ON THE CLASS NUMBER OF THE CLYCLOTOMIC NUMBER FIELD

$k(e^{2\pi i/p^n})$*

BY

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Introduction.—The object of the present paper is to investigate the relation between the class numbers of the cyclotomic number fields $k(e^{2\pi i/p^n})$ and $k(e^{2\pi i/p^{n-1}})$ when p is any odd prime and $n \geqq 2$. The method is similar to the method used by WEBER † for the case $p = 2$.

Let $m = p^n$, $m' = p^{n-1}$, $\mu = \phi(m) = p^{n-1}(p-1)$, $\mu' = \phi(m') = p^{n-2}(p-1)$, $r = e^{2\pi i/p^n}$, $r' = e^{2\pi i/p^{n-1}}$. Denote by h and h' the class numbers of $k(r)$ and $k(r')$ respectively, and set

$$h = h'H.$$

We also set $h = kh_1$, $h' = k'h_1'$, $k = k'A$, $h_1 = h_1'B$ and hence

$$H = AB,$$

where h_1 and h_1' are the class numbers of the real fields $k(r + r^{-1})$ and $k(r' + r'^{-1})$ respectively. Also let

$$E = DE',$$

E and E' being the regulators of $k(r + r^{-1})$ and $k(r' + r'^{-1})$ respectively.

I. *Expressions for A and B.*

If we set $\theta = e^{2\pi i/\mu}$ and $t \equiv g^\gamma$, mod. m, g being a primitive root of m, we have the following expressions for k and h_1: ‡

$$(1) \qquad k = \frac{p^{\frac{1+p^{n-1}[n(p-1)-1]}{4}+n}}{2^{\frac{p^{n-1}(p-1)}{2}-1}\pi^{\frac{p^{n-1}(p-1)}{2}}} \prod_s X_1^{(s)},$$

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† WEBER, *Lehrbuch der Algebra*, 2d ed., vol. 2, pp. 796-818.

‡ WEBER, *Lehrbuch der Algebra*, 2d ed., vol. 2, p. 793.

(2)
$$h_1 = \frac{p^{\frac{-1+p^{n-1}[n(p-1)-1]}{4}}}{2^{\frac{p^{n-1}(p-1)}{2}-1}E} \prod_s X_2^{(s)},$$

where

$$X_1^{(s)} = \frac{\pi i}{m^2} \sum_{\lambda}^{1,\, m-1} \lambda f_s(r^\lambda), \qquad X_2^{(s)} = -\frac{1}{2m} \sum_{\lambda}^{1,\, m-1} f_s(r^\lambda) \log\left(\sin\frac{\lambda\pi}{m}\right)^2,$$

$$f_s(r^\lambda) = \sum_t \theta^{s\gamma} r^{\lambda t}.$$

Here t runs through a complete residue system with respect to the modulus m except multiples of p. In (1), s takes all odd and in (2), all even values less than μ except zero.

If we denote by Y the function X corresponding to the field $k(r')$, then we have for every $s < \mu'$

(3)
$$X^{(ps)} = Y^{(s)}.$$

For

$$f_{ps}(r^\lambda) = \sum_t \theta^{ps\gamma} r^{\lambda t} = \sum_t \theta'^{s\gamma} r^{\lambda t} \qquad (\theta' = \theta^p)$$

$$= \sum_{t'} \theta'^{s\gamma} r^{\lambda t'} (1 + \rho^\lambda + \cdots \rho^{(p-1)\lambda}) \qquad (\rho = e^{2\pi i/p}),$$

where t' runs through a complete residue system, mod. m', except multiples of p.

Hence

$$f_{ps}(r^\lambda) = 0, \quad \text{if} \quad \lambda \not\equiv 0, \quad \text{mod. } p$$

and

$$f_{ps}(r^\lambda) = p \sum_{t'} \theta'^{s\gamma'} r'^{\lambda' t'}, \quad \text{if} \quad \lambda \equiv 0, \quad \text{mod. } p.$$

In the last expression $\lambda = p\lambda'$ and $g^{\gamma'} \equiv t'$ mod. m'. From this (3) follows directly.

