ON THE SYSTEM OF A BINARY CUBIC AND QUADRATIC AND THE REDUCTION OF HYPERELLIPTIC INTEGRALS OF GENUS TWO TO ELLIPTIC INTEGRALS BY A TRANSFORMATION OF THE FOURTH ORDER*

BY

JOHN HECTOR McDONALD

Introduction.

1. If a hyperelliptic integral of genus 2 is reducible to an elliptic integral by a transformation of order k then there always exists a second integral † with the same irrationality which is reducible by a transformation of the same order. In the algebraic treatment of the problem the difficulty lies in the determination of the second integral after the first has been constructed.

For $k = 4$ the problem has been treated by Professor Bolza ‡ and in the sixth section of his dissertation the formulæ for the two reducible integrals are established algebraically. Professor Bolza used a particular system of variables x_1, x_2. These are chosen so that if

$$y_1 = a_0x_1^4 + 4a_1x_1^3x_2 + 6a_2x_1^2x_2^2 + 4a_3x_1x_2^3 + a_4x_2^4$$

and

$$y_2 = b_0x_1^4 + 4b_1x_1^3x_2 + 6b_2x_1^2x_2^2 + 4b_3x_1x_2^3 + b_4x_2^4$$

are the two forms of order 4 which are equal to the homogeneous variables in the elliptic integral, then $a_1 = 0$ and $b_1 = 0$, and the numerators of the two reducible integrals turn out to be x_1 and x_2. The derivation of the second integral depends on the use of this particular system of variables and the connection of these variables with the irrationality and with the reducing substitutions is unexplained.

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† This is a consequence of the Weierstrass-Picard Theorem: Acta Mathematica, vol. 4 (1884), p. 400; Bulletin de la Société Mathématique de France, vol. 11 (1882–1883), p. 25.

‡ *Inaugural Dissertation*, Göttingen, 1886; Mathematische Annalen, vol. 28 (1887), p. 447.

The object of this paper is to supply the following desiderata: (*a*) a solution of the problem independently of a special system of variables, by using theorems on biquadratic involutions, (*b*) a methodical deduction of the second integral from the first, (*c*) the introduction of the system of variables and the normal form which are appropriate to the problem.

The integral

$$\int \frac{(ax+b)\,dx}{\sqrt{R_6(x)}}$$

is a hyperelliptic integral of the first kind of genus 2 if $R_6(x)$ is a polynomial of the 6th degree, and it is reducible to an elliptic integral by a transformation of order 4 if

$$\int \frac{(ax+b)\,dx}{\sqrt{R_6(x)}} = M \int \frac{dy}{\sqrt{R_4(y)}},$$

where $R_4(y)$ is a polynomial of order 3 or 4 and $y = U(x)/V(x)$, U and V being polynomials, the higher of whose orders is 4 and M a multiplier not depending on x or y.

In homogeneous variables $x = x_1/x_2$, $y = y_1/y_2$ and, with the usual notations, the above relation becomes

$$\int \frac{(xd)(x\,dx)}{\sqrt{R_6(x_1x_2)}} = M \int \frac{(y\,dy)}{\sqrt{R_4(y_1y_2)}},$$

where

$$y_1 = \rho\, U(x_1x_2),$$

$$y_2 = \rho\, V(x_1x_2),$$

$$(xd) = x_1d_2 - x_2d_1.$$

Denoting more explicitly $R_6(x_1x_2)$ by $(xa)(x\beta)(x\gamma)(x\delta)(x\kappa)(x\lambda)$ and $R_4(y_1y_2)$ by $(y\mu)(y\nu)(y\pi)(y\omega)$ we see by reasoning similar to that used by JACOBI * for the transformation of elliptic integrals that it is possible to break up R_6 into quadratic factors $(xa)(x\beta)$, $(x\gamma)(x\delta)$, $(x\kappa)(x\lambda)$ and to determine linear forms (xd'), (xd''), (xd''') and a quadratic form $(x\xi)(x\eta)$ so that

$$\mu_2 U - \mu_1 V = (xa)(x\beta)(xd')^2,$$

$$\nu_2 U - \nu_1 V = (x\gamma)(x\delta)(xd'')^2,$$

$$\pi_2 U - \pi_1 V = (x\kappa)(x\lambda)(xd''')^2,$$

$$\omega_2 U - \omega_1 V = (x\xi)^2(x\eta)^2;$$

and if we denote by ϑ the Jacobian of U and V then

$$\vartheta = (xd)(xd')(xd'')(xd''')(x\xi)(x\eta).$$

* JACOBI, *Fundamenta Nova*, *Werke*, vol. 1, p. 57.

Conversely, if these relations are satisfied, the integral

$$\int \frac{(xd)(x\,dx)}{\sqrt{R_6(x_1x_2)}}$$

is reducible by the substitution $y_1 : y_2 = U : V$; that is, these conditions are sufficient as well as necessary.

Accordingly with every reducible hyperelliptic integral is associated a biquadratic involution $I_4 = \lambda U + \nu V$ which is special since it contains a complete square, viz., $(x\xi)^2(x\eta)^2$. Every such involution contains, aside from the square $(x\xi)^2(x\eta)^2$, four forms having double roots. The quadratic factor which goes along with the double factor to make up any one of these four forms is called a branch quadratic and the forms themselves branch forms. It follows from the above equations that the product of three branch quadratics of an involution containing a square gives a sextic belonging to a reducible integral and that the numerator is furnished by the double element in the fourth branch form. To every reducible integral belongs an involution I_4 in which one of the four branch forms is distinguished from the other three, each of which plays the same rôle.

In section I it is shown that there exists one and only one I_4 which contains the square of an arbitrary quadratic f, and has the factors of an arbitrary cubic ϕ for three of its double elements. An I_4 determined in such a way has evidently one branch form distinguished from the rest and the three which contain the factors of ϕ are not distinguished from each other.

In sections II, III it is shown that if ϕ, f be a system (Σ) of a cubic and a quadratic there exists a covariant system $(\bar\Sigma)$ of a cubic and a quadratic $\bar\phi, \bar f$, such that ϕ, f are the same covariants of $\bar\phi, \bar f$ as $\bar\phi, \bar f$ are of ϕ, f, or in other words, the relation between (Σ) and $(\bar\Sigma)$ is mutual. The forms $\bar\phi$ and $\bar f$ are obtained as follows. Using transvectant notation put $\phi = l_x m_x n_x$, the product of linear factors, and let p and q be two linear forms and also let

$$D_{pq}(\phi) = 4^3(p^3q, l)_1(p^3q, m)_1(p^3q, n)_1 p^{-6}.$$

Then $D_{pq}(\phi)$ is an integral covariant of ϕ, p, q. If we let p and q be the two linear covariants of ϕ and f so denoted in the notation of CLEBSCH (*Binäre Formen* *) $D_{pq}(\phi)$ will be a covariant of ϕ and f. Also let $Q(\phi)$ be the cubic covariant of ϕ, $D_{pq}\big(Q(\phi)\big) = \phi' = c_x^3$ and $f' = d_x^2 =$ the Jacobian of pq and $(cq)c_x^2$. Further, let

$$\sigma = \frac{(cd)^2 c_x}{q}, \quad \rho = \frac{(cd)^2(cd')d'_x}{p};$$

σ and ρ are invariants of ϕ, f. In these notations the following theorem holds.

* If $\phi = a_x^3$, $f = b_x^2$, $p = (ab)^2 a_x$, $q = (ab)^2(ab')b'_x$.

