ON THE FORMATION OF THE DERIVATIVES OF THE LUNAR COÖRDINATES WITH RESPECT TO THE ELEMENTS*

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In the determination of the action of the planets on the moon, it is usual to use the method of the variation of arbitrary constants. This method requires the knowledge of the derivatives of the lunar coördinates with respect to the angular elements ϵ, π, θ the epochs of mean longitude, perigee and node—and also with respect to the other three elements n, e, γ—the mean motion, eccentricity and sine of the orbital inclination. If we have a theory in which the coördinates, so far as the actions of the sun and moon only are concerned, are expressed literally in terms of these six elements, it is a simple matter to find the derivatives.

But this is practically not the case. The convergence of many of the coefficients when expressed in powers of $m = n/n'$—the ratio of the mean motions—is so slow that a literal theory with the required accuracy seems almost impossible on account of the labor required for its development; the slowness of convergence does not occur as far as powers of e, γ are concerned. The theory of Hansen is entirely numerical and therefore only the derivatives with respect to the angular elements can be obtained from it. The theory † which I hope shortly to bring to a conclusion is semi-numerical, i. e., the numerical value of m is inserted, the other constants being left in a literal form. ‡

In considering how this theory was to be adapted so as to be of use when considering planetary action, the only difficulty was, therefore, the formation of the derivatives with respect to n. It is this difficulty which has been solved in the following pages. The semi-numerical theory once completed, it is shown that the derivatives with respect to n can be made to depend on quadratures with respect to the time. The formulæ are put into a form ready for computation and it is shown how a simple transformation will permit us to use the advantages of canonical sets of constants without being hampered by their disadvantages for purposes of calculation.

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‡ A detailed statement will be found in arts. 4–6.

§ (i) *Transformation of the integrals of the equations of variations.*

1. Let x, y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$, be the coördinates and velocities of a particle moving under the action of a force function F, so that

$$(1) \qquad \frac{d^2x}{dt^2} = \frac{\partial F}{\partial x}, \qquad \frac{d^2y}{dt^2} = \frac{\partial F}{\partial y}, \qquad \frac{d^2z}{dt^2} = \frac{\partial F}{\partial z},$$

the axes being fixed. Denote by α_p ($p = 1, 2, \cdots, 6$) the arbitrary constants of this motion and put

$$x_p = \frac{dx}{d\alpha_p}, \qquad \dot{x}_p = \frac{d^2x}{dt d\alpha_p} = \frac{d^2x}{d\alpha_p dt} = \frac{d\dot{x}}{d\alpha_p}, \cdots, \cdots,$$

where the solution is supposed to have been found and expressed in terms of the arbitrary constants and the time. It is more convenient at the outset to take the α_p to be *constants* and not some of them to be variables of the form $b_p t + \beta_p$ as is usually done in the applications; certain changes to the latter form will be made later.

The equations of variations are

$$(2) \qquad \ddot{x}_p = \frac{\partial^2 F}{\partial x^2} x_p + \frac{\partial^2 F}{\partial x \partial y} y_p + \frac{\partial^2 F}{\partial x \partial z} z_p, \qquad \ddot{y}_p = \cdots, \qquad \ddot{z}_p = \cdots.$$

They possess the 15 integrals,

$$(3) \quad (p, q) = \dot{x}_p x_q - \dot{x}_q x_p + \dot{y}_p y_q - \dot{y}_q y_p + \dot{z}_p z_q - \dot{z}_q z_p = C_{pq} \quad (p, q = 1, 2, \cdots, 6),$$

where C_{pq} is a constant and $C_{pq} = - C_{qp}$, $C_{pp} \equiv 0$.

