DETERMINATION OF ALL THE GROUPS OF ORDER p^m WHICH CONTAIN THE ABELIAN GROUP OF TYPE $(m-2, 1)$, p BEING ANY PRIME*

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Introduction.

It is well known that there are just three abelian groups of order p^m which contain the group of type $(m-2, 1)$; viz. the groups of types $(m-1, 1)$, $(m-2, 2)$, $(m-2, 1, 1)$.† In what follows we shall therefore assume that the groups under consideration are non-abelian. The common abelian subgroup of type $(m-2, 1)$ will be represented by H. The operators of the required groups must transform those of H according to a subgroup of order p in the group of isomorphisms (I) of H. Our first problem is to determine all the operators of order p which are contained in I. Since the groups of order p^4 are well known we shall always assume $m > 4$. The groups which contain a cyclic invariant subgroup of order p^{m-2} are also known ‡, but we shall re-determine them in connection with the other groups since this will not materially affect the work.

The group H contains one cyclic characteristic subgroup of each order of the form p^a $(a = 1, 2, \cdots, m-3)$. It also contains one non-cyclic characteristic subgroup of each order of the form p^{a_1} $(a_1 = 2, 3, \cdots, m-2)$. Its other subgroups are cyclic and non-characteristic when p is odd. There are $p-1$ of these of each order of the form p^{a_2} $(a_2 = 2, 3, \cdots, m-3)$ while there are p such subgroups of each of the orders p and p^{m-2}. When $p = 2$ these subgroups of orders 2 and 2^{m-2} are the only non-characteristic subgroups of H. All the non-characteristic subgroups of the same order are conjugate under the holomorph of H. In fact, all the operators of the highest order in the non-characteristic subgroups of the same order are conjugate under this holomorph.

We proceed to prove several theorems applying to every abelian group (A), which will be employed in what follows. The operator of the group of isomor-

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† FROBENIUS and STICKELBERGER, Crelle's Journal, vol. 84, 1879, p. 217.

‡ BURNSIDE, *Theory of groups of finite order*, 1897, p. 75.

phisms (A') of A, which transforms each operator of highest order in A into its β-th power must transform every operator of A into its β-th power since A is generated by its operators of highest order. Let s_1, s_2 be two operators of highest order (γ) in A and let $t^{-1}s_1t = s_1^{\alpha}$, $t^{-1}s_2t = s_2^{\beta}$ ($\beta \not\equiv \alpha$, mod γ). The independent generators of the smallest group containing s_1 and s_2 may be represented by s_1 and s_3. From the equations just mentioned we have $t^{-1}s_3t = s_3^{\beta'}$ ($\beta' \not\equiv \alpha$, mod the order of s_3). Since $t^{-1}s_1s_3t = s_1^{\alpha}s_3^{\beta'}$ is not a power of s_1s_3, t does not transform each subgroup of A into itself. Hence it follows that *the only operators of A' which transform each subgroup of A into itself are those which transform each operator of A into the same power.*

It will now be proved that the only invariant operators of A' are those which transform each operator of A into the same power of itself.* Let t' be any operator of A' which does not transform each operator of A into the same power. If t' permutes some of the subgroups of highest order in A it is not commutative † with the operator of A' which transforms into itself a generator of one of these subgroups (K') without transforming into itself a generator of the subgroup into which t' transforms K'. If t' does not permute any of the subgroups of highest order in A it must transform the generator of at least two of them into different powers and hence it is not commutative with the operator which transforms one of these two generators into the other. *The number of invariant operators of A' is therefore equal to the number of natural numbers which do not exceed the highest order of the operators of A and are prime to this order*, and the quotient group of A' with respect to these invariant operators is the group according to which the subgroups of A are transformed by A'.

§ 1. *Group of isomorphisms of H.*

Since each of the $p^{m-2}(p-1)$ operators of order p^{m-2} in H can be made to correspond with every other operator of this order ‡ and any holomorphism of H is completely determined by such a correspondence and a correspondence between two of the $p(p-1)$ operators of order p which are not found in the characteristic subgroup of order p, the order of I is $p^{m-1}(p-1)^2$. By raising all the operators of H to each one of the $p^{m-3}(p-1)$ powers which are prime to p and not greater than p^{m-2} we obtain the holomorphisms of H which correspond to the cyclic subgroup of I all of whose operators are invariant under I.§ To determine the remaining operators of I we employ a method which

* Cf. Transactions of the American Mathematical Society, vol. 1, 1900, p. 396.

† Cf. BOCHERT, Mathematische Annalen, vol. 49, 1897, p. 134.

‡ Transactions of the American Mathematical Society, vol. 1, 1900, p. 397.

