ON THE EXISTENCE OF A MINIMUM OF THE INTEGRAL

$$\int_{x_0}^{x_1} F(x, y, y')\, dx$$

WHEN x_0 AND x_1 ARE CONJUGATE POINTS, AND THE GEODESICS ON AN ELLIPSOID OF REVOLUTION: A REVISION OF A THEOREM OF KNESER'S*

BY

W. F. OSGOOD

In a paper entitled *Zur Variationsrechnung* KNESER enunciates the theorem † that the integral

$$I = \int_{x_0}^{x_1} F(x, y, y')\, dx$$

ceases to be a minimum, not only when the interval (x_0, x_1) contains a point x' conjugate ‡ to x_0; but when x_1 coincides with x'. This theorem is true in general, but not in all cases, and it is the object of this paper to correct and extend the theorem and to give a complete solution of the problem that KNESER proposed to himself. The geodesics on an ellipsoid of revolution afford an example of a case in which the theorem as enunciated by KNESER is not correct.

The methods and results of this paper admit of extension to multiple integrals.

The contents are as follows: §§ 1, 2 are introductory; § 3 gives an account of KNESER'S memoir and obtains his results in a simple manner; §§ 4–6 contain the new material of the paper.

§ 1. *Necessary Conditions.*

A necessary condition that the integral I be made a minimum by the function y is that y satisfy LAGRANGE'S equation

* Presented to the Society December 28, 1900. Received for publication January 30, 1901.

† Mathematische Annalen, vol. 50 (1897), p. 50. Cf. also § 6 of this paper. This theorem had however been stated and proved for the general case twenty years earlier by ERDMANN: Zeitschrift für Mathematik und Physik, vol. 22 (1877), p. 324, who showed that, when $\delta y = \varepsilon \phi_\gamma(x, \gamma_0)$ (in the notation of the present paper) the second variation of the integral I vanishes while

$$\delta^3 I = -\varepsilon^3 (F_{y'y'})_{x=x'} \phi_\gamma'(x', \gamma_0)\, \phi_{\gamma\gamma}(x', \gamma_0),$$

and this is not in general zero. For this reference I am indebted to Professor BOLZA.

‡ The meaning of the word *conjugate* will appear below, § 3.

$$(1) \qquad F_y - \frac{dF_{y'}}{dx} = 0\,, \qquad \left(F_y = \frac{\partial F}{\partial y}\,,\ \text{etc.}\right)$$

or

$$F_y - F_{xy'} - y'F_{yy'} - y''F_{y'y'} = 0\,,$$

and furthermore fulfill the boundary conditions

$$y(x_0) = y_0\,, \quad y(x_1) = y_1\,,$$

where y_0, y_1 are fixed quantities.* Here the function $F(x, y, p)$, regarded as a function of the three independent variables (x, y, p), is to be continuous, together with its partial derivatives of the first and second orders, throughout a region R:

$$A_1 \leqq x \leqq B_1\,, \quad A_2 < y < B_2\,, \quad A_3 < p < B_3\,,$$

and $A_1 \leqq x_0 < x_1 \leqq B_1$. For simplicity we will assume $A_3 = -\infty$, $B_3 = +\infty$, and require that $F_{y'y'}(x, y, p)$ shall vanish at no point of R. The function $F_{y'y'}(x, y, p)$ will, then, not change sign in R.

The functions y admitted to consideration are single valued and continuous throughout the interval $x_0 \leqq x \leqq x_1$ and assume the prescribed boundary values, and they have a first derivative which, at least within the interval $x_0 < x < x_1$, is finite and continuous.

We will denote with KNESER † any solution of (1), or the curve that represents such a function, as an *extremal.*

§ 2. *Sufficient Conditions.*‡

WEIERSTRASS'S sufficient condition that such a function y, represented by a curve C, actually makes I a minimum consists (a) in the existence of a field of extremals surrounding C, and (b) in the relation $F_{y'y'}(x, y, p) > 0$, where (x, y) is any point of C and p may have any value. The notion of the field was first published by SCHWARZ.§ That which is essential may be embraced in the following requirements:

---

* We confine ourselves for simplicity to the case of integrals with fixed limits; but the method here set forth applies equally well to integrals with variable limits. In fact, one of the first examples that I formed was the following. Reversing the usual order of the processes, I started with the envelope, which I assumed as the semicubical parabola $y^2 = x^3$, and took then as extremals the right lines that envelop it. I then constructed a problem of the calculus of variations, of which these elements afford the solution and which, since the above family of right lines is cut orthogonally by a certain parabola P $(y^2 = ax + \beta)$, may be formulated as follows: *To find the shortest line that can be drawn from the point* $x = 0$, $y = 0$ *to the parabola* P.

† *Lehrbuch der Variationsrechnung*, Braunschweig, 1900, p. 24.

‡ For an exposition of WEIERSTRASS'S sufficient condition for a minimum, together with KNESER'S and HILBERT'S proofs of its validity, cf. an article by the writer, Annals of Mathematics, ser. 2, vol. 2, no. 3 (1901), p. 105.

§ Festschrift on the occasion of WEIERSTRASS'S seventieth birthday: *Ueber ein die Flächen kleinsten Inhalts betreffendes Problem der Variationsrechnung*, Acta soc. sci. Fennicae, vol. 15 (1885), p. 315 = Werke, vol. 1, p. 223. Cf. also KNESER, *Variationsrechnung*, chap. 3.

(1) Through each point (x, y) of the neighborhood S of the extremal C there passes one and only one extremal

$$y = \phi(x, \gamma),$$

γ being a parameter. The function ϕ, regarded as a function of the two independent variables (x, γ), is together with its partial derivatives $\phi_x = \phi'$, ϕ_γ, $\phi_{x\gamma} = \phi'_\gamma$ continuous at each point (x, γ) corresponding to an interior point (x, y) of S; and $\phi'(x, \gamma)$ remains finite throughout that part of the neighborhoods of (x_0, y_0), (x_1, y_1) that is bounded by any two curves of the class admitted to consideration in § 1. The equation of C is $y = \phi(x, \gamma_0)$.