THEOREM: *The forms*

$$\frac{\rho}{\sigma^2}\phi' = \bar{\phi}, \quad \frac{\sigma}{\rho} f' = \bar{f}$$

are covariants of ϕ, f *and if* $\bar{\bar{\phi}}, \bar{\bar{f}}$ *are the same covariants of* $\bar{\phi}, \bar{f}$ *as* $\bar{\phi}, \bar{f}$ *were of* ϕ, f *then*

$$\bar{\bar{\phi}} = \phi, \quad \bar{\bar{f}} = f.$$

Also, if the covariants $\bar{p}$, $\bar{q}$ *are formed from* $\bar{\phi}$ *and* $\bar{f}$;

$$\bar{p} = q, \quad \bar{q} = p.$$

It is on the existence of these mutual relations between the systems (Σ) and $(\bar{\Sigma})$ that the second reducible integral depends.

In section IV the involutions I_4 and $\bar{I}_4$ belonging to (Σ) and $(\bar{\Sigma})$ are determined and it is shown that the Jacobian of $I_4 = \phi f q$ and of $\bar{I}_4 = \bar{\phi}\bar{f}\bar{q} = \bar{\phi}\bar{f}p$; moreover the three branch quadratics belonging to the three branch forms whose double elements are the factors of ϕ are identical with those in $\bar{I}_4$ whose double elements are the factors of $\bar{\phi}$, except as to sign. If these branch quadratics are denoted by R, S, T then

$$\int \frac{q\,(x\,dx)}{\sqrt{RST}}, \quad \int \frac{\bar{p}\,(x\,dx)}{\sqrt{\bar{R}\bar{S}\bar{T}}}$$

are both reducible integrals with the same irrationality, the first by I_4, the second by $\bar{I}_4$. The statement of this theorem constitutes section V.

In section VI, the variables $x_1' = q$, $x_2' = p$ are introduced as leading to an appropriate canonical form, which is identical with that used by Professor BOLZA in his solution of the problem.

In section VII a number of miscellaneous results are collected relating to biquadratic involutions containing a complete square and their geometrical representations on a rational quartic and a twisted cubic.

I.

THEOREMS ON THE BIQUADRATIC INVOLUTION HAVING A COMPLETE SQUARE.

2. *A biquadratic involution* I_4 *containing a square is determined without ambiguity by the quadratic form whose square it contains and by any three of its other double elements, or in other words, by a cubic and a quadratic. These forms are independent and their introduction singles out one of the branch forms of the involution from the others.*

Before proving these statements another result must be proved.

LEMMA.—*Let* U *and* f^2 *be two biquadratic forms. Let the Jacobian of* U

and f^2, which must contain f, be $f\theta$. The form θ is apolar to the forms of the involution $\lambda U + \mu f^2$.*

Proof.—Let U and f be taken in a normal form found by introducing as variables the linear factors of f. Then

$$f = x_1 x_2$$

and

$$U = a_0 x_1^4 + 4a_1 x_1^3 x_2 + 6a_2 x_1^2 x_2^2 + 4a_3 x_1 x_2^3 + a_4 x_2^4.$$

The Jacobian of f^2, U is found on calculation to be:

$$\tfrac{1}{2}\, x_1 x_2 (a_0 x_1^4 + 2a_1 x_1^3 x_2 - 2a_3 x_1 x_2^3 - a_4 x_2^4),$$

and so

$$\theta = \tfrac{1}{2}(a_0 x_1^4 + 2a_1 x_1^3 x_2 - 2a_3 x_1 x_2^2 - a_4 x_2^4).$$

Then on calculation one finds the desired relations

$$(u, \theta)_4 = 0, \qquad (f^2, \theta)_4 = 0.$$

3. We now prove the following theorem:

THEOREM.—*Let ϕ denote any cubic and f any quadratic not apolar to ϕ. There exists one and but one I_4 containing the form f^2 and having three of its double elements given by $\phi = 0$.*

Proof.—We can always determine a linear form $(x\delta)$ so that $\phi(x\delta)$ is apolar to f^2, and in but one way. For $(\phi(x\delta)f^2)_4 \not\equiv 0$, since f^2 is not apolar to ϕ;† hence $\big(\phi(x\delta)f^2\big)_4 = 0$ is an equation of the first degree to determine δ_1/δ_2.

Let $f = x_1 x_2$, the factors of f being introduced as variables; then we may take

$$\phi(x\delta) = c_0 x_1^4 + 4c_1 x_1^3 x_2 + 6c_2 x_1^2 x_2^2 + 4c_3 x_1 x_2^3 + c_4 x_2^4;$$

but $c_2 = 0$, since $\phi(x\delta)$ is apolar to $f^2 = x_1^2 x_2^2$, hence

$$\phi(x\delta) = c_0 x_1^4 + 4c_1 x_1^3 x_2 + 4c_3 x_1 x_2^3 + c_4 x_2^4,$$
$$f^2 = x_1^2 x_2^2.$$

First. It appears at once that there is one I_4 satisfying the required conditions, viz.:

$$I_4 = \lambda x_1^2 x_2^2 + \mu(c_0 x_1^4 + 8c_1 x_1^3 x_2 - 8c_3 x_1 x_2^3 - c_4 x_2^4).$$

For calculating the form θ we find it is equal to

$$\tfrac{1}{2}(c_0 x_1^4 + 4c_1 x_1^3 x_2 + 4c_3 x_1 x_2^3 + c_4 x_2^4),$$

hence I_4 contains f^2 and ϕ gives three of its double elements.

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* Two forms $f = a_x^n$, $\phi = b_x^n$ are apolar if

$$(f, \phi)_n = (ab)^n = \sum_{i=0 \cdots n} a_i b_{n-i} (-1)^i {}_nC_i = 0.$$

† If ϕ and f are apolar, that is, if $(\phi f)_2 \equiv 0$, then $\phi(x\delta)$ is apolar to f^2 whatever $(x\delta)$ may be.

Second. There can be only one such I_4. For suppose another to exist:

$$I'_4 = \lambda x_1^2 x_2^2 + \mu(c'_0 x_1^4 + 8c'_1 x_1^3 x_2 - 8c'_3 x_1 x_2^3 - c'_4 x_2^4),$$

we should have

$$x_1 x_2 \theta = \sigma x_1 x_2 \theta',$$

since the Jacobians of I_4 and I'_4 must both be $\phi(x\delta)f$.

Hence

$$c'_0 x_1^4 + 4c'_1 x_1^3 x_2 + 4c'_3 x_1 x_2^3 + c'_4 x_2^4 = \sigma(c_0 x_1^4 + 4c_1 x_1^3 x_2 + 4c_3 x_1 x_2^3 + c_4 x_2^4),$$

or

$$c'_0 = \sigma c_0, \quad c'_1 = \sigma c_1, \quad c'_3 = \sigma c_3, \quad c'_4 = \sigma c_4;$$

or

$$c'_0 x_1^4 + 8c'_1 x_1^3 x_2 - 8c'_3 x_1 x_2^3 - c'_4 x_2^4 = \sigma(c_0 x_1^4 + 8c_1 x_1^3 x_2 - 8c_3 x_1 x_2^3 - c_4 x_2^4);$$

that is

$$c'_0 x_1^4 + 8c'_1 x_1^3 x_2 - 8c'_3 x_1 x_2^3 - c'_4 x_2^4$$

belongs to I_4; hence I'_4 is identical with I_4, which is contrary to hypothesis.

In virtue of this theorem we may consider a special I_4 containing a square as determined uniquely by a cubic and quadratic ϕ and f.

4. The following theorem determines $(x\delta)$ more exactly. In symbolic notation let $\phi = a_x^3$, $f = b_x^2$.

THEOREM: $(x\delta)$ *is proportional and may be taken equal to* $(ab)^2(ab')b'_x$.