Let the solution be such that x, y, z each consists of sines or cosines of sums of multiples of the angles

$$b_j t + \alpha_j \qquad (j = 4, 5, 6),$$

where b_j and the coefficients depend only on the arbitraries α_1, α_2, α_3. The solution may also contain other known angles and constants present in F; it is unnecessary to specify these here. Then it is well known that

$$(4) \quad C_{ii'} = 0, \quad C_{jj'} = 0, \quad C_{ij} = \frac{dc_{j-3}}{d\alpha_i}, \quad b_j = - \frac{dB}{dc_{j-3}} \quad (i, i' = 1, 2, 3; j, j' = 4, 5, 6),$$

where c_1, c_2, c_3 are functions of α_1, α_2, α_3 and the known constants, and B is a constant expressed in terms of c_1, c_2, c_3 (instead of α_1, α_2, α_3) and the known constants.

2. Consider the determinant of the sixth order

$$\Delta = | \dot{x}_p, x_p, \dot{y}_p, y_p, \dot{z}_p, z_p | \qquad (p = 1, 2, \cdots, 6),$$

and form $d\Delta / dt$. The result contains six determinants obtained by differentiating the elements in each of the six columns. The determinants formed by differentiating the elements in the second, fourth and sixth columns will vanish owing to their possession of two identical columns. Those formed by differentiating the elements in the first, third and fifth columns will also vanish when we substitute for $\ddot{x}_p$, $\ddot{y}_p$, $\ddot{z}_p$ their values from equations (2). Hence $d\Delta/dt = 0$ and

$$\Delta = \text{constant} = K.$$

To find K write Δ in the form

$$\Delta = |x_p, -\dot{x}_p, y_p, -\dot{y}_p, z_p, -\dot{z}_p|,$$

and multiply the two forms for Δ by the ordinary rule for the multiplication of determinants. Using the equations (3) it is evident that we obtain

$$\Delta^2 = |C_{pq}| \qquad (p, q = 1, 2, \cdots, 6),$$

Using (4), this gives

$$\Delta = \begin{vmatrix} 0 & 0 & 0 & C_{14} & C_{15} & C_{16} \\ 0 & 0 & 0 & C_{24} & C_{25} & C_{26} \\ 0 & 0 & 0 & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & 0 & 0 & 0 \\ C_{51} & C_{52} & C_{53} & 0 & 0 & 0 \\ C_{61} & C_{62} & C_{63} & 0 & 0 & 0 \end{vmatrix}^{\frac{1}{2}} = \begin{vmatrix} C_{14} & C_{15} & C_{16} \\ C_{24} & C_{25} & C_{26} \\ C_{34} & C_{35} & C_{36} \end{vmatrix},$$

a comparison of terms showing that the positive square root is to be taken.

Let k_{ij} denote the first minor of C_{ij} in this determinant with the sign so taken that a cyclical interchange of the first three or of the second three suffixes does not require a change of sign in the minor. Then

(5) $\qquad \Sigma_j k_{ij} C_{i'j} = \Delta \quad \text{or} \quad 0 \qquad \text{according as} \qquad i' = i \quad \text{or} \quad \neq i.$

3. Next, denote by Δ_p the first minors of the elements in the first column of Δ and by Δ_{pq} the second minors of those in the first two columns. Then

$$\begin{aligned} \dot{x}_4 \Delta_1 &= |\dot{x}_4 x_p, \dot{y}_p, y_p, \dot{z}_p, z_p| && (p = 2, 3, \cdots, 6) \\ &= |x_4 \dot{x}_p - C_{4p}, \dot{y}_p, y_p, \dot{z}_p, z_p| && \text{by (3)} \\ &= x_4 |\dot{x}_p, \dot{y}_p, y_p, \dot{z}_p, z_p| - C_{42}\Delta_{42} - C_{43}\Delta_{43} && \text{by (4).} \end{aligned}$$

The first determinant in the right hand member is, as before, easily seen to be equal to $d\Delta_1/dt$. Hence, writing down the two similar equations, we have

$$\begin{aligned} \dot{x}_4 \Delta_1 - x_4 \dot{\Delta}_1 &= -C_{42}\Delta_{42} - C_{43}\Delta_{43}, \\ \dot{x}_5 \Delta_1 - x_5 \dot{\Delta}_1 &= -C_{52}\Delta_{52} - C_{53}\Delta_{53}, \\ \dot{x}_6 \Delta_1 - x_6 \dot{\Delta}_1 &= -C_{62}\Delta_{62} - C_{63}\Delta_{63}. \end{aligned}$$