To obtain expressions for k' and h_1', we replace, in (1) and (2), n by $n-1$ and X by Y. Making use of (3) we then get after a few reductions the following expressions for the factors A and B :

(4)
$$A = \frac{p^{\frac{p^{n-2}(p-1)^2 n}{4}+1}}{2^{\frac{p^{n-2}(p-1)^2}{2}} \cdot \pi^{\frac{p^{n-2}(p-1)^2}{4}}} \prod_s X_1^{(s)},$$

(5)
$$B = \frac{p^{\frac{np^{n-2}(p-1)^2}{4}}}{2^{\frac{p^{n-2}(p-1)^2}{2}} D} \prod_s X_2^{(s)}.$$

In (4), s takes all odd and in (5), all even values less than μ except multiples of p.

II. *The factor A.*

1. *Simplification of the expression for A.* We will now show how the expression for A given above may be simplified so as to make it more convenient for numerical computation and also prove that A is an integer.

Consider the function $f_s(r^\lambda)$. Two cases present themselves: $\lambda \not\equiv 0$, mod. p, and $\lambda \equiv 0$, mod. p.

1°. $\lambda \not\equiv 0$, *mod.* p. In this case, observing that

$$\theta^{s\gamma} = \theta^{s \text{ ind } t} = \frac{\theta^{s \text{ ind } (\lambda t)}}{\theta^{s \text{ ind } \lambda}},$$

we get, after replacing λt by t,

(6) $$f_s(r^\lambda) = \theta^{-s \text{ ind } \lambda} \sum_t \theta^{s \text{ ind } t} r^t = \theta^{-s\gamma_1}(\theta^s, r),$$

where $\gamma_1 = \text{ind } \lambda$.

2°. $\lambda \equiv 0$, *mod.* p. In this case set $\lambda = p\lambda_1$, and we have

(7) $$f_s(r^\lambda) = \sum_t \theta^{s\gamma} r^{p\lambda_1 t} = \sum_\gamma^{1,\mu} \theta^{s\gamma + (p-1)\lambda_1 t}.$$

Let a be the greatest common divisor of s and $p-1$, and set $p-1 = ab$. Then the exponents in (7) fall into a groups which are congruent to each other mod. μ, the elements of each group being incongruent mod. μ. Hence

$$f_s(r^\lambda) = a \sum_\gamma \theta^{s\gamma + (p-1)\lambda_1 t},$$

where

$$\gamma = 1, 1 + a, 1 + 2a, \cdots, 1 + (p^{n-1}b - 1)a.$$

But the μ/a terms under the summation sign are the roots of the equation

$$x^{p^{n-1}b} - 1 = 0,$$

and hence

(8) $$f_s(r^\lambda) = 0.$$

Making use of (6) and (8), which hold for both even and odd values of s, we get

(9) $$X_1^{(s)} = \frac{\pi i}{m^2}(\theta^s, r)\phi(\theta^s);$$

if we set

(10) $$\phi(\theta^s) = \sum_\lambda \lambda\theta^{-s\gamma},$$

where $\gamma = \text{ind } \lambda$ and $\lambda = 1, 2, \cdots, m-1$ except multiples of p.

The function $\phi(\theta^s)$ may however be simplified. Since

$$(m-\lambda)\theta^{-s\,\mathrm{ind}\,(m-\lambda)} = -(m-\lambda)\theta^{-s\,\mathrm{ind}\,\lambda},$$

we get

$$\phi(\theta^s) = \sum_\lambda (2\lambda - m)\theta^{-s\gamma}, \tag{11}$$

where $\lambda = 1, 2, \cdots, (m-1)/2$ except multiples of p. For A we then obtain, observing that

$$\theta^{\mu-s} = \theta^{-s}(\theta^s, r)(\theta^{-s}, r) = (-1)^s p^n, *$$

the following expression

$$A = \frac{\prod_s \phi(\theta^s)}{2^{\frac{p^{n-2}(p-1)^2}{2}} p^{\frac{p^{n-2}(p-1)^2 n}{2}-1}}, \tag{12}$$

where s takes all odd values less than μ.