§ Transactions of the American Mathematical Society, vol. 1, 1900, p. 396. When $p = 2$ this group of order 2^{m-3} is of type $(m-4, 1)$. In what follows p is assumed to be odd unless the contrary is stated.

was explained in a recent number of the Bulletin of the American Mathematical Society.*

The group I contains an invariant subgroup (I_2) of order p^2 which is composed of all the operators of I which are commutative with each operator of the characteristic subgroup (in H) of order p^{m-2}. These isomorphisms can be obtained by making H isomorphic with its subgroups of order p, the characteristic subgroup of order p^{m-2} corresponding to identity. Hence I_2 is of type (1, 1). By making H isomorphic with its characteristic subgroup of order p with respect to one of its cyclic subgroups of order p^{m-2} we observe that I contains operators of order p which are not contained in I_2. This proves that I contains a subgroup (I_3) of order p^3 which includes no operator of order p^2. We shall now prove that I_3 is the non-abelian group of order p^3 which contains no operator of order p^2.

Let P_1, P_2 represent two independent generators of H, the order of P_1 being p^{m-2}, and let $P_1^{p^{m-3}} = P_3$. The first four of the following equations result directly from the given holomorphisms of H while the last two show that two operators (t_1, t_2) of I_3 are not commutative and hence that I_3 is non-abelian.

$$t_1^{-1}P_2t_1 = P_2, \quad t_1^{-1}P_1t_1 = P_2P_1, \quad t_2^{-1}P_1t_2 = P_1, \quad t_2^{-1}P_2t_1 \doteq P_3P_2,$$
$$t_2^{-1}t_1^{-1}P_1t_1t_2 = P_3P_2P_1, \quad t_1^{-1}t_2^{-1}P_1t_2t_1 = P_2P_1.$$

It has been observed that I contains a cyclic group of order $p^{m-3}(p-1)$ all of whose operators are invariant. Each operator of this group transforms every subgroup of H into itself. The corresponding quotient group of I is of order $p^2(p-1)$. We proceed to prove that this group can be represented as an intransitive substitution group of degree $2p$, formed by establishing a (p, p) correspondence between two metacyclic groups of degree p. Each of the two sets of p non-characteristic subgroups of the same order in H is transformed by I according to the metacyclic group of degree p. As cyclic groups of order p^{m-2} are transformed into each other by operators of I which transform each one of the cyclic groups of order p into itself, the subgroup of H must be transformed according to the said intransitive substitution group. Since the order of this group is the order of the quotient group of I with respect to its operators which transform each subgroup of H into itself, this intransitive group is the quotient group in question. As this quotient group contains only one subgroup of order p^2, I contains only one subgroup (K) of order p^{m-1}.

The subgroup K contains no invariant operators besides those of the given cyclic subgroup of order p^{m-3} since it must include I_3. The group of cogredient isomorphisms of K is therefore the same as that of I_3. Hence it contains the abelian subgroup of type $(m-3, 1)$ which includes I_2. The remaining operators of K are the products of this abelian subgroup and the first $p-1$ powers

* Vol. 6, 1900, p. 337.

of any operator (t_4) of I_3 which is not included in I_2. Let t_5 be any operator of the group of type $(m-3, 1)$ first mentioned. From the properties of the operators of I_3 it follows that

$$(t_5 t_4)^p = t_5 t_4 t_5 t_4 \cdots p \text{ times} = t_5 t_4 t_5 t_4^{-1} t_4^2 t_5 t_4^{-2} \cdots t_4^{-1} t_5 t_4 = t_5^p u u^2 u^3 \cdots u^{p-1},$$

u being an invariant operator of I_3 which may be identity. Since $t_5 t_4$ is of the same order as t_5 whenever t_5 is not identity *I contains no operator of order p besides those of* I_3.

As it is necessary to consider all the operators of I_3 in the construction of the groups under consideration we proceed to determine its sets of operators which are conjugate under I. It has been observed that $p-1$ of its operators are invariant under I and that its subgroup I_2 is also invariant under I. It is clear that I_3 contains p other subgroups of order p^2. It will be proved that one of these is always invariant under I while the other $p-1$ are conjugate when $p > 3$. From the quotient group of I with respect to its invariant operators and from the properties of I_3 it follows that each of the $p^3 - p$ non-invariant operators of I belongs to a set of $p(p-1)$ conjugates under I. Hence all the non-invariant operators of the invariant subgroups of order p^2 are conjugate.