Multiply these equations by k_{14}, k_{15}, k_{16}, respectively and add. The right hand member vanishes by (5) and the left hand member becomes integrable, giving, since we have similar forms for Δ_2, Δ_3,

$$\Delta_i = (k_{i4}x_4 + k_{i5}x_5 + k_{i6}x_6)A_i \qquad (i=1, 2, 3),$$

where A_i is a constant. Similarly

$$\Delta_j = (k_{1j}x_1 + k_{2j}x_2 + k_{3j}x_3)A_j \qquad (j=4, 5, 6).$$

Put

$$X_i = k_{i4}x_4 + k_{i5}x_5 + k_{i6}x_6, \qquad X_j = k_{1j}x_1 + k_{2j}x_2 + k_{3j}x_3.$$

Then

$$\Delta = \Sigma_i A_i \dot{x}_i X_i - \Sigma_j A_j \dot{x}_j X_j \quad (i=1, 2, 3; j=4, 5, 6).$$

To determine the constants A, we notice that if any two suffixes be interchanged in the determinant Δ, it merely changes sign; hence $A_i = A_j = 1$, and

$$\Delta = \Sigma_i \dot{x}_i X_i - \Sigma_j \dot{x}_j X_j,$$

which may also be written

$$\Sigma_i (\dot{x}_i X_i - x_i \dot{X}_i) = K \qquad (i=1, 2, 3).$$

Again, by the properties of determinants,

$$0 = \Sigma x_p \Delta_p = \Sigma y_p \Delta_p = \Sigma \dot{y}_p \Delta_p = \Sigma z_p \Delta_p = \Sigma \dot{z}_p \Delta_p \quad (p=1, 2, \cdots, 6),$$

so that, if Y_i, Z_i be defined with reference to y_j, z_j in the same way as X_i was with reference to x_j, and if we write down the corresponding expressions for Δ obtained by interchanging x, y, z, we have the following set of 15 equations:

$$(6) \qquad K = \Sigma(\dot{x}_i X_i - x_i \dot{X}_i) = \Sigma(\dot{y}_i Y_i - y_i \dot{Y}_i) = \Sigma(\dot{z}_i Z_i - z_i \dot{Z}_i),$$

$$(7) \qquad \begin{aligned} 0 &= \Sigma(x_i Y_i - y_i X_i) = \Sigma(x_i Z_i - z_i X_i) \\ &= \Sigma(\dot{x}_i Y_i - y_i \dot{X}_i) = \Sigma(\dot{x}_i Z_i - z_i \dot{X}_i), \end{aligned}$$

with the eight other equations formed by a cyclical interchange between x, y, z; the summations refer in all cases to the values $i = 1, 2, 3$. Of these fifteen equations, not more than ten are easily seen to be independent of one another, whatever be the values given to x_i, y_i, z_i, X_i, Y_i, Z_i.

§ (ii) *Statement of the problem and its formal solution.*

4. The main object of this paper is to determine the derivatives of the coördinates of the moon with respect to the mean motion n, it being supposed that the derivatives with respect to the other arbitraries have been obtained. The results found above are true whenever the coördinates satisfy the specified conditions and the same may be said of most of those which follow; but it will be

convenient to put them into a form which will be applicable directly to the lunar theory, since they are required for the determination of the action of the planets on the moon.

It is well known that the coördinates of the moon referred to fixed axes are expressible by means of the five angles

$$(8)\quad nt+\epsilon, \qquad n't+\epsilon', \qquad cnt+\epsilon-\pi, \qquad gnt+\epsilon-\theta, \qquad n't+\epsilon'-\pi',$$

where n, n' are the mean motions of the sun and moon; $(1-c)n$, $(1-g)n$, those of the perigee and node of the moon; ϵ, π, θ, the usual angular constants of integration referring to the moon; ϵ', π', the known angular constants of the sun's motion. Also, c, g and the coefficients are expansible in powers of