2. *Proof that A is an integer.* It is evident that $\phi(\theta^s)$ is an algebraic integer in the field $k(\theta)$. Now we have

$$\phi(\theta^s) = \sum_i^{1,\,\mu} \lambda_i \theta^{-is} = \sum_i^{1,\,\mu/2} (\lambda_i - \lambda_{\mu/2+i})\theta^{-si} \qquad (i = \mathrm{ind}\,\lambda_i).$$

But since

$$\lambda_{\mu/2+i} \equiv g^{\mu\,2+i} \equiv -g^i \equiv -\lambda_i \qquad \text{mod. } m,$$

we have

$$\lambda_{\mu/2+i} = m - \lambda_i;$$

and hence

$$\phi(\theta^s) = 2\sum_i^{1,\,\mu/2} \lambda_i \theta^{-is} - m\sum_\lambda^{1,\,\mu/2} \theta^{-is} = 2\sum_i^{1,\,\mu/2} \lambda_i\theta^{-is} + \frac{2m}{1-\theta^s}$$

or

$$(1-\theta^s)\phi(\theta^s) = 2\left[(1-\theta^s)\sum_i^{1,\,\mu/2}\lambda_i\theta^{-is} + m\right].$$

But

$$\prod_s (1-\theta^s) = \frac{\prod_t (1-\theta^t)}{\prod_{t'}(1-\theta'^{t'})},$$

where $\theta' = \theta^p$, and t and t' take all odd values less than μ and μ' respectively. Hence, the quantities θ^t and $\theta'^{t'}$, being the roots of the equations,

$$x^{\mu/2} + 1 = 0 \qquad \text{and} \qquad x^{\mu'/2} + 1 = 0$$

* WEBER, *Lehrbuch der Algebra*, 2d ed., vol. 2, p. 71.

respectively, it follows that

$$\prod_s (1-\theta^s) = 1.$$

Hence we see that $\Pi\,\phi(\theta^s)$ is divisible by $2^{p^{n-2}(p-1)^2/2}$.

To prove that $\Pi\,\phi(\theta^s)$ is divisible by p we have

$$(g-\theta^s)\phi(\theta^s) = \sum_i^{1,\mu} (g\lambda_i - \lambda_{i+1})\theta^{-is}.$$

But

$$g\lambda_i \equiv \lambda_{i+1}, \text{ mod. } m;$$

hence

$$(g-\theta^s)\phi(\theta^s) = m\psi(\theta^s),$$

where $\psi(\theta^s)$ is an algebraic integer, and therefore

$$\prod_s (g-\theta^s)\phi(\theta^s) = m^{\frac{p^{n-2}(p-1)^2}{2}} \prod_s \psi(\theta^s).$$

But

$$\prod_s (g-\theta^s) = \frac{\prod_t (g-\theta^t)}{\prod_{t'} (g-\theta'^{t'})},$$

where t and t' take all odd values less than μ and μ' respectively. Hence, reasoning as above, we find

$$\prod_s (g-\theta^s) = \frac{g^{\mu/2}+1}{g^{\mu'/2}+1}$$

$$= g^{\frac{\mu'}{2}(p-1)} - g^{\frac{\mu'}{2}(p-2)} + \cdots + 1,$$

or, since $g^{\mu'/2} \equiv -1$, mod. m',

$$\Pi_s (g-\theta^s) \equiv p, \text{ mod. } m.$$

We thus see that $\Pi(g-\theta^s)$ is divisible by p and by no higher power of p. Therefore $\Pi\,\phi(\theta^s)$ is divisible by $p^{p^{n-2}(p-1)^2n/2-1}$ and hence A is an integer.

If we now denote by A_n the factor A corresponding to $m = p^n$, we get the following expression for the first factor k of the class number of $k(r)$:

$$(13) \qquad k = k_1 A_2 A_3 \cdots A_n,$$

where k_1 is the first factor of the class number of $k(e^{2\pi i/p})$.