We proceed to consider the holomorphisms obtained by making H isomorphic with its non-cyclic subgroup of order p^2, the cyclic characteristic subgroup of order p^{m-3} corresponding to identity and the non-cyclic subgroup of order p^{m-2} corresponding to the group generated by P_3. From the following equations it may be observed that each one of these $p(p-1)^2$ holomorphisms gives rise to an operator of order p in I. In these equations t_6 is the operator of I which corresponds to the holomorphism under consideration.

$$t_6^{-1} P_1 t_6 = P_2 P_1, \quad t_6^{-2} P_1 t_6^2 = P_3 P_2^2 P_1, \quad t_6^{-3} P_1 t_6^3 = P_3^3 P_2^3 P_1,$$

$$t_6^{-n} P_1 t_6^n = P_3^{1+2+\cdots+n-1} P_2^n P_1, \; t_6^{-p} P_1 t_6^p = P_1, \; t_6^{-1} P_2 t_6 = P_3 P_2, \; t_6^{-p} P_2 t_6^p = P_3^p P_2.$$

We have now found all the holomorphisms of H which correspond to the operators of I_3. In the p^2 holomorphisms which correspond to I_2 each of the operators of the non-cyclic subgroup of order p^{m-2} corresponds to itself. In $p(p-1)$ others each operator of a cyclic group of order p^{m-2} corresponds to itself while in $p(p-1)^2$ just found the only operators which correspond to themselves are those of the cyclic characteristic subgroup of order p^{m-3}. The $p(p-1)$ operators, each of which is commutative with some operator of order p^{m-2}, together with the invariant operators of I_3 must correspond to an invariant subgroup in the quotient group of I with respect to its invariant operators. Hence *they form an invariant subgroup of order* p^2.

We shall now prove that the $p(p-1)$ subgroups, generated by the $p(p-1)^2$ operators of I_3 which transform only p^{m-3} operators of H into themselves, are conjugate under I whenever $p > 3$. Let t_7 be any operator of I which satisfies the two conditions

$$t_7^{-1} P_1 t_7 = P_1, \quad t_7^{-1} P_2 t_7 = P_3^{\alpha} P_2^{\beta}, \qquad \beta \not\equiv 0, \quad \bmod p.$$

The transform of t_6 with respect to t_7 satisfies the following condition:

$$(t_7^{-1} t_6 t_7)^{-1} P_1 t_7^{-1} t_6 t^7 = P_3^{\alpha} P_2^{\beta} P_1.$$

If we assign to t_7 all its values subject to the given conditions we obtain the $p(p-1)$ holomorphisms which correspond to the conjugates of t_6. It remains to prove that the one which corresponds to t_6^2 is not in this set. If it were we should have $P_3^{\alpha} P_2^{\beta} = P_3 P_2^2$ and hence $a = 1$, $\beta = 2$. This would lead to the following equations which are impossible when $p > 3$:

$$(t_7^{-1} t_6 t_7)^{-1} P_2^2 t_7^{-1} t_6 t_7 = P_3 P_2^2 = P_3^4 P_2^2 = t_6^{-2} P_2^2 t_6^2.$$

Since I does not transform t_6 into t_6^2, its quotient group with respect to its invariant operators can be obtained by establishing such a (p, p) correspondence between two metacyclic groups of degree p as leads to a group having only two invariant subgroups of order p.* This is clearly impossible when $p = 3$. In this special case each of the subgroups of order p^2 in I_3 is therefore invariant under I while, in all other cases, I_3 contains only two invariant subgroups of order p^2 and a set of $p - 1$ conjugate subgroups of this order.

The rest of this section will be devoted to the study of I when $p = 2$.

We may assume $m > 5$ since all the groups of order 32 are known. Now I is of order 2^{m-1} and it is isomorphic to the abelian group of type $(m - 5, 1)$ with respect to its operators which are commutative with each operator of the cyclic subgroup of order 2^{m-3} in H which is generated by P_1. Hence I_2 and I_3 may be found in the same manner as when p is odd, the latter being the group of order 8 containing five operators of order 2 and two of order 4 since it is non-abelian and contains I_2.

It has been observed that the 2^{m-3} invariant operators of I form a group of type $(m - 4, 1)$. Since this cannot contain more than 2 operators of I_3, I and I_3 must have the same group of cogredient isomorphisms. Hence I contains an abelian group of type $(m - 4, 1, 1)$ including I_2, and the rest of its operators transform just half of the operators of this group into themselves multiplied by the invariant operator (s_4) of I_3. Since s_4 is commutative with all the operators

* By establishing a (p, p) correspondence between two metacyclic groups of degree p, written in distinct sets of letters, we obtain as many intransitive groups of order $p^2(p-1)$ as there are operators in the group of isomorphisms of the cyclic group of order $p-1$ diminished by one-half of the number of its operators whose orders exceed two. Cf. American Journal of Mathematics, vol. 21, 1899, pp. 292 and 293. In one of these groups each subgroup of order p is invariant while each of the others contains only two invariant subgroups of this order. The latter exists only whenever $p > 3$.