To prove this we must show that

$$\left(a_x^3 \cdot (a'b)^2(a'b')b'_x, \quad b''^2_x b'''^2_x\right)_4 = 0.$$

Proof.—We have

$$(a_x^3, b''^2_x b'''^2_x)_3 = \tfrac{4}{3}(ab'')^2(ab''')b'''_x,$$

and therefore

$$\left((ab'')^2(ab''')b'''_x, \quad (a'b)(a'b')b'_x\right)_1$$

must be zero, or

$$(ab'')^2(ab''')(a'b)(a'b')(b'b''') = 0.$$

If we interchange b', b'''; a, a'; b, b'', which has no effect on the value of the expression, we find

$$\begin{aligned}(ab'')^2(ab''')(a'b)(a'b')(b'b''') &= (ab'')^2(ab''')(a'b)(a'b')(b'''b') \\ &= -(ab'')^2(ab''')(a'b)(a'b')(b'b'''),\end{aligned}$$

or

$$(ab'')^2(ab''')(a'b)(a'b')(b'b''') = 0.$$

II.

The System of a Cubic and two Linear Forms and their Conjugate System.

5. We consider next a system of a cubic form ϕ resolved into its linear factors, $\phi = l_x m_x n_x$, and two linear forms p_x and q_x (denoting them for brevity by l, m, n, p, q), returning later to the system of a cubic and quadratic. The discussion will be much facilitated by three theorems.

Let

$$D_{pq}(\phi) = 4^3 (p^3q, l)_1 (p^3q, m)_1 (p^3q, n)_1 p^{-6}.$$

Here $D_{pq}(\phi)$ is an integral covariant of ϕ, p, q since $(p^3q, l)_1$ is divisible by p^2.

Theorem I: *If*

$$D_{pq}(\phi) = 4^3 (p^3q, l)_1 (p^3q, m)_1 (p^3q, n)_1 p^{-6},$$

where l, m, n *are the linear factors of* ϕ, *then*

$$D_{qp}\big(D_{pq}(\phi)\big) = 27\,(pq)^6\phi.$$

Proof.—We have

$$4\,(p^3q, l)_1 = p^2\big(p\,(ql) + 3q\,(pl)\big),$$

and find that

$$D_{pq}(\phi) = \prod_{l,m,n}\big(p\,(ql) + 3q\,(pl)\big) = l'_x m'_x n'_x = l'm'n', \quad 4\,(q^3p, l')_1 = 3\,(pq)^2 q^2 l$$

by a short computation, making use of the identity

$$l_x(pq) + p_x(ql) + q_x(lp) = 0.$$

Hence

$$D_{qp}(l'm'n') = D_{qp}\big(D_{pq}(\phi)\big) = 27\,(pq)^6\phi.$$

Theorem II: *Let* $Q(\phi)$ *denote the cubic covariant of* ϕ; *then*

$$Q\big(D_{pq}(\phi)\big) = -\,27\,(pq)^6 D_{pq}\big(Q(\phi)\big).$$

Let

$$L = (ml)\,n_x + (nl)\,m_x = 2\,(l, nm)_1,$$

$$M = (nm)\,l_x + (lm)\,n_x = 2\,(m, ln)_1,$$

$$N = (ln)\,m_x + (mn)\,l_x = 2\,(n, lm)_1,$$

so that Ll is harmonic to mn. Then $LMN = \sigma Q$, since the cubic covariant of a cubic may be found by taking the conjugate of each factor with respect to the other two and forming the product. The factor σ is a numerical constant since LMN and Q are both of the 3d degree in the coefficients of ϕ; this appears at once from the expressions of L, M, N in l, m, n.

We have

$$D_{pq}\big(Q(\phi)\big) = D_{pq}(\sigma LMN) = \sigma \prod_{L,\,M,\,N}\big(p(qL) + 3q(pL)\big)$$

$$= \sigma \prod_{l,\,m,\,n} p\{(ml)(qn) + (nl)(qm)\} + 3q\{(ml)(pn) + (nl)(pm)\},$$

and

$$2(l',\, m'n') = \big(p(qm) + 3q(pm)\big)\big((pl')(qn) + 3(pn)(ql')\big)$$
$$+ \big(p(qn) + 3q(pn)\big)\big((pl')(qm) + 3(pm)(ql')\big);$$

but

$$(pl') = 3(pl)(pq), \quad (ql') = (qp)(qm).$$

Substituting in $(l',\, m'n')_1$, and making use of the relation

$$(pl)(qn) + (pq)(nl) + (pn)(lq) = 0,$$

derivable from $l_x(qn) + q_x(nl) + n_x(lq) = 0$ by putting $x_1 : x_2 = -p_2 : p_1$, we find

$$2(l',m'n')_1 = -3(pq)^2\big\{p\big((ml)(qn) + (nl)(qm)\big) + 3q\big((ml)(pn) + (nl)(pm)\big)\big\}.$$

Put

$$\phi' = l'm'n';$$

then

$$Q(\phi') = \sigma 2^3(l',\, m'n')_1(m',\, l'n')_1(n',\, m'l')_1$$
$$= \sigma \prod_{l,\,m,\,n} -3(pq)^2\big\{p\big((ml)(qn) + (nl)(qm)\big)$$
$$+ 3q\big((ml)(pn) + (nl)(pm)\big)\big\}$$
$$= -27(pq)^6 D_{pq}\big(Q(\phi)\big),$$

or

$$Q\big(D_{pq}(\phi)\big) = -27(pq)^6 D_{pq}\big(Q(\phi)\big).$$

6. THEOREM III: *Let R be the discriminant of ϕ; then*

$$D_{qp}\big(Q(\phi')\big) = -\frac{R^2}{4}\, 27^2(pq)^{12}\phi.$$

Proof.—By definition

$$D_{qp}\big(Q(\phi')\big) = D_{qp}\big(Q\{D_{pq}[Q(\phi)]\}\big);$$

but we may interchange the two inside operators by theorem II; therefore

$$D_{qp}\big(Q(\phi')\big) = -27(pq)^6 D_{qp}\big(D_{pq}\{Q[Q(\phi)]\}\big);$$

the factor $-27(pq)^6$ may be taken out because $D_{qp}(\phi)$ is linear in the coefficients of ϕ, viz., $D_{qp}(a\phi) = aD_{qp}(\phi)$. By theorem I,

$$D_{qp}\big(D_{pq}\{Q[Q(\phi)]\}\big) = 27(pq)^6 Q\big(Q(\phi)\big);$$

hence

$$D_{qp}\big(Q(\phi')\big) = -27^2(pq)^{12}Q\big(Q(\phi)\big),$$

and

$$Q\big(Q(\phi)\big) = -4^{-1}R^2\phi$$

(CLEBSCH, *Binäre Formen*); therefore

$$D_{qp}\big(Q(\phi')\big) = -4^{-1}R^2 27^2(pq)^{12}\phi.$$

III.

THE SYSTEM OF A CUBIC AND QUADRATIC AND THEIR CONJUGATE SYSTEM.

7. It is necessary now to recall two properties of the form system of a cubic and quadratic c_x^3 and d_x^2: 1°. The first polar of c_x^3 with respect to the linear covariant $(cd)^2c_x$ of c_x^3 and d_x^2 is a quadratic harmonic to d_x^2; 2°. The linear covariant $(cd)^2(cd')\,d'_x$ is the polar of $(cd)^2c_x$ with respect to d_x^2, or the product of these two linear covariants is harmonic to d_x^2.

Proof.—1°. Let (xy) be any linear form; then $c_x^2c_y$ is harmonic to d_x^2 if $(c_x^2c_y,\, d_x^2)_1 = 0$ or $(cd)^2c_y = 0$; hence $(xy) = (cd)^2c_x$.

2°. If we form $\big(d_x^2,\,(cd)^2c_x\big)_1$ we find $(cd)^2(cd')\,d'_x$. Hence the truth of the above statements.

8. Consider the form $\phi' = D_{pq}\big(Q(\phi)\big) = c_x^3$ and also $(c_x^3,\, q)_1 = (cq)\,c_x^2$. Then there is a quadratic form $f' = d_x^2$ which is the Jacobian of $(cq)\,c_x^2$ and pq. Then from the first property of the previous paragraph $(cd)^2c_x = \sigma q$ and $(cd)^2(cd')\,d'_x = \rho p$, where σ and ϕ are proportionality factors,* for q and p have just the properties of the two linear covariants; which properties define them except as to multipliers.