III. *The factor* B.

1. *Simplification of the expression for* B. Making use of (6) and (8), $X_2^{(s)}$ may be written

$$(14) \qquad X_2^{(s)} = -\frac{(\theta^s, r)}{m} \sum_{\lambda} \theta^{-s\gamma} \log \sin \frac{\lambda\pi}{m},$$

where $\gamma = \operatorname{ind} \lambda$ and $\lambda = 1, 2, \cdots, m-1$ except multiples of p. But since

$$\theta^{-s \operatorname{ind}(m-\lambda)} \log \sin \frac{(m-\lambda)\pi}{m} = \theta^{-s \operatorname{ind} \lambda} \log \sin \frac{\lambda\pi}{m},$$

we obtain

$$(15) \qquad X_2^{(s)} = -\frac{2(\theta^s, r)}{m} \sum_{\lambda} \theta^{-s\gamma} \log \sin \frac{\lambda\pi}{m},$$

for $\lambda = 1, 2, \cdots, (m-1)/2$ except multiples of p. From this we get after a few reductions the following expression for B:

$$(16) \qquad BD = \prod_{s} \psi(\theta^s),$$

where s takes all even values, less than μ, not divisible by p and

$$(17) \qquad \psi(\theta^s) = \sum_{i}^{0, \frac{\mu}{2}-1} \theta^{-si} \log \sin \frac{\lambda_i \pi}{m} \qquad (i = \operatorname{ind} \lambda_i).$$

We will now show how the product $\Pi \psi(\theta^s)$ can be expressed in the form of a determinant. We have

$$(18) \qquad \begin{aligned} \psi(\theta^s) &= \sum_{i}^{0, \frac{\mu}{2}-1} \theta^{-si} \log \frac{\sin \frac{\lambda_i \pi}{m}}{\sin \frac{\pi}{m}} + \log \sin \frac{\pi}{m} \sum_{i}^{0, \frac{\mu}{2}-1} \theta^{-si} \\ &= \sum_{i}^{0, \frac{\mu}{2}-1} \theta^{-si} \log \tau_i = \sum_{i}^{0, \frac{\mu}{2}-1} \theta^{-si} l_i, \end{aligned}$$

where $l_i = \log \tau_i$ and

$$\tau_i = \frac{\sin \frac{\lambda_i \pi}{m}}{\sin \frac{\pi}{m}} = r^{\frac{1-\lambda_i}{2}} \frac{1 - r^{\lambda_i}}{1 - r};$$

or, if we set $\theta^s = \theta_1$, then, since s is even, $\theta_1^{-\mu/2} = 1$ and

$$(19) \qquad \psi(\theta^s) = \sum_{i}^{0, \frac{\mu}{2}-1} \theta_1^{-i} l_i.$$

Now consider the system of equations:

$$
(20)\quad
\begin{aligned}
\psi(\theta_1) &= l_0 + l_1\theta_1^{-1} + \cdots + l_{\frac{\mu}{2}-1}\theta_1^{-\left(\frac{\mu}{2}-1\right)},\\
\theta_1^{-1}\psi(\theta_1) &= l_{\frac{\mu}{2}-1} + l_0\theta_1^{-1} + \cdots + l_{\frac{\mu}{2}-2}\theta_1^{-\left(\frac{\mu}{2}-1\right)},\\
&\cdots\cdots\cdots\cdots\\
\theta_1^{-\left(\frac{\mu}{2}-1\right)}\psi(\theta_1) &= l_1 + l_2\theta_1^{-1} + \cdots + l_0\theta_1^{-\left(\frac{\mu}{2}-1\right)};
\end{aligned}
$$

from which we get, by eliminating the powers of θ_1,

$$
(21)\quad
\begin{vmatrix}
l_0-\psi & l_1 & l_2 & \cdots & l_{\frac{\mu}{2}-1}\\
l_{\frac{\mu}{2}-1} & l_0-\psi & l_1 & \cdots & l_{\frac{\mu}{2}-2}\\
\cdot & \cdot & \cdot & \cdots & \cdot\\
l_1 & l_2 & l_3 & \cdots & l_0-\psi
\end{vmatrix} = 0.
$$

This equation is of degree $\mu/2$ and its roots are the quantities $\psi(\theta^s)$ for $s = 0, 2, 4, \cdots \mu - 2$. The product of these roots is then expressed by the following determinant whose absolute value we denote by T_1:

$$
(22)\quad
\prod_s \psi(\theta^s) = \pm
\begin{vmatrix}
l_0 & l_1 & \cdots & l_{\frac{\mu}{2}-1}\\
l_1 & l_2 & \cdots & l_0\\
\cdot & \cdot & \cdots & \cdot\\
l_{\frac{\mu}{2}-1} & l_0 & \cdots & l_{\frac{\mu}{2}-2}
\end{vmatrix} = \pm T_1.
$$