of H whose order is less than 2^{m-2} it is the square of the operators of order four in the given group of type $(m-4, 1, 1)$. We shall soon observe that all the operators of order 4 in I have the same square. Let s_5 be any operator of the mentioned group of type $(m-4, 1, 1)$ and suppose that s_5 is not invariant in I. Also let s_6 represent an operator of order 2 in I_3 that is not also contained in I_2. From the equation $(s_5 s_6)^2 = s_5 s_6 s_5 s_6 = s_4 s_5^2$ it follows that we obtain additional operators of order two when s_5 is of order four and only then. When s_5 is of order two we obtain the additional operators of order four. Since s_4 is the square of all of these, *all the operators of order four in I have the same square.*

From the results of the preceding paragraph it follows that all the operators of order two in I are contained in a subgroup (I_5) of order 32 which includes all the operators of order four in I. Also I_5 contains 16 operators of order four and 15 of order two. As the latter are not all commutative to each other they must generate I_5. Moreover it is clear that I_5 is the direct product of its operator of order two which transforms each operator of H into its inverse and the group (I_4) of order 16 which contains just 7 operators of order two that generate it.* Since I transforms the four non-characteristic subgroups of H according to the intransitive substitution group of degree and order four, each of its non-invariant operators has just two conjugates. Hence I_5 *contains three invariant operators of order two and six pairs of conjugates.*

§ 2. *Outline of the method employed to construct the groups in question.*

If a group of order p^m is represented as a regular substitution group, every subgroup of order p^{m-1} contains p systems of intransitivity. We may therefore suppose that each of the groups under consideration contains the intransitive substitution group formed by writing H in the regular form in p distinct sets of letters and establishing a simple isomorphism between these p regular groups $H_1, H_2, \cdots, H_p$. In what follows H will represent this intransitive group of type $(m-2, 1)$ while t will be used to represent the substitution of order p which merely permutes the corresponding letters of $H_1, H_2, \cdots, H_p$. Hence t is commutative with every substitution of H. Also, I will represent the intransitive substitution group which is formed by writing the maximal subgroup of degree $p^{m-1} - a$ $(a > 0)$ of the holomorph of H_1† in the p distinct sets of letters involved in H; so that t is also commutative with each substitution of I. Finally, i will be used to represent some substitution of order p that is contained in I, while I_2, I_3, I_4, and I_5 will represent the same subgroups of I as in the preceding section.

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*Quarterly Journal of Mathematics, vol. 28, 1896, p. 271, no. 9.

†Bulletin of the American Mathematical Society, vol. 5, 1899, p. 245.

Now H and ti clearly generate a group (G) of order p^m. All the other groups of this order which transform H in the same manner as G does, may be generated by H and sti, where s is commutative with each substitution of H; for, if such a group is generated by H and r, where r permutes the systems of H in the same way as t does, then $rt^{-1}i^{-1} = s$ must be commutative with each substitution of H, i. e., $r = sti$. Hence s is a substitution of the direct product of $H_1, H_2, \cdots, H_p$. We may assume that s involves no letters except those of H_1; for if it involved the letters of $H_\alpha (p \geqq \alpha > 1)$ but not those of $H_{\alpha+\beta}$ $(\beta > 0)$, the transform of the group generated by H and sti with respect to the component of s which involved the letters of H_α only would be a conjugate group in which s would be replaced by a substitution involving the letters of $H_{\alpha-1}$ but not those of $H_{\alpha-1+\beta}$. Since the other conditions named would not be affected it is proved that *we may assume that s involves the letters of H_1 only.* On this account it will hereafter be denoted by s_1 and its conjugates under the group generated by t will be denoted by $s_1, s_2, \cdots, s_p$.

By hypothesis $(s_1ti)^p$ is some substitution of H, but

$$(s_1ti)^p = s_1ti \cdot s_1ti \cdots = s_1is_pis_{p-1}i \cdots s_2i = s_1s'_ps'_{p-1} \cdots s'_2,$$

where s'_α is the transform of s_α with respect to some power of i. From this it follows that s_1 *and i are always commutative.* Let r_1 represent a substitution of H that is transformed by i^{-1} into itself multiplied by a substitution u_1 of order p which is commutative to i. The conjugates of r_1 and u_1 with respect to the powers of t will be represented by $r_1, r_2, \cdots, r_p$; $u_1, u_2, \cdots, u_p$ respectively. It is easy to verify that

$$(r_1^{p-1}r_3r_4^2\cdots r_p^{p-2})^{-1}tir_1^{p-1}r_3r_4^2\cdots r_p^{p-2} = r_1^{1-p}r_3^{-1}r_4^{-2}\cdots r_p^{2-p}r_2r_3^2\cdots r_p^{p-1}u_2u_3^2\cdots u_p^{p-1}ti$$
$$= r_1^{1-p}r_2r_3\cdots r_pu_2u_3^2\cdots u_p^{p-1}ti.$$