(2) The function $\phi_\gamma(x, \gamma) \neq 0$, when $x_0 < x < x_1$.

By the "neighborhood" of the extremal C is meant the interior of a strip S of the xy-plane bounded by the lines $x = x_0$, $x = x_1$, $y = \phi(x, \gamma_0) + \epsilon$, $y = \phi(x, \gamma_0) - \epsilon$, where ϵ is a positive constant, together with the points (x_0, y_0), (x_1, y_1).

The definition of the field here given is broader than the one employed by Kneser in his treatise or the one that I used in the paper above referred to. Both Kneser's and Hilbert's proofs for Weierstrass's sufficient condition, given in the latter article, can be extended without difficulty to the present case. Thus, in the case of Hilbert's proof, it is sufficient to show that the value of the integral

$$J = \int_{x_0}^{x_1} \{F(x, Y, y') + (Y' - y')F_{y'}(x, Y, y')\}dx,$$

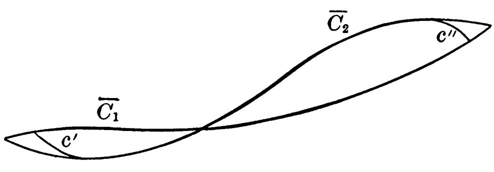
where x, Y are the coördinates of a point of an arbitrary curve $\bar{C}$ lying in the field, and where

$$y' = \phi'(x, \gamma), \quad Y = \phi(x, \gamma),$$

is independent of $\bar{C}$, i. e., is the same for any two such curves, $\bar{C}_1$, $\bar{C}_2$, and this can be proved as follows. Let the values of J for $\bar{C}_1$ and $\bar{C}_2$ be denoted respectively by J_1 and J_2, and suppose

$$J_1 - J_2 = h > 0.$$

Let (x', y') and (x'', y'') be two points of $\bar{C}_1$ chosen respectively arbitrarily near to the points (x_0, y_0) and (x_1, y_1); and let C' be a curve connecting (x', y')



and (x'', y'') and coinciding throughout the greater part of its extent with $\bar{C}_2$. More precisely, let curves c' and c'' be drawn respectively from (x', y') and

(x'', y'') tangent to $\bar{C}_2$ and meeting $\bar{C}_2$ at points that lie near to (x', y') and (x'', y'') respectively, and let c' and c'' be so chosen that the curve $\bar{C}'$, consisting of c', c'' and the part of $\bar{C}_2$ lying between the points of tangency of c' and c'', will be a curve of the class admitted to consideration, § 1. Denote the values of J taken along $\bar{C}_1$ from (x', y') to (x'', y'') and along $\bar{C}'$ respectively by J_1' and J'. Then

$$J_1' - J' = 0.$$

For, all of the conditions of the lemma on which HILBERT'S proof rests are fulfilled for the interval $x' \leqq x \leqq x''$. Now it is clear that, by a proper choice of the points (x', y'), (x'', y'') and the curves c', c'', the integrals J_1' and J' can be made to differ respectively from J_1 and J_2 by quantities each numerically less than $\frac{1}{2}h$. For, though the function $y' = \phi'(x, \gamma)$ may become infinite in the neighborhood of the point (x_0, y_0), still it is possible to cut out from S an angle bounded by the lines $y - y_0 = \pm \lambda(x - x_0)$, where λ is an arbitrarily chosen constant, within which $\phi'(x, \gamma)$ remains finite and from which the curves $\bar{C}_1$, $\bar{C}_2$, $\bar{C}'$ never emerge; and since a similar remark applies to the neighborhood of the point (x_1, y_1), this is sufficient for the proof. Hence the supposition that $h > 0$ leads to a contradiction, and

$$J_1 = J_2.$$

§ 3. *Kneser's Investigation.**

KNESER'S results may be briefly obtained as follows. Let there exist an extremal C connecting the points (x_0, y_0) and (X_1, Y_1). Consider the integral I formed for an arbitrary value of x_1 between x_0 and X_1, the point (x_1, y_1) lying on C. Further assume

(1) That through the point (x_0, y_0) and each point (x, y) of that part of the neighborhood of (x_0, y_0) for which $x > x_0$ (i. e., $x_0 < x < x_0 + \mu$, $|y - y_0| < \mu$), one and only one extremal $y = \phi(x, \gamma)$ passes and that $y = \phi(x, \gamma_0)$ coincides with C. The function $\phi(x, \gamma)$, regarded as a function of the two independent variables (x, γ), shall, together with its partial derivatives $\phi_x = \phi'$, ϕ_γ, $\phi_{x\gamma} = \phi_\gamma'$, be continuous at each of the points (x, γ) corresponding to points (x, y) of the above neighborhood; and furthermore at the points (x, γ) of the region

$$x_0 + \mu \leqq x \leqq X_1, \qquad \gamma_0 - \kappa < \gamma < \gamma_0 + \kappa;$$

and $\phi'(x, \gamma)$ shall remain finite throughout that part of the neighborhood of (x_0, y_0) bounded by any two curves of the class considered in § 1.

(2) $$\phi_\gamma(x, \gamma) \neq 0 \quad \text{when} \quad x_0 < x < x_0 + \mu, \quad |y - y_0| < \mu].$$
$$\phi_\gamma(x, \gamma_0) \neq 0 \quad \text{when} \quad x_0 + \mu \leqq x < x' < X_1; \quad \phi_\gamma(x', \gamma_0) = 0.$$

* Mathematische Annalen, loc. cit.

Since all the extremals $y=\phi(x,\gamma)$ go through the point (x_0, y_0) it follows that $\phi_\gamma(x_0,\gamma)\equiv 0$.

When $x_1<x'$, a field exists about the extremal C, as is readily proved by the ordinary methods of analysis.* We will assume once for all that $F_{y'y'}(x, y, p)$ is positive, and hence the integral I is a minimum.

When $x_1=x'$, x' is called the *conjugate point* of x_0, and it is the study of this case to which KNESER's paper is devoted. It is desirable to distinguish three cases. We will assume that $\phi_{\gamma\gamma}(x,\gamma)$ exists and is continuous throughout the neighborhood of the point (x', γ_0).