Further, it follows that ρ and σ are invariants of the system $\phi,\, p,\, q$. For ϕ' is a covariant of this system and so consequently is also $f' = d_x^2$, and therefore $(cd)^2c_x$, $(cd)^2(cd')\,d'_x$, and finally, $(cd)^2c_xq^{-1} = \sigma$, $(cd)^2(cd')\,d'_xp^{-1} = \rho$.

Take then the covariants $\rho\sigma^{-2}\phi' = C_x^3 = \bar{\phi}$, $\sigma\rho^{-1}f' = D_x^2 = \bar{f}$ of $\phi,\, p,\, q$; then $(CD)^2C_x = \rho\sigma^{-2}\cdot\sigma\rho^{-2}(cd)^2c_x = q$ and

$$(CD)^2(CD')\,D'_x = \rho\sigma^{-2}\cdot\sigma^2\rho^2\cdot(cd)\,(cd')\,d'_x = p.$$

Hence q and p are these two linear covariants of $\bar{\phi},\bar{f}$.

From the reasoning just used it is clear that if ϕ' is any cubic and q,p any two linear forms there is but one solution to the problem to find a quadratic $\bar{f}$ and a multiplier k so that the two linear covariants defined above of $k\phi'$ and $\bar{f}$ should be q and p.

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* We determine σ and ρ in § 11.

For let $\bar{\phi}$ and $\bar{f}$ be one solution, $\bar{\phi} = C_x^3, \bar{f} = D_x^2$; and let $u\bar{\phi} = \Gamma_x^3$, $v\bar{f} = \Delta_x^2$ be any solution. Then

$$(\Gamma\Delta)^2\Gamma_x = (CD)^2 C_x, \quad (\Gamma\Delta)^2(\Gamma\Delta')\Delta'_x = (CD)^2(CD')D'_x,$$

or

$$uv(CD)^2 C_x = (CD)^2 C_x, \quad uv^2(CD)^2(CD')D'_x = (CD)^2(CD')D'_x;$$

hence $uv = 1$, $uv^2 = 1$, the only solution of which is $u = 1$, $v = 1$, which was to be proved.

This remark will be useful in the sequel.

9. Suppose given a system (Σ) of a cubic ϕ and quadratic f, in symbolic notation a_x^3, b_x^2; let $(ab)^2 a_x = p$ and $(ab)^2(ab')b'_x = q$ where ϕ, p, q are to be used as in the preceding paragraphs. Let $\bar{\phi}, \bar{f}$ be formed according to the prescription given there, calling the system of $\bar{\phi}, \bar{f}$, $(\bar{\Sigma})$. Then $\bar{\phi}, \bar{f}$ are two covariants of (Σ).

THEOREM.—*If we form the same two covariants $\bar{\bar{\phi}}, \bar{\bar{f}}$, of $(\bar{\Sigma})$ as $\bar{\phi}, \bar{f}$ are of (Σ) then*

$$\bar{\bar{\phi}} = \phi, \quad \bar{\bar{f}} = f;$$

that is (Σ) may be derived from $(\bar{\Sigma})$ as $(\bar{\Sigma})$ was from (Σ).

Proof.—Remembering that the covariants of (Σ) which we call p, q are for $(\bar{\Sigma})$ equal to q, p we operate with $\bar{\phi}, q, p$ to produce $\bar{\bar{\phi}}$ as we operated with ϕ, p, q to produce $\bar{\phi}$.

Since $Q(a\phi) = a^3 Q(\phi)$ and $D_{pq}\big(a^3 Q(\phi)\big) = a^3 D_{pq}\big(Q(\phi)\big)$ and $\bar{\phi} = \rho\sigma^{-2}\phi'$ it follows that

$$D_{qp}\big(Q(\bar{\phi})\big) = \frac{\rho^3}{\sigma^6} D_{qp}\big(Q(\phi')\big) = -\frac{R^2}{4}27^2(pq)^2\frac{\rho^3}{\sigma^6}\phi = \phi''.$$

From ϕ'', p, q deduce $\bar{\bar{\phi}}, \bar{\bar{f}}$ as $\bar{\phi}, \bar{f}$ were deduced from ϕ', q, p. Then $\bar{\bar{\phi}}$ is proportional to ϕ'' and if $\bar{\bar{\phi}} = A_x^3, \bar{\bar{f}} = B_x^2$, we have

$$(AB)^2 A_x = p, \quad (AB)^2(AB')B'_x = q.$$

But ϕ is proportional to ϕ'' and $\phi = a_x^3$, $f = b_x^2$ have the property $(ab)^2 a_x = p$, $(ab)^2(ab')b'_x = q$; therefore, by the remark at the end of § 8,

$$\bar{\bar{\phi}} = \phi, \quad \bar{\bar{f}} = f.$$

We have then *complete reciprocity* between two systems of a cubic and quadratic (Σ) and $(\bar{\Sigma})$. If we denote forms derived analogously from the two systems by a stroke we have the scheme:

(Σ)	$(\bar{\Sigma})$
ϕ, f	$\bar{\phi}, \bar{f}$
p, q	$\bar{p}, \bar{q}$

and $\bar{p} = q$, $\bar{q} = p$.

This duality has some resemblance to the relations existing between a binary cubic and its cubic covariant, the pencil of a binary biquadratic and its hessian and the hessians of the pencil,* and the pencil of a ternary cubic and its hessian and the hessians of the pencil.†

10. Let ψ, ϕ_1, $\cdots \phi_n$ denote any covariants of (Σ) and $\bar{\psi}$, $\bar{\phi}_1$, $\cdots \bar{\phi}_n$ the corresponding covariants of $(\bar{\Sigma})$ and suppose we have a relation

$$\chi\left((\psi\bar{\psi})_\lambda, \phi_1 \cdots \phi_n\right) = 0;$$

then because of the duality of (Σ), $(\bar{\Sigma})$ we have also

$$\chi\left(\pm (\psi\bar{\psi})_\lambda, \bar{\phi}_1 \cdots \bar{\phi}_n\right) = 0,$$

since

$$(\bar{\psi}\psi)_\lambda = \pm (\psi\bar{\psi})_\lambda,$$

according as λ is even or odd.

By means of such a relation we are able to show the duality which exists between the involutions I_4, $\bar{I}_4$ determined by (Σ), $(\bar{\Sigma})$.

11. It is easy to calculate $\bar{\phi}$ and $\bar{f}$ in terms of the fundamental concomitants of ϕ and f since the two covariants p and q have been used to give a typical representation of ϕ and f. We start from the formulæ to be found in GORDAN's *Invariantentheorie*, vol. 2, p. 327 :

$$fF = \tfrac{1}{2} A_{ff} p^2 + q^2,$$

$$-F^3\phi = \left(F(pq) - \tfrac{1}{2}A_{ff}(sq)\right)p^3 - \tfrac{3}{2}MA_{ff}p^2q + \tfrac{3}{2}(A_{ff}L - A_{f\Delta}F)pq^2 + Mq^3,$$

where the meanings of M, A_{ff}, etc., are given in the preceding pages of GORDAN.

In calculating covariants, using the typical representation, we may proceed as if p and q were the variables x_1, x_2 provided we multiply the result by a power of (pq) equal to the weight of the covariant. For if I be a covariant of weight λ so that $I' = r^\lambda I$, r being the determinant of substitution, we may make the substitution $x_1' = p$, $x_2' = q$ of determinant $1/(pq)$; then $I' = [1/(pq)^\lambda]\ I$ or $I = (pq)^\lambda I'$ and I' is expressed in x_1', x_2' but $x_1' = p$, $x_2' = q$; hence the truth of the above statement.

Suppose then the cubic covariant $Q(\phi)$ of ϕ to be calculated from the typical representation; then the operation $D_{pq}\left(Q(\phi)\right)$ may be effected.

