first real, the second purely imaginary.

Let $t_1$ be the parameter $t$ corresponding to $u$; $t_2$ that corresponding to $v$. Then $t_1$ varies from $\omega_3$ to $0$; $t_2$ from $\omega_1$ to $\omega_3 + \omega_1$.

We can easily write:

$$(8) \qquad \begin{cases} e_1 = \frac{1}{12}\{(D^2 - a^2)(c^2 - b^2) + (b^2 - a^2)(c^2 - D^2)\} > 0, \\ e_2 = \frac{1}{12}\{(D^2 - a^2)(b^2 - c^2) + (c^2 - a^2)(b^2 - D^2)\} > 0, \\ e_3 = \frac{1}{12}\{(b^2 - a^2)(D^2 - c^2) + (c^2 - a^2)(D^2 - b^2)\} < 0. \end{cases}$$

Also

$$\frac{d\theta}{dt} = \frac{\wp'(\frac{1}{2}\lambda)\wp'(t)}{[\wp(t) - \wp(\frac{1}{2}\lambda)]^2}, \qquad \pm\sqrt{R(\theta)} = i\frac{d\theta}{dt}.$$

Therefore

$$\frac{du}{\sqrt{R(u)}} = idt_1, \quad \frac{dv}{\sqrt{R(v)}} = -idt_2.$$

On substituting the above values into (6) and integrating we have

$$(9) \qquad e^{2c_1 + (t_1 - t_2)\left\{-\frac{\wp'(\frac{1}{2}\lambda)}{\wp(\frac{1}{2}\lambda) - e_3} - 2\zeta(\frac{1}{2}\lambda)\right\}} = \frac{\sigma(t_1 - \frac{1}{2}\lambda)\sigma(t_2 + \frac{1}{2}\lambda)}{\sigma(t_1 + \frac{1}{2}\lambda)\sigma(t_2 - \frac{1}{2}\lambda)},$$

$$-2Ui = t_1 - t_2.$$

Introduce another parameter $a$, such that

$$2ia = t_1 + t_2;$$

then

$$(10) \qquad \begin{cases} t_1 = (a - U)i, \\ t_2 = (a + U)i. \end{cases}$$

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* HALPHEN, *Fonctions Elliptiques*, vol. I, p. 131.

Put also

$$-\frac{\wp'(\frac{1}{2}\lambda)}{\wp(\frac{1}{2}\lambda)-e_3}-2\zeta(\tfrac{1}{2}\lambda)=-2\frac{\sigma_3'(\frac{1}{2}\lambda)}{\sigma_3(\frac{1}{2}\lambda)}=K.$$

Then (9) can be reduced to

$$\wp(ai)=\frac{\wp(Ui+\frac{1}{2}\lambda)\sigma^2(Ui+\frac{1}{2}\lambda)-e^{2c_1-2UiK}\wp(Ui+\frac{1}{2}\lambda)\sigma^2(Ui-\frac{1}{2}\lambda)}{\sigma^2(Ui+\frac{1}{2}\lambda)-e^{2c_1-2UiK}\sigma^2(Ui-\frac{1}{2}\lambda)}.$$

§7. We shall use this equation to calculate the coördinates $x$, $y$, $z$ of the points of the "$D$" lines as uniform functions of the parameter $U$.

Without going into calculations which should not trouble anyone familiar with elliptic functions we merely write the final results:

$$(11)\quad\begin{cases}\dfrac{x}{a}=\pm\dfrac{1}{2\sigma_3(\frac{1}{2}\lambda)}\cdot\dfrac{1}{\sigma(2Ui)}\left(\sigma(2Ui+\tfrac{1}{2}\lambda)\cdot e^{-2Ui\frac{\sigma_3'(\frac{1}{2}\lambda)}{\sigma_3(\frac{1}{2}\lambda)}-c_1}\right.\\ \qquad\qquad\left.+\sigma(2Ui-\tfrac{1}{2}\lambda)\cdot e^{2Ui\frac{\sigma_3'(\frac{1}{2}\lambda)}{\sigma_3(\frac{1}{2}\lambda)}+c_1}\right),\\ \dfrac{y}{b}=\pm\dfrac{1}{2\sigma_3(\frac{1}{2}\lambda)\sqrt{e_2-e_1}}\cdot\dfrac{1}{\sigma(2Ui)}\left(\sigma_2(2Ui+\tfrac{1}{2}\lambda)\cdot e^{-2Ui\frac{\sigma_3'(\frac{1}{2}\lambda)}{\sigma_3(\frac{1}{2}\lambda)}-c_1}\right.\\ \qquad\qquad\left.-\sigma_2(2Ui-\tfrac{1}{2}\lambda)\cdot e^{2Ui\frac{\sigma_3'(\frac{1}{2}\lambda)}{\sigma_3(\frac{1}{2}\lambda)}+c_1}\right),\\ \dfrac{z}{c}=\pm\dfrac{1}{2\sigma_3(\frac{1}{2}\lambda)\sqrt{e_1-e_2}}\cdot\dfrac{1}{\sigma(2Ui)}\left(\sigma_1(2Ui+\tfrac{1}{2}\lambda)\cdot e^{-2Ui\frac{\sigma_3'(\frac{1}{2}\lambda)}{\sigma_3(\frac{1}{2}\lambda)}-c_1}\right.\\ \qquad\qquad\left.-\sigma_1(2Ui-\tfrac{1}{2}\lambda)\cdot e^{2Ui\frac{\sigma_3'(\frac{1}{2}\lambda)}{\sigma_3(\frac{1}{2}\lambda)}+c_1}\right).\end{cases}$$

From (7) we obtain:

$$(12)\quad\begin{aligned}\frac{1}{D}\cdot\frac{ds}{d(2Ui)}=\frac{1}{2\sigma_3(\frac{1}{2}\lambda)}\cdot\frac{1}{\sigma(2Ui)}&\left(\sigma_3(2Ui+\tfrac{1}{2}\lambda)\cdot e^{-2Ui\frac{\sigma_3'(\frac{1}{2}\lambda)}{\sigma_3(\frac{1}{2}\lambda)}-c_1}\right.\\ &\left.-\sigma_3(2Ui-\tfrac{1}{2}\lambda)\cdot e^{2Ui\frac{\sigma_3'(\frac{1}{2}\lambda)}{\sigma_3(\frac{1}{2}\lambda)}+c_1}\right).\end{aligned}$$

The above expressions contain two arbitrary constants $\lambda$ and $c_1$.

§8. The derivative $ds/d(2Ui)$ satisfies Lamé's equation:

$$\frac{1}{\phi}\frac{d^2\phi}{d(2Ui)^2}=2\wp(2Ui)+\wp(\tfrac{1}{2}\lambda-\omega_3).$$

§ 9. We have seen that $t_1$ and $t_2 - \omega_1$ must be purely imaginary and vary between $\omega_3$ and 0. Hence $t_2 - t_1 = 2Ui$ must be a complex number of the form:

$$\omega_1 + k\omega_3 + iX,$$

where $k$ is any whole number and $X$ varies from 0 to $\pm\,\omega_3/i$.

Substituting this value for $2Ui$ in the formulæ (11) and distinguishing the cases of $k$ even and odd we have the two following sets of expressions, in which $A$ and $C$ represent $-\frac{1}{2}K$ and $ic_1/A$ respectively:

1) $k$ even:

$$\frac{x}{a} = \pm\frac{1}{2\sigma_3(\frac{1}{2}\lambda)}\cdot\frac{e^{-A(X-C)i}\sigma_1(iX+\frac{1}{2}\lambda)+e^{A(X-C)i}\sigma_1(iX-\frac{1}{2}\lambda)}{\sigma_1(iX)},$$

$$\frac{y}{c} = \pm\frac{1}{2\sigma_3(\frac{1}{2}\lambda)}\cdot\frac{e^{-Ai(X-C)i}\sigma_3(iX+\frac{1}{2}\lambda)-e^{A(X-C)i}\sigma_3(iX-\frac{1}{2}\lambda)}{\sigma_1(iX)},$$