Case I. $\phi_{\gamma\gamma}(x',\gamma_0)\neq 0$. This is the general case.

Case II. $\phi_{\gamma\gamma}(x',\gamma)\equiv 0$, $|\gamma-\gamma_0|<\delta$.

Case III. $\phi_{\gamma\gamma}(x',\gamma_0)=0$, $\phi_{\gamma\gamma}(x',\gamma)\not\equiv 0$.

Cases I and II are considered by KNESER and his results are as follows. In Case I the extremals $y=\phi(x,\gamma)$ have an envelope † which passes through the point x', $y'=\phi(x',\gamma_0)$ and this point is an ordinary point of the envelope. The proof may be based on DINI's existence theorem for implicit functions of real variables,‡ according to which the equation

$$0=\phi_\gamma(x,\gamma)$$

can be solved for γ in the neighborhood of the point (x',γ_0). The function

$$\gamma=\psi(x).$$

thus defined is single valued in the neighborhood of the point x' and it has a continuous derivative throughout this region, given by the formula

$$\frac{d\gamma}{dx}=-\frac{\phi'_\gamma(x,\gamma)}{\phi_{\gamma\gamma}(x,\gamma)}.$$

Moreover, the pairs of values $\big(x, \gamma=\psi(x)\big)$ are the only pairs of values (x,γ) in the neighborhood of the point (x',γ_0) which cause $\phi_\gamma(x,\gamma)$ to vanish.

The equation of the envelope is

$$y=\phi(x,\gamma) \qquad \text{where} \qquad \gamma=\psi(x).$$

Its slope is given by the formula

$$\frac{dy}{dx}=\phi'(x,\gamma)+\phi_\gamma(x,\gamma)\,\frac{-\phi'_\gamma(x,\gamma)}{\phi_{\gamma\gamma}(x,\gamma)}=\phi'(x,\gamma).$$

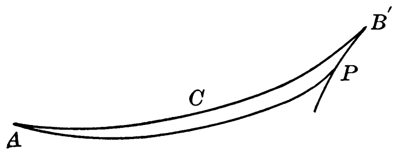
---

*The details of the proof are given in a similar case in the writer's paper, Annals of Mathematics, loc. cit., § 2.

† That this *in general* is the case is essentially JACOBI's criterion. The introduction of sufficient restrictions so that one is able to prove that, in the problem thus restricted, an envelope does exist which has an ordinary point at (x', γ_0), is characteristic of the progress of analysis since Jacobi's time.

‡ Cf. PEANO-GENOCCHI, *Calcolo differenziale*, etc., § 110, or JORDAN, *Cours d'Analyse*, vol. 1, 2d ed., p. 80. KNESER uses the method of power series.

Thus in Case I an envelope in the proper sense of the term exists, and it *enters the interval* (x_0, x'). This fact is the essential characteristic for Case I. It can now be shown that, when $x_1 = x'$, I ceases to be a minimum. Let $P = (x'', y'')$ be an arbitrary point of the envelope lying in the neighborhood of



$B = B' = (x', y')$ and let $x'' < x'$. Then the value of I taken from A to B along C is the same as its value taken first along the extremal AP and then along the envelope from P to B. The proof of this theorem is readily deduced from the analysis by which WEIERSTRASS's sufficient condition is established.* Thus the function $f(s)$ (loc. cit., § 3) formed for the curve $\overline{C}$ consisting of APB is easily shown to be constant throughout the entire interval $0 \leqq s \leqq l$. If we employ HILBERT's method (loc. cit., § 5), the proof is still simpler. Let $\overline{C}'$ be a curve connecting A with B' and lying within the region bounded by C and the curve APB'. Then it follows, as in §2 of this article, that HILBERT's integral J has the same value for $\overline{C}'$ as for C. When $\overline{C}'$ goes over continuously into APB', the integral J goes over continuously into this integral taken along APB', and the proof is complete.

Thus far it has only been shown that the integral I loses the minimum property, inasmuch as there are curves other than C lying in the neighborhood of C for which I has the same value as for C. By making use of the relation †

$$\phi'_\gamma(x', \gamma_0) \neq 0,$$

we can show that the envelope PB' is not an extremal, and hence that there exist curves in the neighborhood of C for which I has a smaller value than for C. In fact, for the envelope,

$$\frac{dy}{dx} = \phi'(x, \gamma), \qquad \frac{d^2y}{dx^2} = \phi''(x, \gamma) - \frac{\{\phi'_\gamma(x, \gamma)\}^2}{\phi_{\gamma\gamma}(x, \gamma)} \neq \phi''(x, \gamma),$$

and so the envelope does not satisfy LAGRANGE's equation (1).

In Case II no envelope in the proper sense of the term exists. All the extremals $y = \phi(x, \gamma)$ for which $\gamma_0 - \delta < \gamma < \gamma_0 + \delta$ go through the point (x', y'). The value of I taken along each of these extremals is the same, as can be shown by reasoning similar to that employed in Case I, and hence I ceases to be a

* Cf. Annals of Mathematics, loc. cit., §§ 3, 5.

† Cf. KNESER's memoir, p. 40, formula (32).

minimum. If the function $\phi(x, \gamma)$ satisfies all of the conditions requisite for a field about the arc AB' of C with the one exception that through the point (x', y') more than one extremal passes, there will be no curve $\overline{C}$ in the neighborhood of C for which I has a smaller value than for C. The great circles of a sphere, regarded as shortest lines on the sphere, are typical for this case, A and B' being two points situated diametrically opposite each other.

§ 4. *The Revision of Kneser's Theorem.*

In Case III an envelope may exist which has a singular point (e. g., cusp) at B' and *does not enter the interval* (x_0, x'). This case KNESER appears to have overlooked. * All of the conditions laid down for a field in § 2 may be fulfilled, so that the integral I will still be a minimum, as is shown by the example of the following section. On the other hand, there are cases in which the conditions of Case III are fulfilled, but in which the envelope enters the region (x_0, x') and the reasoning employed in Case I to show that I ceases to be a minimum can be repeated.