In a similar manner, making use of the fact that in this case $\theta_1^{-\mu'/2} = 1$, we obtain $\Pi_s \Psi(\theta^s)$ where s runs through the even multiples of p, from the following system of equations:

$$
(23)\quad
\begin{aligned}
\psi(\theta_1) &= L_0 + L_1\theta_1^{-1} + \cdots + L_{\frac{\mu'}{2}-1}\theta_1^{-\left(\frac{\mu'}{2}-1\right)},\\
\theta_1^{-1}\psi(\theta_1) &= L_{\frac{\mu'}{2}-1} + L_0\theta_1^{-1} + \cdots L_{\frac{\mu'}{2}-2}\theta_1^{-\left(\frac{\mu'}{2}-1\right)},\\
&\cdots\cdots\cdots\cdots\\
\theta_1^{-\left(\frac{\mu'}{2}-1\right)}\psi(\theta_1) &= L_1 + L_2\theta_1^{-1} + \cdots + L_0\theta_1^{-\left(\frac{\mu'}{2}-1\right)},
\end{aligned}
$$

where

$$
(24)\quad L_i = l_i + l_{i+\frac{\mu'}{2}} + \cdots + l_{i+(p-1)\frac{\mu'}{2}} \qquad \left(i=0, 1, \cdots, \frac{\mu'}{2}-1\right).
$$

Then as above $\Pi\,\psi(\theta^s)$ is expressed by the following determinant whose absolute value we denote by T_2:

$$(25)\qquad \prod_s \psi(\theta^s) = \pm \begin{vmatrix} L_0 & L_1 & \cdots & L_{\frac{\mu'}{2}-1} \\ L_1 & L_2 & \cdots & L_0 \\ \cdot & \cdot & \cdot & \cdot \\ L_{\frac{\mu'}{2}-1} & L_0 & \cdots & L_{\frac{\mu'}{2}-2} \end{vmatrix} = \pm T_2.$$

Hence from (22) and (25) we get

$$(26)\qquad BD = \frac{T_1}{T_2}.$$

From (24) it is seen that T_1 may be written

$$\pm T_1 = \begin{vmatrix} L_0 & \cdots & L_{\frac{\mu'}{2}-1} & l_{\frac{\mu'}{2}} & \cdots & l_{\frac{\mu}{2}-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ L_{\frac{\mu'}{2}-1} & \cdots & L_{\frac{\mu'}{2}-2} & l_{\mu'-1} & \cdots & l_{\frac{\mu'}{2}-2} \\ L_0 & \cdots & L_{\frac{\mu'}{2}-1} & l_{\mu'} & \cdots & l_{\frac{\mu'}{2}-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ L_0 & \cdots & L_{\frac{\mu'}{2}-1} & l_0 & \cdots & l_{\frac{\mu'}{2}-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ L_{\frac{\mu'}{2}-1} & \cdots & L_{\frac{\mu'}{2}-2} & l_{\frac{\mu'}{2}-1} & \cdots & l_{\frac{\mu}{2}-2} \end{vmatrix}.$$

Introducing the set of units

$$\tau'_i = r^{\frac{\lambda_i - \lambda_{i+\mu'/2}}{2}} \cdot \frac{1 - r^{\lambda_{i+\mu'/2}}}{1 - r^{\lambda_i}} = \frac{\sin \frac{\lambda_{i+\mu'/2} \cdot \pi}{m}}{\sin \frac{\lambda_i \pi}{m}},$$

with $l'_i = \log \tau'_i$, and making use of the fact that $l'_i = l_{i+\mu'/2} - l_i$, we get

$$T_1 = T_2 \cdot T_3,$$

where

$$T_3 = \pm \begin{vmatrix} l'_0 & l'_1 & \cdots & l'_{(p-1)\frac{\mu'}{2}-1} \\ l'_1 & l'_2 & \cdots & l'_{(p-1)\frac{\mu'}{2}} \\ \cdot & \cdot & \cdot & \cdot \\ l'_{(p-1)\frac{\mu'}{2}-1} & l'_{(p-1)\frac{\mu'}{2}} & \cdots & l'_{(p-1)\frac{\mu'}{2}-2} \end{vmatrix},$$