$$\frac{z}{c} = \pm\frac{\sqrt{e_1-e_3}}{2\sigma_3(\frac{1}{2}\lambda)}\cdot\frac{e^{-Ai(X-C)}\sigma(iX+\frac{1}{2}\lambda)+e^{Ai(X-C)}\sigma(iX-\frac{1}{2}\lambda)}{\sigma_1(iX)},$$

$$\frac{1}{D}\cdot\frac{ds}{dX} = \frac{\sqrt{e_1-e_3}}{2\sigma_3(\frac{1}{2}\lambda)}\cdot\frac{e^{-Ai(X-C)}\sigma_2(iX+\frac{1}{2}\lambda)-e^{Ai(X-C)}\sigma_2(iX-\frac{1}{2}\lambda)}{\sigma_1(iX)}.$$

2) $k$ odd:

$$\frac{x}{a} = \pm\frac{1}{2\sigma_3(\frac{1}{2}\lambda)}\cdot\frac{e^{-Ai(X-C)}\sigma_2(iX+\frac{1}{2}\lambda)+e^{Ai(X-C)}\sigma_2(iX-\frac{1}{2}\lambda)}{\sigma_2(iX)},$$

$$\frac{y}{b} = \pm\frac{\sqrt{e_2-e_3}}{2\sigma_3(\frac{1}{2}\lambda)}\cdot\frac{e^{-Ai(X-C)}\sigma(iX+\frac{1}{2}\lambda)+e^{Ai(X-C)}\sigma(iX-\frac{1}{2}\lambda)}{\sigma_2(iX)},$$

$$\frac{z}{c} = \pm\frac{i}{2\sigma_3(\frac{1}{2}\lambda)}\cdot\frac{e^{-Ai(X-C)}\sigma_3(iX+\frac{1}{2}\lambda)-e^{Ai(X-C)}\sigma_3(iX-\frac{1}{2}\lambda)}{\sigma_2(iX)},$$

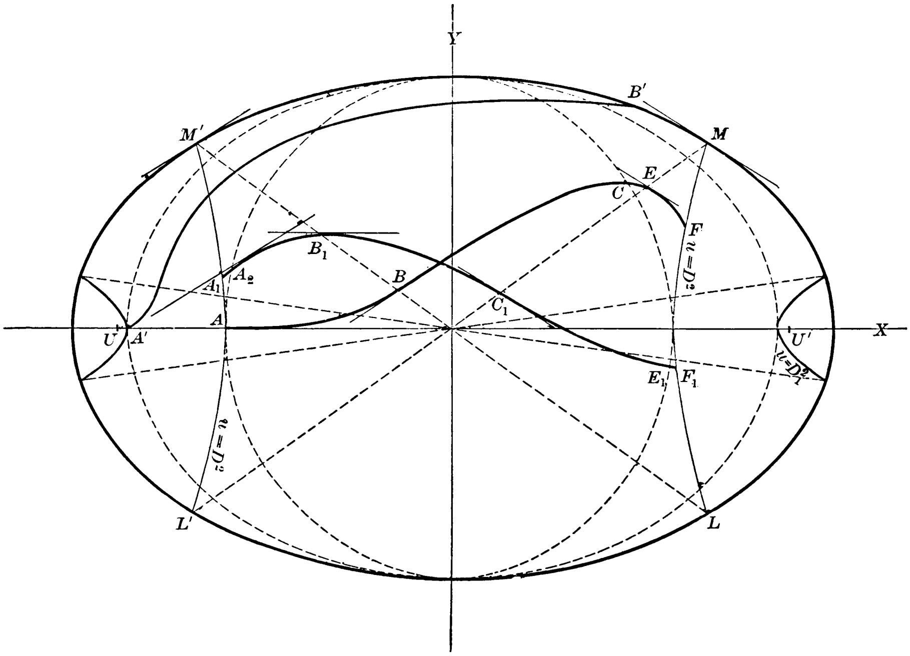
$$\frac{1}{D}\cdot\frac{ds}{dX} = \frac{\sqrt{e_2-e_3}}{2\sigma_3(\frac{1}{2}\lambda)}\cdot\frac{e^{-Ai(X-C)}\sigma_1(iX+\frac{1}{2}\lambda)-e^{Ai(X-C)}\sigma_1(iX-\frac{1}{2}\lambda)}{\sigma_2(iX)}.$$

In these formulæ $C$ and $X$ are real, $C$ varying from 0 to $\pm\infty$, and $X$ from 0 to $\pm\,\omega_3/i$.