Let us, then, examine the possibilities which Case III presents with some detail. Denote the set of points (x, y) lying in the neighborhood of B', for which

$$\phi_\gamma(x, \gamma) = 0$$

by (E). The point B' is a member of (E). We will distinguish two cases.

(a) The set (E) contains a point (x, y) for which $x < x'$.

(b) The set (E) contains no such point.

In case (a) the integral I will cease to be a minimum whenever the points of (E)—or some of these points—for which $x \leqq x'$ lie on a curve E which has a continuously turning tangent and goes through B'; which, furthermore, at each point of E in the neighborhood of B', is tangent to the extremal $y = \phi(x, \gamma)$ that passes through that point; and when lastly $\phi_\gamma(x, \gamma) \neq 0$ at all points on the segment of each line parallel to the y-axis that lies between E and C. For, the proof given in Case I applies here.

A sufficient condition for the existence of case (a) is that in the neighborhood of the point $\gamma = \gamma_0$, the derivatives $\phi_{\gamma^k}(x', \gamma)$, $k = 1, 2, \cdots, n$, exist; that $\phi_{\gamma^k}(x', \gamma_0)$ vanishes when $k < n$, but does not vanish when $k = n$; and that n is even. Case (a) may, however, exist when n is odd.

For, let $\phi_{\gamma^n}(x', \gamma_0) > 0$; n even. Then the function $y = \phi(x', \gamma)$ is a minimum when $\gamma = \gamma_0$, and hence in the neighborhood of the point $\gamma = \gamma_0$, $\phi(x', \gamma) > \phi(x', \gamma_0)$ both for values of γ larger and for values of γ smaller than γ_0. Now at any point $(\overline{x}, \overline{y})$ of C intermediate between A and B', $\phi_\gamma(\overline{x}, \gamma_0) \neq 0$, and if $\phi_\gamma(\overline{x}, \gamma_0) > 0$, the curves $y = \phi(x, \gamma)$ for which

* Cf., however, below, § 6.

$\gamma_0 - \eta < \gamma < \gamma_0$ all lie below C at the point $x = \bar{x}$ and hence cut C between $\bar{x}$ and x'. The equation

$$\phi(x, \gamma) - \phi(x, \gamma_0) = (\gamma - \gamma_0)\phi'_\gamma\big(x, \gamma_0 + \theta(\gamma - \gamma_0)\big) = 0$$

has a solution $x = X$, $\bar{x} < X < x'$, and $\big(X, \gamma_0 + \Theta(\gamma - \gamma_0)\big)$ is a point of (E).

Thus for example if $\phi(x, \gamma)$ is an analytic function of (x, γ) in the neighborhood of (x', γ_0) and if

$$\phi_{\gamma^k}(x', \gamma_0) = 0 \quad (k = 1, 2, \cdots, \quad n-1); \quad \phi_{\gamma^n}(x', \gamma_0) \neq 0,$$

n being even, we shall certainly have case (a). But more than this: with the present restrictions on the function $\phi(x, \gamma)$ we can infer that all the conditions named above are fulfilled which suffice that I cease to be a minimum. In fact, the equation

$$0 = \phi_\gamma(x, \gamma)$$

now defines a set of points (E) that may be grouped together so as to constitute a finite number of curves passing through B' and not intersecting one another elsewhere in the neighborhood of B'. Through that part of the neighborhood of B' for which $x < x'$ at least one of these curves will pass, and if we denote by E that one of these curves that lies next to C (on either side), then E will satisfy all of the above requirements. For, it is possible to write $\phi_\gamma(x, \gamma)$ in the form

$$\phi_\gamma(x, \gamma) = \bar{\phi}\,\Phi,$$

where Φ is an analytic function of (x, γ) in the neighborhood of the point (x', γ_0), and does not vanish there; and where $\bar{\phi}$ is of the form:

$$\bar{\phi} = \gamma^n + A_1\gamma^{n-1} + \cdots + A_n,$$

the coefficients A_i being analytic functions of x in the neighborhood of the point $x = x'$.* The irreducible factors † of this polynomial, set equal to zero, yield the curves which constitute the set (E). In particular, let $\bar{\bar{\phi}}$ be the factor which yields the curve lying next to C. This curve satisfies all the conditions imposed on E. For, since $\bar{\bar{\phi}}$ is irreducible, $\bar{\bar{\phi}}_\gamma$ does not vanish simultaneously with $\bar{\bar{\phi}}$ in the neighborhood of (x', γ_0) when $x < x'$, and hence

$$\frac{d\gamma}{dx} = -\frac{\bar{\bar{\phi}}_x}{\bar{\bar{\phi}}_\gamma}$$

exists and is finite at all such points. The slope of the curve is given by the equation

* This is a fundamental theorem of WEIERSTRASS'S; Werke, vol. 2, p. 135.

† "Irreducible" in the sense in which I have used this term in reporting WEIERSTRASS'S theorem, Encyclopädie der Mathematischen Wissenschaften, II., B. 1.

$$\frac{dy}{dx} = \phi' + \phi_\gamma \frac{d\gamma}{dx} = \phi',$$

where $x < x'$, and it is readily shown from the continuity of $\phi'(x, \gamma)$ that at B' the slope of E is given by the same formula.

If all of the foregoing conditions hold except the one that n be *even*, and if now n is odd, it would appear that we may still have case (a) and hence the existence of an envelope E with the above properties. For, the function

$$\phi(x, \gamma) = (x+1)(x-\gamma)^3, \quad x_0 = -1, \quad x' = 0, \quad \gamma_0 = 0,$$

for example, satisfies all the conditions imposed on the function γ except, possibly, that of being an extremal. It seems probable that there are families of extremals for which this example is typical.

In case (b) a field will exist and I will continue to be a minimum provided that through B' and each point of its neighborhood for which $x < x'$ one and only one extremal $y = \phi(x, \gamma)$ passes.*

This will always be the case when it is the case for the points of the line $x = x'$ that lie in the neighborhood of B'. For then γ will be a single valued function of y along this line, continuous, monotonic and never constant, since the inverse function, $y = \phi(x', \gamma)$ is continuous and single valued.