If we transform the last substitution successively by $u_p^{p-1}, u_{p-1}^{2p-3}, \cdots, u_3^{\frac{(p+1)(p-2)}{2}}$ and multiply the result by $r_1^{-1}, r_2^{-1}, \cdots, r_p^{-1}$, which is in H, we observe that the group generated by H and ti is conjugate to the one which is generated by H and $r_1^{-p}ti$; i. e., *no new group is obtained when s_1 is the pth power of a substitution of H_1 which is transformed by i into itself multiplied by a substitution of order p that is commutative with i.*

By interchanging systems of intransitivity of H, t can be transformed into any power while each of the substitutions i and s_1 is transformed into itself. Hence the $p-1$ powers of i which are prime to the order of i give rise to the same groups. It is also clear that each of the conjugates of i under I leads to the same groups. Moreover, if each of two groups generated by ti_α and ti_β respectively, contains only one subgroup that is simply isomorphic with H and if i_α is not conjugate with some power of i_β under I, then must the two groups be distinct; for if they were simply isomorphic they could be transformed by operators of I so that the identical operators of H would correspond. The remaining operators of the group would have to transform H in the same manner; i. e., the group generated by i_α would be conjugate to the one generated by i_β. The main results of this section may be summarized as follows:

1. Only one operator from each set of subgroups of order p which are conjugate under I is to be used for i.

2. s_1 is a substitution of H_1 which is commutative with i.

3. s_1 is not the pth power of a substitution of H_1 which is either commutative with i or is transformed by i into itself multiplied by a substitution of order p that is commutative with i.

4. All the powers of s_1 which are prime to its order lead to the same group since it can be transformed into these powers by operators that are commutative with t and i.

§ 3. *Determination of the groups when p is odd.*

It was observed in § 1 that i is contained in the non-abelian group I_3 of order p^3 which contains no operator of order p^2. We shall begin with the case when i is one (i_1) of the $p-1$ operators of the invariant subgroup of order p. Hence i_1 is commutative with every operator of the non-cyclic subgroup of order p^{m-2} in H. From the equations

$$(P_1ti_1)^p = P_1ti_1 \cdot P_1ti_1 \cdots p \text{ times} = P_1i_1 \cdot P_1i_1 \cdots t^p$$
$$= P_1i_1 \cdot P_1i_1^{-1}i_1^2P_1i_1^{-2}i_1^3 \cdots i_1^{p-1}P_1i_1^{1-p}i_1^p = P_3P_3^2 \cdots P_3^{p-1}P_1^p = P_1^p,$$

it follows that the product of ti_1 and any operator of H besides identity is of the same order as the operator of H. The group (G_1) generated by H and ti_1 has therefore the same number of operators of each order as the abelian group of type $(m-2, 1, 1)$. Two groups which have the same number of operators of each order may be called conformal.* Now G_1 contains just $p+1$ abelian subgroups of order p^{m-1}, p of these are of type $(m-2, 1)$ while the remaining one is of type $(m-3, 1, 1)$. As each of the former is transformed in the same manner by all the operators of G_1, we shall not arrive at G_1 when we use in place of i_1 an operator from another set of conjugates in I_3.

From § 2 it follows that s_1ti_1 and H generate G_1 whenever s_1 is an operator of the characteristic cyclic subgroup of order p^{m-3} in H_1. If s_1 is the constituent of P_2 which involves the letters of H_1 only we clearly obtain a group (G_2) which contains no operator of order p besides those of H. Then G_2 contains the same number of operators of each order $> p^2$ as G_1 does. It also includes the same

*Bulletin of the American Mathematical Society, vol. 2, 1896, p. 140.

number of abelian subgroups of type $(m-2, 1)$ as G_1 does, but the group of type $(m-3, 1, 1)$ in G_1 is replaced by one of type $(m-3, 2)$ in G_2. These are the only abelian subgroups of order p^{m-1} that are contained in G_2 and all those of type $(m-2, 1)$ are transformed in the same way by all the operators of G_2. Since we obtain G_2 if we replace s_1 by any one of its $p-1$ powers multiplied into any operator of the cyclic characteristic subgroup of order p^{m-3} in H_1, i_1 gives rise to only the two groups G_1 and G_2.

We shall now consider the possible groups when i is one (i_2) of the $p(p-1)$ conjugate operators of I_2. i_2 and i_1 are commutative with the same operators of H. Just as in the preceding case we find that i_2 leads to two groups (G_3 and G_4) which are conformal with G_1 and G_2 respectively. In G_1 and G_2 all the cyclic subgroups of order p^{m-2} are invariant but in G_3 and G_4 these subgroups are conjugate in sets of p. Each of the groups G_3 and G_4 contains $p+1$ abelian subgroups of order p^{m-1}, p of these are of type $(m-2, 1)$ and are transformed by the remaining operators in the same way as H is transformed. Hence we cannot obtain any of the four groups just found by using for i an operator of I_3 which is not contained in I_2. By using for i one (i_3) of the $p(p-1)$ conjugate operators of the other invariant subgroup of I_3, when $p > 3$, we clearly obtain only one additional group (G_5) that involves no operator whose order exceeds p^{m-2}. The group G_5 contains $p+1$ abelian subgroups of type $(m-2, 1)$ and it transforms each of its cyclic subgroups of order p^{m-2} into itself. It is the only group involving H that has invariant operators of order p^{m-2} without containing any operator of order p^{m-1}. It is clear that i_3 leads also to the non-abelian group (G_6) of order p^m * which involves an operator of order p^{m-1} and to no other group.