§ 10. It is easy to establish the following properties of these curves:

1) At the points of intersection of these curves with the lines of curvature $u = D^2$, they are orthogonal to these lines of curvature.

2) They do not extend beyond the two branches of the lines of curvature $u = D^2$.




3) The points corresponding to the final values 0 and $\omega_3/i$ of the parameter $X$ lie upon two plane central sections tangent to the line $u = D^2$ at the points $u = D^2$, $v = c^2$ and $u = D^2$, $v = b^2$, respectively.

4) The tangents at these points are parallel to the elements of the cylinders touching the ellipsoid along the two plane sections.

5) Two curves corresponding to the values $C$ and $C + (A\omega_3 - \frac{1}{2}\eta_3\lambda)/A$, have a common point and a common tangent on one of these plane sections.

6) Four such curves form a continuous "$D$" line between the two branches of $u = D^2$, having an inflectional tangent plane.

7) The curves $C = 0$ have a cuspidal point for $X = 0$.

8) Lines of curvature passing through the point of intersection of two lines corresponding to the same $D$ bisect the angle between them.

In the figure $ML$, $M'L'$ are the projections on the $XY$-plane of the lines of curvature $u = D^2$; $ABCEF$ is the "$D$" line corresponding to $C = 0$; $M'L$, $ML'$, $AA_2CE_1$ are plane sections; and $A'B'$ is a "$D$" line corresponding to a value of $D$ nearly equal to $b$.

§ 11. The third case, in which $D^2 < b^2$, can be treated in exactly the same manner. The expressions for the coördinates will be

1) $k$ even:

$$\frac{x}{a}=\pm\frac{1}{2\sigma_2(\frac{1}{2}\lambda)}\cdot\frac{e^{-A(X-C)i}\sigma(\frac{1}{2}\lambda+iX)+e^{A(X-C)i}\sigma(\frac{1}{2}\lambda-iX)}{\sigma_1(iX)},$$

$$\frac{y}{b}=\pm\frac{i}{2\sigma_2(\frac{1}{2}\lambda)}\cdot\frac{e^{-A(X-C)i}\sigma_2(\frac{1}{2}\lambda+iX)-e^{A(X-C)i}\sigma_2(\frac{1}{2}\lambda-iX)}{\sigma_1(iX)},$$

$$\frac{z}{c}=\pm\frac{\sqrt{e_1-e_3}}{2\sigma_2(\frac{1}{2}\lambda)}\cdot\frac{e^{-A(X-C)i}\sigma_1(\frac{1}{2}\lambda+iX)+e^{A(X-C)i}\sigma_1(\frac{1}{2}\lambda-iX)}{\sigma_1(iX)}.$$

2) $k$ odd:

$$\frac{x}{a}=\pm\frac{1}{2\sigma_2(\frac{1}{2}\lambda)}\cdot\frac{e^{-A(X-C)i}\sigma_2(\frac{1}{2}\lambda+iX)-e^{A(X-C)i}\sigma_2(\frac{1}{2}\lambda-iX)}{\sigma_3(iX)},$$

$$\frac{y}{b}=\pm\frac{\sqrt{e_2-e_3}}{2\sigma_2(\frac{1}{2}\lambda)}\cdot\frac{e^{-A(X-C)i}\sigma(\frac{1}{2}\lambda+iX)+e^{A(X-C)i}\sigma(\frac{1}{2}\lambda-iX)}{\sigma_3(iX)},$$

$$\frac{z}{c}=\pm\frac{i}{2\sigma_2(\frac{1}{2}\lambda)}\cdot\frac{e^{-A(X-C)i}\sigma_3(\frac{1}{2}\lambda+iX)+e^{A(X-C)i}\sigma_3(\frac{1}{2}\lambda-iX)}{\sigma_3(iX)}.$$

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