Consider the region bounded by the quadrilateral whose sides are

$$y = \phi(x, \gamma_0 + \epsilon), \quad y = \phi(x, \gamma_0 - \epsilon), \quad x = x', x = x' - \epsilon > x_0,$$

where ϵ is so chosen (1) that the points $y' = \phi(x', \gamma_0 \pm \epsilon)$ are points of the above neighborhood of y' and (2) that $\phi_\gamma(x, \gamma)$ does not vanish within the quadrilateral. Let $(\bar{x}, \bar{y})$ be any point lying within the quadrilateral. Then the equation

$$\bar{y} = \phi(\bar{x}, \gamma)$$

admits one and only one solution $\gamma_0 - \epsilon < \gamma < \gamma_0 + \epsilon$. For

$$\phi(\bar{x}, \gamma_0 - \epsilon) < \bar{y} < \phi(\bar{x}, \gamma_0 + \epsilon)$$

and $\phi(\bar{x}, \gamma)$ is a continuous function of γ in the interval $\gamma_0 - \epsilon \leqq \gamma \leqq \gamma_0 + \epsilon$. Hence $\phi(\bar{x}, \gamma)$ assumes the value $\bar{y}$ for at least one value of γ in the interval. Since $\phi_\gamma(\bar{x}, \gamma)$ does not vanish in the interval, it follows from ROLLE'S theorem that the equation $\phi(\bar{x}, \gamma) = \bar{y}$ has only one root in the interval. Thus the sufficiency of the above condition is established.

If $\phi(x, \gamma)$ is an analytic function of (x, γ) in the neighborhood of the point (x', γ_0) and we have case (b), then the integral will always be a minimum. For, only a finite number of the derivatives $\phi_{\gamma^k}(x', \gamma_0)$ can vanish. The first one

* More precisely: provided that, if $(\bar{x}, \bar{y})$ is the point B' or any point of its neighborhood, for which $x < x'$, the equation $\bar{y} = \phi(\bar{x}, \gamma)$ has one and only one root γ lying in the neighborhood of γ_0.

that does not vanish cannot be of even order, otherwise we should have case (a). The order being odd, it is readily seen that the equation $y = \phi(x', \gamma)$ admits, for each value y in the neighborhood of y', one and only one root γ in the neighborhood of γ_0.

The foregoing results may be summarized in the following theorems, which may be regarded as a correction and extension of KNESER'S theorem.

THEOREM A. *The integral*

$$I = \int_{x_0}^{x_1} F(x, y, y')\, dx$$

ceases to be a minimum when x_1 is a conjugate point, $x_1 = x'$, of x_0 and the conditions of Case I *or* II *are fulfilled.*

This is KNESER'S theorem accurately enunciated.

THEOREM B. *In Case* III, *the integral ceases to be a minimum when we have case (a), i. e., when some of the points (E) in the neighborhood of B' at which $\phi_\gamma(x, \gamma) = 0$ lie in the interval (x_0, x'), and when, besides, such points, or a part of them, can be so grouped as to form the arc of an envelope E entering the region (x_0, x') and, together with the extremal C and the line $x = x' - \epsilon$, bounding a region within which $\phi_\gamma(x, \gamma) \neq 0$.*

If $\phi(x, \gamma)$ is an analytic function of (x, γ) in the neighborhood of the point (x', γ_0), and if we have case (a), then the further conditions of this theorem will be fulfilled. Here, a sufficient, but apparently not a necessary, condition that we have case (a) is that

$$\phi_{\gamma^k}(x', \gamma_0) = 0 \;\; (k = 1, 2, \cdots, n-1); \quad \phi_{\gamma^n}(x', \gamma_0) \neq 0,$$

where n' is even.

THEOREM C. *In Case* III, *the integral continues to be a minimum when we have case (b), i. e., when none of the points (E) in the neighborhood of B' at which $\phi_\gamma(x, \gamma) = 0$ lie within the interval (x_0, x'); and when, besides, through B' and each point (x, y) of its neighborhood for which $x < x'$ one and only one extremal $y = \phi(x, \gamma)$ passes.*

If $\phi(x, \gamma)$ is an analytic function of (x, γ) in the neighborhood of the point (x', γ_0) and if we have case (b), then the further conditions of this theorem will be fulfilled.

These theorems exhaust completely the case that $\phi(x, \gamma)$ is an analytic function of (x, γ) in the neighborhood of the point (x', γ_0), the case to which KNESER restricts his investigation.

§ 5. *The Geodesics on an Ellipsoid of Revolution.*

The geodesics on an oblate ellipsoid of revolution furnish an example of an integral I which does not always cease to be a minimum when x_1 is a conjugate point of x_0.

The form of the envelope of the geodesics drawn from a fixed point on an ellipsoid of revolution was given by JACOBI * and the envelope was investigated analytically by VON BRAUNMÜHL.†

Let the equation of a surface of revolution be written:

$$y = f(r), \quad \text{where} \quad r^2 = x^2 + z^2,$$

and let us take as the coördinates of a point on the surface r and the angle ω that the plane of the meridian through the point makes with a fixed plane through the axis of revolution. The length of the arc of a curve joining two points (r_0, ω_0) and (r_1, ω_1) is given by the formula

$$\int_{\omega_0}^{\omega_1} \sqrt{r^2 + (1 + f'(r)^2) r'^2} \, d\omega.$$

This integral is to be made a minimum. Since the integral does not contain the independent variable explicitly, LAGRANGE'S equation (1) admits a first integral and the differential equation of the extremals (geodesics) can be written in the form: ‡

$$r^2 (r^2 - \nu^2) \, d\omega^2 = \nu^2 (1 + f'(r)^2) \, dr^2, \tag{2}$$

where ν is the constant of integration.