Each of the remaining $p(p-1)^2$ operators of I_3 transforms only p^{m-3} operators of H into themselves and the $p(p-1)$ subgroups which they generate are conjugate when $p > 3$. Hence it is only necessary to consider one (i_4) of them when $p > 3$. The group (G_7) generated by ti_4 and H contains only one abelian subgroup of order p^{m-1}. From the equation

$$(P_1 t i_4)^p = (P i_4)^p = P_1 i_4 P_1 i_4^{-1} i_4^2 P_1 i_4^{-2} i_4^3 \cdots i_4^{p-1} P_1 i_4^{1-p} = P_1^p \qquad (\text{when } p > 3)$$

$$\text{or} = P_3 P_1^p \quad (\text{when } p = 3),$$

it follows that G_7 is conformal with G_1, G_3 and G_5. The following equations show that i_4 gives rise to only one group. In these equations r_1 represents the constituent of P_1 that is contained in H_1, while v_1 and w_1 represent the corresponding constituents of P_2 and P_3 respectively. The higher subscripts indicate the conjugates of these substitutions with respect to powers of t.

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* BURNSIDE, *Theory of groups of finite order*, 1897, p. 76.

$$(r_1^{p-1}r_3r_4^2\cdots r_p^{p-2})^{-1}ti_4r_1^{p-1}r_3r_4^2\cdots r_p^{p-2}=r_1^{1-p}r_3^{-1}r_4^{-2}\cdots r_p^{2-p}i_4r_2r_3^2\cdots r_p^{p-1}t$$
$$=r_1^{1-p}r_2r_3\cdots r_pv_2v_3^2\cdots v_p^{p-1}ti=r_1^{1-p}r_2r_3\cdots r_pw_1^{-1-3-6\cdots-\frac{(p-1)(p-2)}{2}}ti_4$$

which, when multiplied by $r^{-1}r_2^{-1}r_3^{-1}\cdots r_p^{-1}$ of H, is equal to $r_1^{-p}ti$ if $p>3$, and to $w_1^{-1}r_1^{-p}ti$ if $p=3$.

When $p=3$, I_3 contains another invariant subgroup of order p^2. If i_5 represents one of its conjugate operators we obtain an additional group G_8 by multiplying H and ti_5. It may be proved in exactly the same manner as in the preceding case that i_5 leads to only one group, and it is, moreover, clear that G_8 and G_7 are conformal. Since each contains only one abelian subgroup of order p^{m-1} and since i_4 and i_5 are not conjugate under I, G_8 and G_7 cannot be conjugate.

§ 4. *Determination of the groups when p is even.*

In § 1 it was observed that all the operators of order two in I are contained in a group of order 32 which has three invariant operators (i_1, i_2, i_3) of order 2; viz., those which transform each operator of H into its $2^{m-3}+1$, $2^{m-3}-1$, $2^{m-2}-1$ powers respectively. The group (G_1) generated by H and ti_1 is conformal with the abelian group of type $(m-2, 1, 1)$ and each of its subgroups, with exception of the four of order two which are not in H, is invariant. The group G_1 contains three abelian subgroups of order p^{m-1}, two being of type $(m-2, 1)$ and the third of type $(m-3, 1, 1)$, and it transforms all the operators of the former into their $2^{m-3}+1$ power. In exactly the same manner as in the preceding section we obtain G_2 by multiplying H and s_1ti_1, s_1 being the constituent of P_2 which is in H_1. The group G_2 contains only three operators of order 2 and each of these is the square of four operators of order 4. The operators of its two abelian subgroups of type $(m-2, 1)$ are transformed in the same way as those of the corresponding subgroups of G_1.

The group (G_3) generated by H and ti_2 contains 2^{m-2} operators of each of the orders two and four in addition to the operators of H. All the operators of order four in G_3 have the same square. Let s_1 be the constituent of P_1 which involves the same letters as H_1. From the equation $s_1^{-1}ti_2s_1=s_1^{-1}s_2^{2^{m-3}-1}ti_2$ it follows that transforms of G_3 include H and $s_1^{2^{m-3}}ti_2$. Hence there is only one additional group (G_4) which transforms H in the same way as i_2 does, viz., the group generated by H and s_1ti_2, where s_1 represents the constituent of P_2 which is in H_1. All the operators of G_4 which are not found in H are of order 4 and the squares of these operators of order 4 are the two non-characteristic operators of order two in H, each of the latter being the square of one-half of the former.