Let

$$Q(\phi) = C_0 p^3 + C_1 p^2 q + C_2 pq^2 + C_3 q^3 = C_0(p - a_1 q)(p - a_2 q)(p - a_3 q),$$

* CLEBSCH, *Binäre Formen.*

† CLEBSCH-LINDEMANN, *Geometrie*, p. 559.

where C_0, C_1, C_2 and C_3 are expressed in terms of the fundamental invariants of ϕ, f.

Then

$$D_{pq}\big(Q(\phi)\big) = (pq)^3 C_0(p + 3a_1q)(p + 3a_2q)(p + 3a_3q),$$

as a short computation shows, or

$$D_{pq}\big(Q(\phi)\big) = (pq)^3(C_0p^3 - 3\,C_1p^2q + 9\,C_2pq^2 - 27\,(C_3q^3)$$
$$= \phi' = c_0p^3 + 3c_1p^2q + 3c_2pq^2 + c_3q^3 = c_x^3,$$

where

$$c_0 = (pq)^3C_0, \quad -c_1 = (pq)^3C_1, \quad c_2 = 3\,(pq)^3C_2, \quad -c_3 = 27\,(pq)^3C_3.$$

Next let f' be the Jacobian of (c_x^3, q), and pq; we find

$$f' = \tfrac{1}{2}(pq)^2(c_0p^2 - c_2q^2) = d_x^2.$$

If we calculate $(cd)^2c_x$ and $(cd)^2(cd')\,d'_x$ which are of weights 2 and 3 we find

$$(cd)^2c_x = \tfrac{1}{2}(pq)^4(c_3c_0 - c_1c_2)q,$$
$$(cd)^2(cd')\,d'_x = \tfrac{1}{4}(pq)^7(c_0c_1c_2 - c_0^2c_3)p,$$

so that

$$\sigma = \tfrac{1}{2}(pq)^4(c_3c_0 - c_1c_2),$$
$$\rho = \tfrac{1}{4}(pq)^7(c_0c_1c_2 - c_0^2c_3),$$

and consequently $\rho\sigma^{-2}\phi'$ and $\sigma\rho^{-2}f'$ are expressed in terms of the form system of (Σ); that is, $\bar{\phi}$ and $\bar{f}$ are so expressed, since

$$\rho\sigma^{-2}\phi' = \bar{\phi}, \quad \sigma\rho^{-1}f' = \bar{f}.$$

THEOREM.—*The forms $\bar{\phi}, \bar{f}$ are expressed in terms of the fundamental concomitants of ϕ, f by the following equations:*

$$\phi' = c_0p^3 + 3c_1p^2q + 3c_2pq^2 + c_3q^3,$$
$$c_0 = (pq)^3C_0, \quad -c_1 = (pq)^3C_1, \quad c_2 = 3\,(pq)^3C_2, \quad -c_3 = 27\,(pq)^3C_3,$$

where C_0, C_1, C_2, C_3 are defined by

$$Q(\phi) = C_0p^3 + C_1p^2q + C_2pq^2 + C_3q^3;$$
$$f' = \tfrac{1}{2}(pq)^2(c_0p^2 - c_2q^2),$$
$$\sigma = \tfrac{1}{2}(pq)^4(c_3c_0 - c_1c_2),$$
$$\rho = \tfrac{1}{4}(pq)^7(c_0c_1c_2 - c_0^2c_3),$$

and

$$\bar{\phi} = \frac{\rho}{\sigma^2}\phi', \quad \bar{f} = \frac{\sigma}{\rho}f'.$$

IV.

The Involutions Belonging to ϕ, f and $\bar{\phi}, \bar{f}$.

12. In this section the involutions I_4 and $\bar{I}_4$ belonging to (Σ) and $(\bar{\Sigma})$ are determined.

Let
$$(pq, l)_1 = l_1, \qquad (pq, \bar{l})_1 = \bar{l}_1,$$
$$(pq, m)_1 = m_1, \qquad (pq, \bar{m})_1 = \bar{m}_1,$$
$$(pq, n)_1 = n_1; \qquad (pq, \bar{n})_1 = \bar{n}_1;$$
also
$$(p\bar{l}_1, ql_1)_1 = R_x^2 = R,$$
$$(p\bar{m}_1, qm_1)_1 = S_x^2 = S,$$
$$(p\bar{n}_1, qn_1)_1 = T_x^2 = T.$$
The three quadratics R, S, T are such that if we interchange the rôles of (Σ) and $(\bar{\Sigma})$ in forming them we find $-R$, $-S$, $-T$. Hence any relation between R, S, T and covariants of (Σ), $(\bar{\Sigma})$ is accompanied by a relation between $-R$, $-S$, $-T$ and the same covariants of $(\bar{\Sigma})$, (Σ). The three quadratics are branch quadratics of both I_4 and $\bar{I}_4$.

Theorem: *The forms Rl^2, Sm^2, Tn^2, f^2 belong to I_4 and the Jacobian of I_4 is proportional to $\phi f q$.*

From this result it follows that $R\bar{l}^2$, $S\bar{m}^2$, $T\bar{n}^2$, $\bar{f}^2$ belong to $\bar{I}_4$ whose Jacobian is proportional to $\overline{\phi f p}$.

13. For the proof of this theorem some results on apolar systems of biquadratic forms are necessary.*

1°. To every involution $(\Pi) = \lambda f_1 + \mu f_2$ there is an apolar linear system $(\bar{\Pi}) = \lambda\phi_1 + \mu\phi_2 + \nu\phi_3$ such that every form of $(\bar{\Pi})$ is apolar to every form of (Π) and (Π) determines $(\bar{\Pi})$ and conversely.

2°. If a biquadratic form is apolar to the three forms ϕ_1, ϕ_2, ϕ_3 it must belong to (Π).

If, therefore, we can show that Rl^2, Sm^2, Tn^2, f^2 are all apolar to three biquadratic forms they must belong to the involution apolar to the three forms.

We are able to find three such forms, viz., $l^3(fl)_1$, $m^3(fl)_1$, $n^3(fn)_1$.

1°. It must be shown that $l^3(fl)_1$, $m^3(fm)_1$, $n^3(fn)_1$ are linearly independent, which is supposed of ϕ_1, ϕ_2, ϕ_3 above. Since $\phi = lmn$ we may regard l, m, n as arbitrary and also $(fl)_1$, $(fm)_1$ since f is so. For let a_x and b_x be any two linear forms, then if we take f to be the Jacobian of $a_x l_x$ and $b_x m_x$ we know that $(fl)_1$ and $(fm)_1$ are proportional to a_x and b_x, that is, we may choose f so that $(fl)_1$ and $(fm)_1$ are any two linear forms.

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* Rosanes, Crelle's Journal, vol. 76.

Hence we may regard $l^3(fl)_1$ and $m^3(fm)_1$ as any two biquadratics having cubic factors.

If then a linear relation existed between $l^3(fl)_1$, $m^3(fm)_1$, $n^3(fn)_1$ it would take the form

$$n^3(fn)_1 = Al^3(fl)_1 + Bm^3(fm)_1$$

or $n^3(fn)_1$ would belong to the involution I_4 of $l^3(fl)_1$ and $m^3(fm)_1$. But a cube factor in a form of I_4 enters as a square in the Jacobian of I_4, hence the Jacobian of I_4 must be proportional to $l^2m^2n^2$. To see that this is not the case make a linear transformation by putting $l = x_1'$, $m = x_2'$; then denote $(fl)_1$ by $\rho(x_1' + 4\lambda x_2')$, $(fm)_1$ by $\sigma(4\mu x_1' + x_2')$. Then the Jacobian ϑ of $\rho x_1'^3(x_1' + 4\lambda x_2')$ and $\sigma x_2^3(4\mu x_1' + x_2')$ must be a square. The computation gives

$$\vartheta = x_1'^2 x_2'^2\{3\mu x_1'^2 + (8\lambda\mu + 1)x_1'x_2' + 3\lambda x_2'^2\},$$

so that $3\mu x_1'^2 + (8\lambda\mu + 1)x_1'x_2' + 3\lambda x_2'^2$ must be a square, which it is not, since $(8\lambda\mu + 1)^2 - 36\lambda\mu \neq 0$ if λ and μ are arbitrary.