and therefore

$$BD = T_3.$$

2. *Normal Units.* In order to investigate the character of D, we have to consider the normal * units of $k(r+r^{-1})$. By a normal unit in $k(r+r^{-1})$ we understand a unit $\epsilon(r)$, different from ± 1, which satisfies

$$\epsilon(r)\epsilon(\rho r)\epsilon(\rho^2 r)\cdots\epsilon(\rho^{p-1}r)=\pm 1, \tag{29}$$

where $\rho = e^{2\pi i/p} = r^{p^{n-1}}$. This means that the relative norm of $\epsilon(r)$ in $k(r+r^{-1})$, with respect to $k(r'+r'^{-1})$, is ± 1.

It is evident that no unit in $k(r'+r'^{-1})$, which is also a unit in $k(r+r^{-1})$, can be a normal unit. The units τ_i', considered above, are normal units. For

$$\tau'_{i+a\cdot\mu'/2}(r)=\pm\tau'_i(\rho^{(-1)^a A_a}r),$$

where

$$g^{a\mu'/2}=(-1)^a + A_a m';$$

and, since $(-1)^a A_a$ runs through a complete residue system with respect to the modulus p when $a=0, 1, \cdots, p-1$, it follows that

$$\tau_i'(r)\tau_i'(\rho r)\cdots\tau_i'(\rho^{p-1}r)=\pm\prod_a^{0,p-1}\tau'_{i+a\cdot\mu'/2}(r)=\pm 1. \tag{30}$$

A system of $\nu=(p-1)\mu'/2$ normal units $\epsilon_0(r), \epsilon_1(r), \cdots, \epsilon_{\nu-1}(r)$ is said to be an *independent* system of normal units if

$$\begin{vmatrix} \log|\epsilon_0(r)| & \cdots & \log|\epsilon_{\nu-1}(r)| \\ \log|\epsilon_0(r^g)| & \cdots & \log|\epsilon_{\nu-1}(r^g)| \\ \cdot & \cdot & \cdot \\ \log|\epsilon_0(r^{g^{\nu-1}})| & \cdots & \log|\epsilon_{\nu-1}(r^{g^{\nu-1}})| \end{vmatrix} \neq 0;$$

and the absolute value of the determinant is called the *regulator* of the system $\epsilon_0, \epsilon_1, \cdots, \epsilon_{\nu-1}$. The units $\tau_0', \tau_1', \cdots, \tau_{\nu-1}'$ form such an independent system of normal units; for its regulator, being the determinant T_3, is evidently different from zero.

Now let $\epsilon_0, \epsilon_1, \cdots, \epsilon_{\nu-1}$ be an independent system of normal units and let $L_{i,\kappa}=\log|\epsilon_i(r^{g^\kappa})|$. Then, if $\epsilon'(r)$ be any normal unit and $L_\kappa'=\log|\epsilon'(r^{g^\kappa})|$ we can determine $\xi_0, \xi_1, \cdots, \xi_{\nu-1}$ from the system of equations

$$L_\kappa' = \xi_0 L_{0,\kappa} + \xi_1 L_{1,\kappa} + \cdots \xi_{\nu-1}L_{\nu-1,\kappa} \quad (\kappa=0, 1, \cdots, \nu-1). \tag{31}$$

That this equation also holds for any value of κ follows immediately from (29) and (30). By applying the same reasoning as for independent systems of units

* WEBER, *Lehrbuch der Algebra*, 2d ed., vol. 2, p. 806.

in any algebraic number field* we can prove that $\xi_0, \cdots, \xi_{\nu-1}$ are rational, and hence that there exists an independent system of normal units whose regulator is the least possible. Such a system we call a *fundamental system* of normal units, and any normal unit can be written in the form

$$\pm \epsilon_0^{m_0} \epsilon_1^{m_1} \cdots \epsilon_{\nu-1}^{m_{\nu-1}},$$

where $\epsilon_0, \epsilon_1, \cdots, \epsilon_{\nu-1}$ are a fundamental system and $m_0, m_1, \cdots m_{\nu-1}$ are integers. The regulator of a fundamental normal system is, therefore, a divisor of the regulator of any (independent) system of normal units.