The group H and ti_3 generate a group (G_5) composed of H and 2^{m-1} operators of order 2 which are conjugate in sets of 2^{m-3}. In fact, it is clear that in all the groups to which i_2 and i_3 lead, all the operators not contained in H are conjugate in sets of 2^{m-3}. The group (G_6) generated by H and s_1ti_3, where s_1 is the component of P_3 which is found in H_1, contains $2^{m-1}+4$ operators of order four. All of these have the same square. If s_1 is the component of P_2 which belongs to H_1, the group (G_7) is conformal with G_6 but its operators of order four which are not found in H have P_2 for their square. As the other operator of H_1 which is commutative with i_3 is conjugate under I with the last s_1, the seven groups just found are all those which depend upon i_1, i_2, i_4. In each of these seven groups every cyclic subgroup whose order exceeds four is invariant. It remains to find the groups which depend upon operators that are not invariant under I.

Let i_4 be one of the two non-invariant operators of I_3 which are commutative with each operator of the non-cyclic subgroup of order 2^{m-2} in H. Since the product of ti_4 into any operator of H is of the same order as this operator of H, the group (G_8) generated by H and ti_4 is conformal with G_1. G_8 contains three abelian subgroups of types $(2^{m-2}, 1)$ and $(2^{m-3}, 1, 1)$, there being two of the former type. The four cyclic subgroups of order 2^{m-2} form two pairs of conjugates. If s_1 represents the component of P_1 which is contained in H_1 we have $s_1^{-1}ti_4s_1 = s_1^{-1}s_2s_2'ti_4$, s_2' being the constituent of P_2 which is in H_2. Hence a transform of G_8 contains H and $s_1's_1^{-2}ti_4$. We therefore obtain only one additional group (G_9) by multiplying H and s_1ti_4, where s_1 is a substitution of H_1. It may be convenient to assume that G_9 is generated by H and s_1ti_4, where s_1 is the constituent of P_2 which is in H_1. From this we observe that G_9 is conformal with G_2. Like G_8 it contains two pairs of conjugate cyclic subgroups of order 2^{m-2} and three abelian subgroups of order 2^{m-1}. One of these is, however, of type $(2^{m-3}, 2)$ instead of type $(2^{m-3}, 1, 1)$ in G_8.

Each of the other two conjugate operators of order two in I_3 is commutative with an operator of order 2^{m-2} in H. If i_5 represents one of these operators it is clear that H and ti_5 generate a group (G_{10}) which is conformal with G_1. Its subgroup of order 16 which includes all the operators of orders two and four is generated by its operators of order two. Its four cyclic subgroups of order 2^{m-2} are invariant, one being generated by an invariant operator while the operators of the others are transformed into their $2^{m-3}+1$ power. The other group (G_{11}) which involves invariant operators of order 2^{m-2} is generated by H and s_1ti_5, where i_5 is the constituent of P_1 which is in H_1. Hence G_{11} contains operators of order 2^{m-1} which it transforms into their $2^{m-2}+1$ power. It includes just two cyclic subgroups of order 2^{m-1}.

The group (G_{12}) generated by H and ti_6, where i_6 is one of the two conjugate operators of order two in I which are commutative with each operator of the cyclic subgroup of order 2^{m-3} generated by $P_1^2P_2$ without being commutative with any other operator of H, is conformal with G_1. Like G_3, G_4, G_5, G_6, and G_7 this group and all those which follow contain only one abelian subgroup of order 2^{m-1} while each of the other six groups (G_1, G_2, G_8, G_9, G_{10}, G_{11}) contains just three abelian groups of this order. From the equation $s_1^{-1}ti_6s_1 = s_2's_1^{-1}s_2ti_6$, s_1 being an operator of order 2^{m-2} in H_1, it follows that i_6 leads to one other group of order 2^m. The four cyclic subgroups of order 2^{m-2} are conjugate in pairs in G_{12}.

We have now considered all the operators of order two which are contained in I_4 as well as the two additional operators i_3 and $i_1i_3 = i_2$. It remains to consider the three operators i_3i_4, i_3i_5, and i_3i_6. We shall represent these respectively by i_7, i_8, and i_9. The group H and ti_7 generate a group (G_{13}) which contains 2^{m-2} operators of each of the orders two and four in addition to H. H and s_1ti_7, where s_1 is the constituent of P_3 which is in H_1, generate a group G_{14} which contains only operators of order four in addition to H. Neither of these two groups contains an invariant cyclic subgroup of order 2^{m-2}. From the equations $s_1^{-1}ti_7s_1 = s_1^{-1}s_2^{-1}s_2'ti_7$, s_2' being the constituent of P_2 in H_2, it follows that i_7 gives rise to no additional group.