2°. To prove Rl^2, Sm^2, Tn^2, f^2 apolar to three forms involves twelve relations but it suffices to prove three of them; thus if $l^3(fl)_1$ is apolar to Sm^2 it must be to Tn^2, etc. It is sufficient to prove $l^3(fl)_1$ apolar to (*a*) Rl^2, (*b*) f^2, (*c*) Sm^2. The proof is in each case facilitated by the use of a proper normal form.

(*a*) $l^3(fl)_1$ is apolar to Rl^2 independently of the particular values of $(fl)_1$ or R. Make the substitution

$$x_1' = l, \quad x_2' = x_2,$$

then

$$l^3(fl)_1 = x_1'^3(a_1x_1' + a_2x_2'), \quad Rl^2 = x_1'^2(b_0x_1'^2 + 2b_1x_1'x_2' + b_2x_2'^2),$$

and these two forms are at once seen to be apolar. This involves the proof of three relations.

(*b*) To prove $l^3(fl)_1$ apolar to f^2b let $l = x_1'$, $(fl)_1 = x_2'$; then since $(fl)_1 l$ is harmonic to f we must have $x_1'x_2'$ harmonic to f, that is $f = A_0x_1'^2 + A_2x_2'^2$ and $l^3(fl)_1 = x_1'^3x_2'$, but $x_1'^3x_2'$ is apolar to

$$(A_0x_1'^2 + A_2x_2'^2)^2 = A_0^2x_1'^4 + 2A_0A_2x_1'^2x_2'^2 + A_2^2x_2'^4.$$

This involves three relations.

(*c*) Since the relation $\big(l^3(fm)_1, Rm^2\big)_1 = 0$ is homogeneous in the coefficients of l, m, n we may in proving it use instead of l, m, n any forms proportional to them.

Let $x_1' = p$, $x_2' = q$, dropping the accents after transforming. Let ϕ be proportional to

$$x_1^3 + 3\lambda x_1^2x_2 + 3\mu x_1x_2^2 + \nu x_2^3 = a\phi = (x_1 + \xi_1x_2)(x_1 + \xi_2x_2)(x_1 + \xi_3x_2),$$

so that

$$\rho_1 l = (x_1 + \xi_1 x_2), \quad \rho_2 m = (x_1 + \xi_2 x_2), \quad \rho_3 n = (x_1 + \xi_3 x_2).$$

We know that f is harmonic to $x_1 x_2$, also to $(\phi p)_2$, that is, to

$$\lambda x_1^2 + \mu x_1 x_2 + \nu x_2^2;$$

so that f must be proportional to the Jacobian of $x_1 x_2$ and $(\phi p)_1$, or

$$\beta f = \lambda x_1^2 - \nu x_2^2,$$

where a and β are quantities which it is not necessary to determine.

If we put

$$\eta_i = \frac{3(\mu\xi_i - \nu)}{\lambda - \xi_i} \qquad (i = 1, 2, 3),$$

and use the method of forming $Q(\phi)$ described in section II we find $Q(\phi)$ proportional to

$$(3\xi_1 x_1 - \eta_1 x_2)(3\xi_2 x_1 - \eta_2 x_2)(3\xi_3 x_1 - \eta_3 x_2),$$

and performing the operation $D_{pq}\big(Q(\phi)\big)$ we find $\bar{\phi}$ proportional to

$$(\xi_1 x_1 + \eta_1 x_2)(\xi_2 x_1 + \eta_2 x_2)(\xi_3 x_1 + \eta_3 x_2);$$

so that R is proportional to the Jacobian of $x_1(\xi_1 x_1 - \eta_1 x_2)$ and $x_2(x_1 - \xi_1 x_2)$, since $x_1 - \xi_1 x_2$, $\xi_1 x_1 - \eta_1 x_2$ are proportional to l_1, $\bar{l}_1$ or finally R, S, T are equal to

$$\sigma_1(x_1^2 - 2\xi_1 x_1 x_2 + \eta_1 x_2^2), \quad \sigma_2(x_1^2 - 2\xi_2 x_1 x_2 + \eta_2 x_2^2), \quad \sigma_3(x_1^2 - 2\xi_3 x_1 x_2 + \eta_3 x_2^2),$$

respectively.

Also $(fl)_1$ is proportional to

$$x_1 + \frac{\nu}{\xi_1 \lambda} x_2;$$

hence

$$(x_1 + \xi_1 x_2)^3 \left(x_1 + \frac{\nu}{\xi_1\lambda} x_2\right)$$

must be apolar to

$$(x_1^2 - 2\xi_1 x_1 x_2 + \eta_2 x_2^2)(x_1 + \xi_2 x_2)^2.$$

To show this we note that if $a_x^4 = (xa)(x\beta)(x\gamma)(x\delta)$ is apolar to b_x^4 then $b_a b_\beta b_\gamma b_\delta = 0$, so that if we take the third polar of Sm^2 with respect to $x_1 + \xi_1 x_2$ and the first polar of the result with respect to

$$x_1 + \frac{\nu}{\xi_1 \lambda} x_2,$$

the final result must be zero.

Doing this we obtain

$$\tfrac{1}{2}(\xi_2-\xi_1)\{(\xi_1^2+2\xi_1\xi_2+\eta_2)(\lambda\xi_1\xi_2-\nu)$$
$$+(\xi_2-\xi_1)\big(\nu(\xi_1+\xi_2)+(\xi_1\xi_2+\eta_2)\lambda\xi_2\big)\}$$

which must be zero, or since $\xi_1 \neq \xi_2$,

$$(\xi_1^2+2\xi_1\xi_2+\eta_2)(\lambda\xi_1\xi_2-\nu)+(\xi_2-\xi_1)\big(\nu(\xi_1+\xi_2)+(\xi_1\xi_2+\eta_2)\lambda\xi_2\big)$$

must be zero; or solving for η_2 and remembering that

$$3\lambda=\xi_1+\xi_2+\xi_3,$$
$$3\mu=\xi_2\xi_3+\xi_3\xi_1+\xi_1\xi_2,$$
$$\nu=\xi_1\xi_2\xi_3,$$

η_2 must equal

$$\frac{(\xi_1^2+2\xi_1\xi_2)(\xi_2\lambda-\xi_2\xi_3)-(\xi_1-\xi_2)\big(\lambda\xi_1\xi_2+\xi_1\xi_2(\xi_1+\xi_2)\big)}{\xi_2\xi_3-\lambda(2\xi_2-\xi_1)}.$$