The surface here to be considered is the oblate ellipsoid

$$\frac{r^2}{a^2} + \frac{y^2}{b^2} = 1, \quad b = ca, \quad c < 1, \tag{3}$$

and the point A through which all the extremals pass is to be a point of the equator, for which ω shall vanish. The extremal C is to be the equator, $r = a$. We will choose ω as the independent variable and replace it by x to conform to the earlier notation, and y as dependent variable. The function $F_{y'y'}$ is seen to be positive for all values $-b < y < b$ and $-\infty < y' < \infty$; it does not contain x explicitly. Equation (2) becomes the following:

$$(b^2 - y^2)^2 \{c^2 (a^2 - \nu^2) - y^2\} \, dx^2 = c^2 \nu^2 \{c^2 b^2 + (1 - c^2) y^2\} \, dy^2.$$

To integrate this equation set

$$\gamma = \pm c \sqrt{a^2 - \nu^2}, \qquad y = \gamma \sin \theta.$$

---

* *Vorlesungen über Dynamik*, 6th lecture, Werke, Supplementband, p. 47, where the form of the envelope is indicated by a figure.

† Mathematische Annalen, vol. 14 (1879), p. 557. Cf. also the models made by VON BRAUNMÜHL in the Brill (now Schilling) collection, Katalog mathematischer Modelle, ser. 5, XVIII, a, b = Specialkatalog, 104, 106 = Deutsche Mathematiker-Vereinigung, Katalog mathematische Modelle, München, 1892, 2. Teil, II. Abteilung, 214.

In a second paper by the same author (Mathematische Annalen, vol. 20 (1882), p. 557) the geodesics on an ellipsoid with three distinct axes are treated and a number of bibliographical references are given.

‡ KNESER, *Lehrbuch der Variationsreehnung*, p. 24.

Then x is given as a function of θ by a quadrature and we have the formulas:

$$(4)\qquad \begin{cases} y = \gamma \sin \theta , \\ x = \sqrt{b^2 - \gamma^2} \displaystyle\int_0^\theta \frac{\sqrt{c^2 b^2 + (1 - c^2)\gamma^2 \sin^2 \theta}}{b^2 - \gamma^2 \sin^2 \theta}\, d\theta , \end{cases}$$

where all the radicals are taken positive and $-b < \gamma < b$. For any one of these values of γ, x is a single valued continuous function of θ that increases when θ increases, and hence the inverse function, $\theta(x)$, is also single valued and increases as x increases. Equations (4) thus define y as a single valued function of (x, γ) for all values of (x, γ) in the domain:

$$-\infty < x < \infty, \quad -b < \gamma < b,$$

and it follows from DINI's theorem (cf. p. 170) that this function, $y = \phi(x, \gamma)$ has continuous partial derivatives at each point of this region.

The course of the geodetics on the ellipsoid is shown in VON BRAUNMÜHL'S first paper (loc. cit.), Figs. 1–3, and on the models. It is suggested geometrically by Fig. 3 and the corresponding model that either cusp of the envelope lying on the equator is a point x' conjugate to A for the portion of the equator C included between A and x'; that this portion of C is embedded in a family of extremals having an envelope passing through x'; and that this envelope does not enter the region Ax'. Thus it would appear that this portion of C is surrounded by a field and hence that for it the integral is a minimum. It remains to supply the corresponding analytic proofs.

In the case of the prolate ellipsoid, Fig. 1 of VON BRAUNMÜHL'S paper, it is evidently the cusps on the meridian through A that yield the desired example.

For a given value of γ lying between $-b$ and b, the curve $y = \phi(x, \gamma)$ is an extremal and its course on the ellipsoid is easily traced. Since θ always increases with x, the curve lies within the zone of the ellipsoid bounded by the planes $y = \pm \gamma$. It is a periodic curve and the value of x corresponding to a quarter period is given by the formula

$$\lambda = \sqrt{b^2 - \gamma^2} \int_0^{\frac{\pi}{2}} \frac{\sqrt{c^2 b^2 + (1 - c^2)\gamma^2 \sin^2 \theta}}{b^2 - \gamma^2 \sin^2 \theta}\, d\theta .$$

For this value of x, y is a maximum, $= \gamma$. The curve crosses the equator $y = 0$, when $\theta = \pi$, $x = 2\lambda$. Let γ converge toward 0; then 2λ approaches πc as its limit and, as will appear in the course of the subsequent work,

$$x' = \pi c$$

is the conjugate point to A $(x = 0)$ on the equator, C.

For all values of x that are positive and less than πc, $\phi_\gamma(x, \gamma) \neq 0$ when $\gamma = \gamma_0 = 0$. In act, x is then equal to $c\theta$, so that $0 < \theta < \pi$; and since

$$\phi_\gamma(x, \gamma) = \frac{\partial y}{\partial \gamma} = \sin\theta + \gamma\cos\theta\,\frac{\partial\theta}{\partial\gamma}; \tag{5}$$

it follows that $\phi_\gamma(x, \gamma_0) = \sin\theta > 0$.

At the point $x = \pi c$, $\gamma = 0$, the function $\phi_\gamma(x', \gamma_0)$ vanishes; and the same is true of the function

$$\phi_{\gamma\gamma}(x, \gamma) = \frac{\partial^2 y}{\partial\gamma^2} = 2\cos\theta\,\frac{\partial\theta}{\partial\gamma} - \gamma\sin\theta\left(\frac{\partial\theta}{\partial\gamma}\right)^2 + \gamma\cos\theta\,\frac{\partial^2\theta}{\partial\gamma^2}.$$