The group (G_{15}) generated by H and ti_8 is conformal with G_{13} but all its operators of order four have the same square while in G_{13} they have two distinct squares. Moreover, its invariant operators form a cyclic group of order four while they form a non-cyclic group of this order in G_{13}. The group H and s_1ti_8, where s_1 is the H_1 component of the invariant operator of order four in G_{15}, generate a group (G_{16}) which contains only operators of order eight in addition to those of H. The two cyclic subgroups of order 2^{m-2} are clearly invariant in each one of these two groups. That these two are the only groups to which i_8 gives rise follows from the equation $s_1^{-1}ti_8s_1 = s_1^{-1}s_2^{-1+2^{m-3}}$, s_1s_2 being an operator of order 2^{m-2} in H which is not commutative with i_5.

H and ti_9 generate a group (G_{17}) which contains, besides H, 2^{m-2} operators of order eight and 2^{m-3} operators of each of the orders two and four. Like the preceding four groups it contains just four invariant operators. That this is the only group to which i_9 leads follows from the equations $s_1^{-1}ti_9s_1 = s_1^{-1}s_2^{-1}s_2'$, where s_2' is of order four if s_1 is of order 2^{m-2}, and of order one when s_1 is of order 2^{m-3} and not commutative with i_6. Hence the number of the groups of order 2^m which contain the abelian group of type $(m-2, 1)$ is twenty, three being abelian and seventeen being non-abelian. Eleven of the latter involve invariant cyclic subgroups of order 2^{m-2}.

Summary.

There are three abelian groups of order p^m that contain a subgroup of type $(m-2, 1)$, viz., those of types $(m-1, 1)$, $(m-2, 2)$, $(m-2, 1, 1)$. These exist for every prime p.

When $p > 3$ there are seven other groups. The number of their operators of each order is exhibited by the following table:

Groups	Operators				
	Order	p	p^2	p^α	p^{m-1}
G_1, G_3, G_5, G_7	Number	p^3-1	$p^3(p-1)$	$p^{\alpha+1}(p-1)$	0
G_2, G_4	"	0	p^4-1	$p^{\alpha+1}(p-1)$	0
G_6	"	p^2-1	$p^2(p-1)$	$p^\alpha(p-1)$	$p^{m-1}(p-1)$

$(p>3)$ $(2<\alpha<m-1)$.

The first four groups may be distinguished by the following properties: G_7 contains only one abelian subgroup of order p^{m-1} while each of the others contains three such subgroups; G_5 is the only one which contains invariant operators of order p^{m-2}; G_1 contains cyclic invariant subgroups of order p^{m-2} while G_3 does not; G_2 and G_4 can be distinguished by the same property as was employed to distinguish G_1 and G_3. When $p=3$ there is one more group, which is conformal with G_6. It is not simply isomorphic with G_6 since these two groups contain only one abelian subgroup of order p^{m-1} which they transform according to non-conjugate subgroups of I.

The following table exhibits the orders of the operators of the 17 non-abelian groups of order 2^m which include an abelian group of type $(m-2, 1)$:

Groups	Operators					
	Order	2	4	8	2^α	2^{m-1}
G_1, G_8, G_{10}, G_{12}	Number	7	8	16	$2^{\alpha+1}$	0
G_2, G_9	"	3	12	16	$2^{\alpha+1}$	0
G_3, G_{13}, G_{15}	"	$2^{m-2}+3$	$2^{m-2}+4$	8	2^α	0
G_4, G_6, G_7, G_{14}	"	3	$2^{m-1}+4$	8	2^α	0
G_5	"	$2^{m-1}+3$	4	8	2^α	0
G_{11}	"	3	4	8	2^α	2^{m-1}
G_{17}	"	$2^{m-3}+3$	$2^{m-3}+4$	$2^{m-2}+8$	2^α	0
G_{16}	"	3	4	$2^{m-1}+8$	2^α	0

$(3<\alpha<m-1)$.

The conformal groups may be distinguished by the following properties: G_{10} is the only one of the first four which contains invariant operators of order 2^{m-2}; G_{12} contains only one abelian group of order 2^{m-1} while G_1 and G_8 contain three such groups; in G_1 all the cyclic subgroups of order 2^{m-2} are invariant while they are conjugate in sets of two in G_8; G_2 and G_9 may be distinguished in the same way as G_1 and G_8; G_{13} contains two operators of order two which are squares of operators of order four while in G_3 and G_{15} all the operators of order four have the same square; in G_3 the invariant operators form a non-cyclic subgroup of order four while they form a cyclic subgroup of this order in G_{15}; in G_{14} the two cyclic subgroups of order 2^{m-2} are conjugate while they are invariant in G_4, G_6, and G_7; in G_4 the operators of order 4 have three distinct squares, in G_6 they have the same square, and in G_7 they have two distinct squares.

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