3. *Study of D.* Let $\epsilon_1, \epsilon_2, \cdots, \epsilon_{\mu/2-1}$ be a fundamental system of units in $k(r+r^{-1})$, with the conjugate logarithms $\lambda_{1,\kappa}, \lambda_{2,\kappa}, \cdots, \lambda_{\mu/2-1,\kappa}$ and regulator E, and let $\epsilon_1', \epsilon_2', \cdots, \epsilon_{\mu'/2-1}$ be a fundamental system of units in $k(r'+r'^{-1})$ with the conjugate logarithms $\lambda'_{1,\kappa}, \lambda'_{2,\kappa}, \cdots, \lambda'_{\mu'/2-1,\kappa}$ and regulator E'. Also let $\omega_0, \omega_1, \cdots, \omega_{\nu-1}$ be a fundamental system of normal units in $k(r+r^{-1})$ with the conjugate logarithms $L_{0,\kappa}, L_{1,\kappa}, \cdots, L_{\nu-1,\kappa}$ and regulator T_0. Then the units

$$\epsilon_1', \epsilon_2', \cdots, \epsilon'_{\frac{\mu'}{2}-1}, \omega_0, \omega_1, \cdots, \omega_{\nu-1} \tag{32}$$

form an independent system of units in $k(r+r^{-1})$. For since

$$\lambda'_{i,\kappa+\mu'/2} = \lambda'_{i,\kappa}$$

and

$$L_{i,0} + L_{i,\mu'/2} + \cdots + L_{i,(p-1)\mu'/2} = 0,$$

we get for the regulator R of the system (32),

$$R = p^{\mu'/2-1} E' T_0, \tag{33}$$

which shows that $R \neq 0$ and hence that (32) form an independent system of units.

We can then determine rational numbers $m_{i,\kappa}$ and $M_{i,\kappa}$ such that

$$p\lambda_{i,\kappa} = m_{1,i}\lambda'_{1,\kappa} + \cdots + m_{\mu'/2-1,i}\lambda'_{\mu'/2-1,\kappa} + M_{0,i}L_{0,\kappa} + \cdots + M_{\nu-1,i}L_{\nu-1,\kappa} \tag{34}$$

$$\left(\kappa = 0, \cdots, \frac{\mu}{2}-2;\ i = 1, \cdots, \frac{\mu}{2}-1\right).$$

We now wish to prove that $m_{i,\kappa}$ and $M_{i,\kappa}$ are integers. From (34) we get

$$\lambda_{i,\kappa} + \lambda_{i,\kappa+\mu'/2} + \cdots + \lambda_{i+(p-1)\mu'/2} = m_{1,i}\lambda'_{1,\kappa} + \cdots + m_{\mu'/2-1,i}\lambda'_{\mu'/2-1,\kappa},$$

and, since

$$\epsilon_i(r)\epsilon_i(r^{g^{\mu'/2}}) \cdots \epsilon_i(r^{g^{(p-1)\mu'/2}})$$

* WEBER, *Lehrbuch der Algebra*, 2d ed , vol. 2, § 191.

is a unit in $k(r' + r'^{-1})$, it follows that $m_{1,i}, m_{2,i}, \cdots, m_{\mu'/2-1,i}$ are integers.

We also obtain from (34)

$$\lambda_{i,\kappa} + \cdots + \lambda_{i,\kappa+(p-1)\mu'/2} - p\lambda_{i,\kappa} = -M_{0,i}L_{0,\kappa} - \cdots - M_{\nu-1,\kappa}L_{\nu-1,\kappa},$$

and since
$$\frac{\epsilon_i(r)\epsilon_i(r^{g^{\mu'}})\cdots\epsilon_i(r^{g^{(p-1)\mu'}})}{[\epsilon_i(r)]^p}$$

is a normal unit, it follows that $M_{0,i}, \cdots, M_{\nu-1,i}$ are integers.