For, $\partial\theta/\partial\gamma$ is given by the formula:

$$\begin{aligned} 0 &= \sqrt{b^2-\gamma^2}\,\frac{\sqrt{c^2b^2+(1-c^2)\gamma^2\sin^2\theta}}{b^2-\gamma^2\sin^2\theta}\,\frac{\partial\theta}{\partial\gamma} \\ &\quad - \frac{\gamma}{\sqrt{b^2-\gamma^2}}\int_0^\theta \frac{\sqrt{c^2b^2+(1-c^2)\gamma^2\sin^2\theta}}{b^2-\gamma^2\sin^2\theta}\,d\theta \\ &\quad + \sqrt{b^2-\gamma^2}\int_0^\theta\left\{\frac{(1-c^2)\gamma\sin^2\theta}{(b^2-\gamma^2\sin^2\theta)\sqrt{c^2b^2+(1-c^2)\gamma^2\sin^2\theta}}\right. \\ &\qquad\qquad \left. + \frac{2\gamma\sin^2\theta\sqrt{c^2b^2+(1-c^2)\gamma^2\sin^2\theta}}{(b^2-\gamma^2\sin^2\theta)^2}\right\}d\theta \end{aligned} \tag{6}$$

and hence $\partial\theta/\partial\gamma$ vanishes when $\gamma = 0$. The function $\phi_{\gamma\gamma}(x', \gamma)$ does not, however, vanish identically at the point $\gamma = 0$. In fact, $\phi_{\gamma 3}(x', \gamma_0) \neq 0$. For

$$\phi_{\gamma 3}(x, \gamma) = 2\cos\theta\,\frac{\partial^2\theta}{\partial\gamma^2} + \rho,$$

where ρ is composed of terms that vanish at the point (x', γ_0), and it is readily shown by direct computation that $\partial^2\theta/\partial\gamma^2$ does not vanish at this point.

We proceed now to show that the further conditions of Theorem C, § 4, are fulfilled, and hence that the integral taken along C from A to B' is a minimum. It remains to establish two things: (a) that through each point (x, y) of the neighborhood of A, for which $x > 0$, one and only one extremal $\phi(x, \gamma)$ passes, and that at each of these points $\phi_\gamma(x, \gamma) \neq 0$; and that, furthermore, $\phi'(x, \gamma)$ remains finite in that part of this neighborhood lying between any two curves, y_1 and y_2, of the class considered in § 1; (β) that at the point B' we have case (b) (§ 4), for $\phi(x, \gamma)$ is an analytic function of (x, γ) in the neighborhood of B', and hence the existence of case (b) suffices.

The proof of (β) is direct. At all points of the region

$$x' - \epsilon < x < x', \quad -\kappa < \gamma < \kappa,$$

where ϵ, κ are sufficiently small positive constants, $\phi_\gamma(x, \gamma) > 0$. For, an approximate evaluation of x by (4) in terms of γ near 0 and θ near π shows that when x and γ lie in the above region, θ is less than π. Similarly it appears from (6) that for such values of θ and γ, $\gamma\, \partial\theta/\partial\gamma \leqq 0$. Hence the first term in the right hand member of (5) is always positive and the second is never negative.

Finally, to establish (α), we show (a) that through each point (x, y) of the neighborhood of A for which $x > 0$ at least one extremal $y = \phi(x, \gamma)$ passes; (b) that for all such points $0 < \theta < \delta$, where δ can be made arbitrarily small by choosing the above neighborhood suitably; (c) that $\gamma\, \partial\theta/\partial\gamma \geqq 0$ when $0 < \theta < \delta$, whence it follows by the aid of (5) that $\phi_\gamma(x, \gamma) > 0$; ($d$) that $\phi'(x, \gamma)$ remains finite between any two extremals $y = \phi(x, \gamma_1)$, $y = \phi(x, \gamma_2)$.

ad (a). Let ϵ, η be two positive quantities, the first arbitrarily small, the second, any quantity between 0 and 1, e. g., $\eta = \frac{1}{2}$. Then through each of the points

$$y = Y = \eta b \sin \epsilon, \quad 0 < x \leqq \epsilon',$$

where ϵ' denotes the value of x corresponding to the values $\gamma = \eta b$, $\theta = \epsilon$, there passes at least one extremal $\phi(x, \gamma)$. For, the equation

$$\gamma \sin \theta = \eta b \sin \epsilon$$

defines θ as a single valued continuous function of γ that decreases from the value ϵ to the value $0 < \sin^{-1}(\eta \sin \epsilon) < \epsilon$ when γ increases from the value ηb to b. Substitute this function $\theta(\gamma)$ in the formula for x, (4). Then x becomes a continuous function of γ in the interval $\eta b \leqq \gamma \leqq b$, assuming at the extremities of the interval respectively the values ϵ' and 0, and hence assuming each intermediate value at least once.

Thus the equation

$$Y = \phi(\bar{x}, \gamma),$$

where $\bar{x}$ is chosen arbitrarily between 0 and ϵ', admits at least one solution $\gamma = \Gamma$, where $\eta b < \Gamma < b$. Let $(\bar{x}, \bar{y})$ be any point of the region

$$0 < \bar{x} < \epsilon', \quad 0 \leqq \bar{y} < Y.$$

Then the equation

$$\bar{y} = \phi(\bar{x}, \gamma)$$

admits at least one solution, $\gamma = \bar{\gamma}$, where

$$0 \leqq \bar{\gamma} < b.$$

For

$$\phi(\bar{x}, 0) \leqq \bar{y} < \phi(\bar{x}, \Gamma),$$

and the continuous function $\phi(\bar{x}, \gamma)$ must assume the value $\bar{y}$ at least once when γ passes from 0 to Γ. To show that $\phi(\bar{x}, \gamma)$ assumes the value $\bar{y}$ only once, it is sufficient to show that $\phi_\gamma(\bar{x}, \gamma) > 0$, when $0 < \gamma < b$ and this relation follows from (b) and (c).

ad (b). At any point $(\bar{x}, \bar{y})$ at which $\gamma \geqq \eta b$,

$$\theta \leqq \epsilon, \quad \text{since} \quad \eta b \sin\theta \leqq \gamma \sin\theta = \bar{y} < Y = \eta b \sin\epsilon.$$

On the other hand, when $\gamma < \eta b$, it appears from (4) that

$$\bar{x} > \sqrt{b^2 - \eta^2 b^2}\int_0^\theta \frac{cb}{b^2 - \eta^2 b^2}\, d\theta = \frac{c}{\sqrt{1-\eta^2}}\,\theta,$$

or

$$\theta < c^{-1}\sqrt{1-\eta^2}\,\bar{x} < c^{-1}\sqrt{1-\eta^2}\,\epsilon'.$$

Hence δ may be taken equal to the larger one of the quantities ϵ, $c^{-1}\sqrt{1-\eta^2}\,\epsilon'$.

ad (c). From (6) it is clear that for all values of $-b < \gamma < b$ and for all values of $0 < \theta < \delta$, where δ is chosen sufficiently small, the sign of $\gamma\,\partial\theta/\partial\gamma$ is dominated by the sign of the first integral, and hence is positive.