$$\frac{n}{n'}=m, \quad e, \quad \gamma, \quad e', \quad \frac{a}{a'}=\alpha,$$

where e, γ are "eccentricity" and "inclination" constants of integration referring to the moon; e' (eccentricity), a' (mean distance), known constants of the sun's motion; $n^2a^3=$ sum of the masses of the earth and moon; $n'^2a'^3=$ sum of the masses of the earth, moon and sun. The set of arbitrary constants α_1, α_2, α_3 is taken to be n, e, γ and the set α_4, α_5, α_6 to be ϵ, π, θ; n, $n(1-c)$, $n(1-g)$ are the motions of the corresponding angles.

If the coördinates be referred to axes of which one points to the mean place of the sun or moon, the coördinates are functions of *four* angles only, the first two of (8) being replaced by their difference.

5. When it becomes a question of finding the derivatives of the coördinates with respect to the arbitraries, there is no difficulty as far as ϵ, π, θ are concerned, since these only occur under the signs, sine and cosines, whether the theory has been worked out literally or with numerical values for m, e, γ, e', α. But this is not the case with n, e, γ: a literal theory like DELAUNAY's is necessary. But the use of such a theory gives rise to a practical difficulty. Every coefficient of a periodic term is of the form $aA\lambda$ where λ is the product of the highest positive powers of e, γ, e', α which are factors of the coefficient, and A is expansible in powers of m, e^2, γ^2, e'^2, α^2. If we arrange A according to powers of the last four parameters, the convergence is sufficiently rapid for practical purposes and if, further, the coefficient (a function of m only) of each of these powers is determined with sufficient accuracy, there is no difficulty in finding the derivatives with respect to e, γ. It is quite different when we arrange according to powers of m. In the first place, the rate of convergence is in many cases very slow, corresponding frequently to that of a geometric series with a common ratio $\frac{1}{2}$, and the rate of convergence of the earlier terms is less in the series formed by taking the derivatives with respect to n or

m. The most accurate literal theory is that of DELAUNAY and this is insufficient for modern demands. Moreover, it may be said to be almost a practical impossibility to calculate a literal theory to a sufficient degree of approximation.

6. It was with these facts in view that I set out to obtain a semi-numerical theory in which the parameters e, e', γ, α were left in their literal form, while the numerical value of m (which is known from observation with very high accuracy) was used from the start. The values of the coördinates have been published* as far as the fourth order inclusive in these four parameters and this order is probably sufficiently high to obtain the derivatives of the coördinates with respect to e, γ with the degree of accuracy required to obtain the action of the planets on the moon. But we cannot, of course, directly obtain from it the derivatives with respect to m. It might be possible to use DELAUNAY's theory in conjunction with the numerical results by a somewhat empirical method which I adopted in an earlier paper† and which served well the purpose for which it was intended. But it appears doubtful whether it would be sufficiently accurate for the determination of all the planetary inequalities. I have therefore considered this problem:

Given the derivatives of the various functions with respect to ϵ, π, θ, e, γ, *to find those with respect to n from a theory in which the numerical value of m has been substituted.*

The possibility of solving this problem practically arose from considering a subsidiary result which I gave in an earlier paper,‡ where it was solved so far as the terms independent of e, e', γ, α were concerned; the object and the method were, however, quite different from those of this paper. In the following sections the problem will first be solved in the form which naturally presents itself and afterwards the results will be so transformed that certain difficulties arising from the infinities of the integrals are avoided and that the results of the theory, above referred to, can be directly used.

7. It has just been seen that we require to determine one set of the derivatives in terms of the other five: let this set be x_1, y_1, z_1. For this purpose we have the fifteen equations (6), (7); we have seen, however, that they are not all independent. Let us attempt to determine x_1 without an integration from

$$\Sigma(\dot{x}_i X_i - x_i \dot{X}_i) = K, \quad \Sigma(x_i Y_i - y_i X_i) = 0, \quad \Sigma(\dot{x}_i Y_i - y_i \dot{X}_i) = 0,$$

of which the terms containing the suffix 1 are

$$\dot{x}_1 X_1 - x_1 \dot{X}_1, \qquad x_1 Y_1 - y_1 X_1, \qquad \dot{x}_1 Y_1 - y_1 \dot{X}_1,$$

* T. M. M., parts I, II, III.