The denominator $\xi_2\xi_3-\lambda(2\xi_2-\xi_1)$ is equal to

$$(\lambda-\xi_2)(\xi_1+\xi_2),$$

and the numerator is equal to

$$(\xi_1+\xi_2)\{(\xi_2-\xi_1)\xi_2\xi_3+\xi_1(\xi_2\lambda-\xi_2\xi_3)\}+\xi_1\xi_2(\xi_2\lambda-\xi_2\xi_3)-(\xi_1-\xi_2)\lambda\xi_1\xi_2,$$

and

$$\xi_1\xi_2(\xi_2\lambda-\xi_2\xi_3)-(\xi_1-\xi_2)\lambda\xi_1\xi_2=-(\lambda\xi_2\xi_1^2+\xi_2\nu-2\lambda\xi_2^2\xi_1)$$
$$=-\{\lambda\xi_1\xi_2(\xi_1+\xi_2)-\xi_1^2\xi_2^2-\xi_2^3\xi_1\}=(\xi_1+\xi_2)(\xi_2^2\xi_1-\lambda\xi_1\xi_2);$$

or the numerator of η_2 is equal to

$$(\xi_1+\xi_2)(\xi_2^2\xi_3-\xi_1\xi_2\xi_3+\lambda\xi_1\xi_2-\xi_1\xi_2\xi_3-\lambda\xi_1\xi_2+\xi_2^2\xi_1)$$
$$=(\xi_1+\xi_2)\{\xi_2(\xi_2\xi_3+\xi_3\xi_1+\xi_1\xi_2)-3\xi_1\xi_2\xi_3\}=3(\xi_1+\xi_2)(\mu\xi_2-\nu)$$

or finally, η_2 must equal

$$\frac{3(\xi_1+\xi_2)(\mu\xi_2-\nu)}{(\lambda-\xi_1)(\lambda-\xi_2)}=\frac{3(\mu\xi_2-\nu)}{\lambda-\xi_2},$$

which is the definition of η_2. Hence the relation from which we started must be true. This proves (c) involving six relations and the theorem is proved in all its parts if we show that the Jacobian of I_4 is proportional to $\phi f q$. This follows immediately from section I where it was proved that the Jacobian of I_4 determined from ϕ and f is proportional to $\phi f(x\delta)$, where $(x\delta)=(ab)^2(ab')b'_x$ which is the definition of q.

In virtue of the reasoning at the end of section III it follows that $\bar{R}\bar{l}^2$, $\bar{S}\bar{m}^2$, $\bar{T}\bar{n}^2, \bar{f}^2$ belong to $\bar{I}_4$ whose Jacobian is proportional to $\bar{\phi}\bar{f}\bar{q}$, that is to $\bar{\phi}\bar{f}p$.

It follows that I_4 and $\bar{I}_4$ determined by (Σ) and $(\bar{\Sigma})$ have three branch quadratics in common and the two covariants p, q are the fourth double elements, q in I_4 and p in $\bar{I}_4$.

I_4 derived from (Σ)	$\bar{I}_4$ derived from $(\bar{\Sigma})$
$Rl^2,\ Sm^2,\ Tn^2,\ f^2$	$\bar{R}\bar{l}^2,\ \bar{S}\bar{m}^2,\ \bar{T}\bar{n}^2,\ \bar{f}^2$
with double elements given by	with double elements given by
$\phi f q$.	$\bar{\phi}\bar{f}p$.

14. In section III $\bar{\phi}$ and $\bar{f}$ were expressed in terms of the fundamental concomitants of ϕ and f by means of the typical representation by p and q. Since f^2 is a form of I_4 if we express one other form of I_4 in terms of the concomitants of ϕ, f we have every form of I_4 so expressed. The branch form whose double element is q will be rational in the concomitants of ϕ, f. Let it be equal to $F = q^2(ap^2 + 2\gamma pq + \beta q^2)$ and let f be $Ap^2 + Bq^2$ (there is no term in pq in the typical representation of f). Then the Jacobian of F and f^2 must be proportional to $\phi f q$, or the Jacobian of F, f to ϕq, or finally we may take a, β, γ so that

$$q\begin{vmatrix} Ap, & Bq \\ q(ap+\gamma q), & q(\gamma p + \beta q) + 2(ap^2 + \beta q^2 + \gamma pq) \end{vmatrix} = \phi q,$$

or

$$2Aap^3 + 3A\gamma p^2 q + (3A\beta - aB)pq^2 - B\gamma q^3 = \phi,$$

and since ϕ is expressed in terms of p and q, by equating coefficients a, β, γ may be found and consequently F expressed as concomitant to ϕ, f.

If we express the form of $\bar{I}_4$, which contains p^2, in terms of p, q, we can express every form of $\bar{I}_4$ in terms of p, q, for $\bar{f}^2$ is already so expressed.

V.

The reducible Hyperelliptic Integrals.

15. Since I_4 and $\bar{I}_4$ have three branch quadratics in common and q and p are the fourth double elements, we have from the Introduction the

Theorem: *The integrals*

$$\int \frac{q\,(x\,dx)}{\sqrt{RST}}, \qquad \int \frac{p\,(x\,dx)}{\sqrt{\bar{R}\bar{S}\bar{T}}}$$

are reducible to elliptic integrals by transformations of the fourth order I_4, $\bar{I}_4$.

Having given one reducible integral and its reducing transformation we can solve the problem to find the second one belonging to the same irrationality and its reducing transformation. For the reducing transformation being known the form ϕ is known and also f; from ϕ and f we deduce the covariant $(ab)^2 a_x$ which is the numerator of the second reducible integral; the second reducing transformation being given by the involution $\bar{I}_4$ derived from $\bar{\phi}$ and $\bar{f}$, which can be found from ϕ and f.

We can enunciate the following theorem:

THEOREM: *If $\phi = a_x^3$ denotes the product of the three double elements complementary to the three quadratic factors of the sextic of a reducible integral, and $f = b_x^2$ denotes the form whose square occurs among the forms of the reducing involution, then the numerator of the second reducible integral belonging to the same irrationality is equal to $(ab)^2 a_x$.*

16. It is possible to give another definition of the numerator of the second integral which resembles that given by Professor BOLZA* for the transformation of order 3 and also that for order 2. For we have shown that if l_x denotes a double element of an I_4 containing a complete square f^2, then $l^3(fl)_1$ is apolar to the forms of I_4; but q is a double element of I_4 and $(fq)_1$ is proportional to p, hence $q^3 p$ is apolar to I_4. We can enunciate the theorem:

THEOREM: *If $(x\delta)$ denotes the numerator of the first reducible integral and $(x\bar{\delta})$ of the second and I_4 the reducing involution, then $(x\delta)^3(x\bar{\delta})$ is apolar to the forms of I_4 and this is sufficient to determine $(x\bar{\delta})$.*

The following conjectural theorem suggests itself: *If $(x\delta)$ and $(x\bar{\delta})$ denote as before the numerators of the integrals—reducible by a transformation of order k—and I_k the reducing involution, then $(x\delta)^{k-1}(x\bar{\delta})$ is apolar to the forms of I_k.* This surely holds for $k = 2, 3$ and, as has been shown in this paper, for $k = 4$.

VI.

NORMAL FORM OF REDUCIBLE INTEGRALS.

17. The normal form used by Professor BOLZA † is found by introducing p and q as variables, that is, by introducing the numerators of the two integrals. This has been done for a special purpose in section IV, and the formulæ there developed have only to be supplemented; we found

$$R = \sigma_1(x_1^2 - 2\xi_1 x_1 x_2 + \eta_1 x_2^2),$$
$$S = \sigma_2(x_1^2 - 2\xi_2 x_1 x_2 + \eta_2 x_2^2),$$
$$T = \sigma_3(x_1^2 - 2\xi_3 x_1 x_2 + \eta_3 x_2^2),$$

*Mathematische Annalen, vol. 50, p. 314.