From (5) it now follows that, when ϵ is chosen sufficiently small, $\phi_\gamma(x, \gamma) > 0$ at all points (x, y) of the neighborhood of A for which $x > 0$, $y \geqq 0$. Similar reasoning applies to the region $x > 0$, $y < 0$.

ad (d). The value of $\phi'(x, \gamma)$ is given by the formula

$$\phi'(x, \gamma) = \frac{dy}{dx} = \frac{\gamma}{\sqrt{b^2 - \gamma^2}}\,\frac{\cos\theta\,(b^2 - \gamma^2\sin^2\theta)}{\sqrt{c^2b^2 + (1-c^2)\gamma^2\sin^2\theta}},$$

from which it follows that $|\phi'(x, \gamma)| < b^2c^{-1}(b^2 - \gamma^2)^{-\frac{1}{2}}$. On the other hand, two extremals, $y = \phi(x, \gamma_1)$, $y = \phi(x, \gamma_2)$ can always be so chosen as to include between them that part of the neighborhood in question which lies between the curves y_1 and y_2. Hence the finiteness of $\phi'(x, \gamma)$ follows without difficulty.

§6. *The Integral* $I = \int_{t_0}^{t_1} F(x, y, x', y')\,dt$.*

In the foregoing, the value of the integral taken along C has been compared

* This section was added 7 March. At the time that I sent the MS. to the TRANSACTIONS, I had not observed that KNESER, in his treatise, § 25, notes the case in which the theorem cited in § 1 of this paper fails as an "exceptional case" (p. 95, l. 4). He recognizes that it is conceivable that the envelope may have a cusp, without however showing that this is ever actually the case. Near the close of the section he says: "Eine Ausnahme findet nur statt, wenn die stets vorhandene Enveloppe der Curven des Feldes im Brennpunkte einen Rückkehrpunkt von besonderer Art besitzt." In what the specialization of the cusp consists is not explained, and, in fact, it is not even clear whether the "Ausnahme" is considered to be an exception to the *proof* or an exception to the *theorem*.

with its value taken along only such curves of the neighborhood of C as are cut by a parallel to the y-axis in at most one point. In the case of the geodesic AB' on the ellipsoid it is readily seen that it continues to be a minimum even when compared with any continuous curve having a continuously turning tangent, which can be drawn from A to B' in the neighborhood of C. (The neighborhood of C shall now include the whole neighborhood of the points A and B'.) For, to begin with, the integral being written in the Weierstrassian form with a parameter t:

$$(7) \qquad I = \int_{t_0}^{t_1} F(x, y, x', y')\, dt,$$

it is easily seen that the field may be taken as the neighborhood of C with the exception of that part of this region that is cut out by the envelope. The Weierstrassian proof for the existence of a minimum will apply to any curve connecting A and B' and lying in this field.

Next, consider the family of geodesics through A that lie in the neighborhood of C, and lay off on each of these a length measured from A equal to the length of the geodesic AB'. The locus of the points thus determined is readily seen to be a curve Γ passing through B' and not tangent to C.

Now consider *any* curve lying in the neighborhood of C and connecting A and B'. If this curve meets Γ only at the point B', it lies wholly in the field, and hence is longer than the geodesic AB'. If it meets Γ before it reaches B', then this part of it is at least as long as AB', and hence in this case, too, its total length is greater than that of the arc AB' of the geodesic C.*

It is possible to generalize from this example as follows. Let A, B' be two conjugate points on the extremal C of the general integral of the form (7). If the extremals of the field are given by the formulas†

$$x = \xi(t, a), \qquad y = \eta(t, a),$$

then, to the former function $\phi_\gamma(x, \gamma)$ corresponds now the determinant

$$\Delta = \begin{vmatrix} \xi_t & \xi_a \\ \eta_t & \eta_a \end{vmatrix}.$$

We assume that Δ vanishes only at A and at points in the neighborhood of B'. If, furthermore, $\xi(t, a)$, $\eta(t, a)$ are both analytic in the neighborhood of B', the equation

$$\Delta = 0$$

will define one or more curves going through B' and tangent to C there.

---

* It may furthermore be shown that the arc AB' is shorter than any other curve that can be drawn on the ellipsoid connecting A and B'. For, the curve Γ is a closed curve that does not cut itself, and the field may be so chosen as to include the whole interior of this curve.

† KNESER, *Variationsrechnung*, chap. 3.

Divide the neighborhood S of C into two pieces, S_1 (containing A) and S_2, by a curve through B' cutting C normally. Then to cases (a) and (b) of § 4 correspond here the cases (a) that there are interior points of S_1 at which $\Delta = 0$ (we will also tabulate here the case that $\Delta = 0$ at B', but nowhere else in the neighborhood of B'); (b) that there are no such points, but that there are points within S_2 in the neighborhood of B' at which $\Delta = 0$. In case (a) the integral I will cease to be a minimum. In case (b) the integral will certainly continue to be a minimum when compared with the value that it attains when taken along any curve of the neighborhood of C that meets the locus $\Delta = 0$ only at B', for such a curve will lie in the field; and it will continue to be a minimum even when a curve of the kind just excluded is considered, provided that the integrand

$$F(x_1, y_1, \cos\phi, \sin\phi) > 0,$$

where the coördinates of B' are written as (x_1, y_1) and ϕ has any value. For, the locus of points so taken on the extremals which lie in the neighborhood of C that the value of I along each of these extremals is the same as the value of I along C from A to B' is a curve Γ passing through B' and not tangent to C, and the points at which $\Delta = 0$ lie on a curve or curves tangent to C at B'. The remainder of the proof follows the same lines as the proof of the example above considered.