From (34)

$$E = p^{-\frac{\mu}{2}+1}RM, \tag{35}$$

where M is the determinant of the coefficients $m_{i,\kappa}$ and $M_{i,\kappa}$ and hence an integer. Foamulæ (33) and (35) then give

$$D = p^{(\mu'-\mu)/2}MT_0. \tag{36}$$

We now propose to investigate the character of M. To do this let

$$\lambda'_{i,\kappa} = n_{1,i}\lambda_{1,\kappa} + \cdots + n_{\frac{\mu}{2}-1,i}\lambda_{\frac{\mu}{2}-1,\kappa} \quad \left(i=1, 2, \cdots, \frac{\mu'}{2}-1\right),$$
$$L_{i,\kappa} = N_{1,i}\lambda_{1,\kappa} + \cdots + N_{\frac{\mu}{2}-1,i}\lambda_{\frac{\mu}{2}-1,\kappa} \quad (i=0, 1, \cdots, \nu-1),$$

where $n_{i,\kappa}$ and $N_{i,\kappa}$ are integers. Denoting by N the determinant of the coefficients $n_{i,\kappa}$ and $N_{i,\kappa}$, we get

$$R = EN$$

and hence

$$MN = p^{\frac{\mu}{2}-1}, \tag{37}$$

i. e., M and N are both powers of p. To determine the power of p by which M is divisible, we determine a system of integers $a_1, a_2, \cdots, a_{\mu/2-1}$ without common divisor satisfying the system of equations

$$a_1 m_{i,1} + a_2 m_{i,2} + \cdots + a_{\mu/2-1} m_{i,\mu/2-1} = 0 \quad \left(i=1, 2, \cdots, \frac{\mu'}{2}-1\right). \tag{38}$$

Let

$$a_1 M_{i,1} + a_2 M_{i,2} + \cdots + a_{\mu/2-1} M_{i,\mu/2-1} = \xi_i \quad (i=0, 1, \cdots, \nu-1),$$

and we have

$$p\sum_i^{1,\frac{\mu}{2}-1} a_i\lambda_{i,\kappa} = \xi_0 L_{0,\kappa} + \xi_1 L_{1,\kappa} + \cdots + \xi_{\nu-1} L_{\nu-1,\kappa},$$

from which

$$\sum_i^{1,\frac{\mu}{2}-1} a_i\lambda_{i,\kappa} + \sum_i^{1,\frac{\mu}{2}-1} a_i\lambda_{i,\kappa+\mu'/2} + \cdots + \sum_i^{1,\frac{\mu}{2}-1} a_i\lambda_{i,\kappa+(p-1)\mu'/2} = 0.$$

Hence we infer that $\epsilon_1^{a_1}\epsilon_2^{a_2}\cdots\epsilon_{\mu/2-1}^{a_{\mu/2-1}}$ is a normal unit and that $\xi_0, \xi_1, \cdots, \xi_{\nu-1}$ are integers divisible by p. It is then very easy to show, by applying the same reasoning as in the case $p=2$,* that M is divisible by $p^{(p-1)\mu'/2}$. Hence if we set

$$M = p^{(p-1)\frac{\mu'}{2}+\sigma},$$

we obtain from (36)

(39)
$$D = p^{\sigma} T_0 \qquad \left(\leqq \sigma \leqq \frac{\mu'}{2} - 1\right).$$

From (28) we then have

$$B = p^{-\sigma}\frac{T_3}{T_0}.$$

where T_3/T_0 is an integer, T_0 being the regulator of a fundamental system of normal units.

If we now denote by B_n the factor B corresponding to $m=p^n$, we get the following expression for the second factor of the class number of $k(r)$:

(41)
$$h_1 = h_1'' B_2 B_3 \cdots B_n,$$

where h_1'' is the class number of $k(e^{2\pi i/p} + e^{-2\pi i/p})$.

Comparing our results with those obtained by WEBER for $p=2$, we notice that, for all values of p, A is an integer and $B = p^{-\sigma} T_3/T_0$, where T_3/T_0 is an integer. For $p=2$, WEBER proves that $\sigma = 0$ and that both A and T_3/T_0 and hence B are odd numbers. When p is an odd prime, the question whether A and T_3/T_0 are divisible by p or not, and what the value of σ is, remains unsettled. The writer, however, hopes to be able to come back to this question in a following paper.

PURDUE UNIVERSITY,
August, 1902.

* WEBER, *Lehrbuch der Algebra*, 2d ed., vol. 2, pp. 816, 817.