† Dissertation, Mathematische Annalen, vol. 28.

$$RST = \sigma_1\sigma_2\sigma_3\{\nu' x_1^6 - 6\lambda\nu' x_1^5 x_2 + 3(4\mu\nu' + \lambda\mu')x_1^4 x_2^2$$
$$+ 2(\lambda\lambda' + 5\nu\nu')x_1^3 x_2^3 + 3(4\mu'\nu' + \lambda'\mu)x_1^2 x_2^4 - 6\lambda'\nu x_1 x_2^5 + \nu x_2^6\},$$

where

$$\lambda' = -\tfrac{1}{3}\frac{2\lambda^2\nu - \lambda\mu^2 - \mu\nu}{-\nu^2 + 3\lambda\mu\nu - 2\mu^3},$$

$$\mu' = \tfrac{1}{9}\frac{\lambda^2\mu + \lambda\nu - 2\mu^2}{-\nu^2 + 3\lambda\mu\nu - 2\mu^3},$$

$$\nu' = -\tfrac{1}{27}\frac{2\lambda^3 - 3\lambda\mu + \nu}{-\nu^2 + 3\lambda\mu\nu - 2\mu^3}.$$

Professor Bolza used implicitly a form corresponding to ϕ, viz.:

$$x_2^3 + 3\lambda' x_2^2 x_1 + 3\mu' x_2 x_1^2 + \nu' x_1^3,$$

and made the remark that if the cubic covariant of

$$x_1^3 + 3\lambda x_1^2 x_2 + 3\mu x_1 x_2^2 + \nu x_2^3$$

is

$$C_0 x_1^3 + 3C_1 x_1^2 x_2 + 3C_2 x_1 x_2^2 + C_3 x_2^3,$$

then

$$\lambda' = -\tfrac{1}{3}\frac{C_2}{C_3}, \quad \mu' = \tfrac{1}{9}\frac{C_1}{C_3}, \quad \nu' = -\tfrac{1}{27}\frac{C_0}{C_3},$$

which was the starting point for discovering the relations in sections III and IV.

The reducible integrals are

$$\int \frac{x_1(x\,dx)}{\sqrt{R}}, \quad \int \frac{x_2(x\,dx)}{\sqrt{R}},$$

where $R = RST$ is expressed in x_1, x_2 above.

The reducing substitution for the first integral is given by any two of the forms

$$\sigma_i(x_1^2 - 2\xi_i x_1 x_2 + \eta_i x_2^2)(x_1 + \xi_i x_2)^2 \qquad (i = 1, 2, 3).$$

The form f is equal to $(\lambda x_1^2 - \nu x_2^2)$.

VII.

Miscellaneous Results on Biquadratic Involutions Containing a Complete Square.

18. In section VII are collected some miscellaneous results relating to special biquadratic involutions containing a square and their geometrical representations.

(A) It is a known theorem that the Jacobians of two apolar systems are the same.* Since the roots of the Jacobian are multiple roots in forms of the system from which it is derived this theorem states that if (xa) is a double element of a biquadratic involution (Π) it is a triple element of the apolar system $(\overline{\Pi})$; hence $(xa)^3(x\bar{a})$ is a form of $(\overline{\Pi})$. When (Π) is a special involution we are able to determine $(x\bar{a})$ from section IV for it was there shown that $(x\bar{a}) = \big(f, (xa)\big)_1$, f being the form whose square belongs to (Π).

THEOREM.—*If (xa) is a double element of a special biquadratic involution I_4 containing a square f^2, then $(xa)^3\big(f, (xa)\big)_1$ belongs to the system apolar to I_4.*

(B) The theorem of section I which says that the form there denoted by θ is apolar to the involution $\lambda u + \mu f'^2$ admits of a geometrical interpretation. Let a_x^4, β_x^4, γ_x^4 be three forms apolar to u_x^4 and v_x^4. The rational quartic C_4† given by the parameter representation

$$\rho x_1 = a_x^4,$$

$$\rho x_2 = \beta_x^4,$$

$$\rho x_3 = \gamma_x^4,$$

is intersected by a straight line $\lambda x_1 + \mu x_2 + \nu x_3$ in 4 points whose parameters satisfy the equation:

$$\lambda a_x^4 + \mu \beta_x^4 + \nu \gamma_x^4 = 0,$$

that is, they are given by a form apolar to u_x^4 and v_x^4.

If $\lambda x_1 + \mu x_2 + \nu x_3$ is an inflexional tangent of C_4 the equation

$$\lambda a_x^4 + \mu \beta_x^4 + \nu \gamma_x^4 = 0$$

has a triple root which is the parameter value for the point of inflexion. By the theorem already quoted in this section this must be a root of the Jacobian of u_x^4 and v_x^4. When this is applied to the special involution, $\theta = 0$ must give 4 points of inflexion; but θ is a form of the system apolar to the involution, that is $\theta = 0$ gives 4 collinear points, hence the

THEOREM: *Four of the points of inflexion of a special C_4 are collinear.*

(C) Another definition can be given of the linear factors of $\bar{\phi}$, which is adapted to their determination when a twisted cubic C_3 is used to carry the binary variables.‡ Using the notation of section II, we have the

THEOREM: *Lq^2 is apolar to $(pq, \bar{l})_1\bar{p}^2$.*

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*STEPHANOS: *Sur les faisceaux ayant une même jacobienne*, Mémoires par divers savants, vol. 27.

† MEYER: *Apolarität und Rationale Curven.*

‡ STURM, Crelle's Journal, vol. 86.

Proof.—Referring to the definition, l is proportional to $p(qL)+3q(pL)$, hence $(pq, \bar{l})_1$ is proportional to $p(pq)(qL)-3q(pq)(pL)$.

To express L in a convenient form we notice that $\big(pq, (pq, L)_1\big)_1$ is proportional to L, for we have taken the harmonic $(pq, L)_1$ of L with regard to pq and then of the linear form so found with regard to pq again, which must give the original L up to a factor.

Hence,

$$(pq, L)_1 = \tfrac{1}{2}\big(p(qL)+q(pL)\big),$$

and therefore

$$\big(pq, (pq, L)_1\big)_1 = \tfrac{1}{4}\{-p(pq)(qL)+q(pq)(pL)\},$$

therefore

$$\{-p(pq)(qL)+q(pq)(pL)\}q^2$$

must be apolar to

$$\{(pq)(qL)-3q(pq)(pL)\}p^2;$$

that is, if we put $x_1'=p$, $x_2'=q$,

$$-x_1'x_2'^2(pq)(qL)+x_2'^3(pq)(pL)$$

must be apolar to

$$x_1'^3(pq)(qL)-3x_1'^2x_2'(pq)(pL),$$

that is,

$$(pq)^2(qL)(pL)-(pq)^2(qL)(pL)$$

must be zero. This is true and therefore the theorem is proved.

The apolarity of cubic forms admits of geometrical representation (STURM, loc. cit.). Let a_x^3 be a cubic, to it will correspond a plane having a null point; if b_x^3 is another cubic apolar to a_x^3 then the plane corresponding to b_x^3 will pass through the null point of the plane corresponding to a_x^3, and conversely; if the plane passes through that null point b_x^3 will be apolar to a_x^3.

There is also a geometrical representation of the cubic covariant Q of $\phi = a_x^3$ on C_3. Call A the plane corresponding to ϕ, then A contains one line which is the intersection of two osculating planes of C_3; call these B and C. Then the plane D, which in the pencil containing A, B, C divides A harmonically from B and C, meets C_3 in 3 points given by $Q(\phi)=0$.

Suppose we start with ϕ and f, then we can construct p and q (STURM, loc. cit.); also $Q(\phi)=LMN$. Then by the theorem last proved we can find the point $(pq, \bar{l})_1=0$, viz: the plane containing the tangent to C_3 at $p=0$ and the null point of the plane through the point $L=0$ and the tangent at $q=0$ must meet C_3 in three points among which is the point $p=0$ counted twice and the third of which is the point $(pq, \bar{l})_1=0$.

If we take the harmonic of the point $(pq, \bar{l})_1=0$ with respect to $pq=0$ we reach the point $\bar{l}=0$ and thus the points $\phi=0$ may be found.

THEOREM: *Given the plane corresponding to* $\phi = 0$ *and the line corresponding to* $f = 0$, *it is possible to find on the twisted cubic* C_3 *the plane corresponding to* $\bar{\phi} = 0$.

The harmonic of a point with respect to two others may be found (MEYER, loc. cit.) by considering the hyperboloids through C_3. But if we wish the harmonic of a with respect to b and c it will be found among the points given by the cubic covariant of a, b, c, viz: it is that one which is separated from a by b and c in virtue of the definition of cubic covariant used in section II.

This is a simpler way of reaching it because of the simple geometrical construction of the cubic covariant.