† *On the theoretical values of the secular accelerations in the lunar theory,* Monthly Notices of the Royal Astronomical Society, vol. 57 (1897), p. 346.

‡ *On the solution of a pair of simultaneous differential equations,* etc., Memoirs of the Cambridge Philosophical Society, vol. 18 (Stokes Memorial, 1900), pp. 94–106.

and the capital letters contain the suffices 4, 5, 6 only. The Jacobian of these with respect to x_1, x_1, y_1 is zero; a similar result appears to hold if we attempt to combine any of the equations. Hence we shall use the first two of (6) to determine x_1, y_1, and one of the others to determine z_1. We find

$$(9) \qquad x_1 = X_1 \int \frac{K - \dot{x}_2 X_2 + x_2 \dot{X}_2 - \dot{x}_3 X_3 + x_3 \dot{X}_3}{X_1^2} dt,$$

with a similar equation for y_1; no added constant is necessary since x_1 cannot contain terms of the form AX_1, that is, of the form Ax_j, for the x_j consist of sines and x_1 of cosines. It will be sufficient to consider the equation for x_1 since that for y_1 is exactly similar; the equation for z_1 is treated separately.

8. There are three points to be considered in connection with the equation (9). First, x_2, x_3, $\dot{x}_2$, $\dot{x}_3$ give rise to terms containing the time as a factor owing to the presence of e, γ in c, g, that is, in the arguments of the periodic terms; practical convenience requires this to be avoided. Second, X_2, X_3 contain the derivatives dc_i/dn owing to their presence in certain of the coefficients k_{ij}; these must either be calculated or eliminated. Third, the principal term of X_1, which appears as a denominator under the integral sign, is the derivative of $a \cos(nt + \epsilon)$ with respect to n and it may therefore vanish. In order to show how the first two difficulties may be avoided I shall develop the results with a canonical system of constants and then adapt them for practical use by the adoption of a certain semi-canonical system. The third difficulty easily disappears when complex coördinates are used.

9. Instead of $\alpha_1 = n$, $\alpha_2 = e$, $\alpha_3 = \gamma$ as the first set of arbitrary constants, let us take $\alpha_1 = c_1$, $\alpha_2 = c_2$, $\alpha_3 = c_3$, and suppose that it be required to find the derivatives with respect to c_1, those with respect to c_2, c_3, ϵ, π, θ being supposed known. We have

$$K = 1, \qquad k_{i,i+3} = 1, \qquad k_{ij} = 0 \qquad (j \neq i+3),$$

$$X_i = x_{i+3}, \qquad Y_i = y_{i+3}, \qquad Z_i = z_{i+3};$$

and the first of equations (6) becomes

$$(10) \qquad \Sigma_i (\dot{x}_i x_{i+3} - x_i \dot{x}_{i+3}) = 1 \qquad (i = 1, 2, 3).$$

Denote the three angles by $w_j = b_j t + \beta_j$ ($j = 4, 5, 6$). Then

$$\frac{dx}{dc_i} = \frac{\partial x}{\partial c_i} + t \Sigma_j \frac{db_j}{dc_i} x_j,$$

$$\frac{d\dot{x}}{dc_i} = \frac{d}{dt}\frac{\partial x}{\partial c_i} + \left(1 + t\frac{d}{dt}\right)\left(\Sigma_j \frac{db_j}{dc_i} x_j\right) = \frac{\partial}{\partial c_i}\frac{dx}{dt} + t\frac{d}{dt}\left(\Sigma_j \frac{db_j}{dc_i} x_i\right),$$

where $\partial/\partial c_i$ denotes that the derivative is taken with respect to c_i, only in so far as c_i occurs in the *coefficients* of the periodic terms of the function on which it operates.

Substitute in (10) and equate separately to zero the terms which do and do not contain t as a factor. The latter give the well known results

$$\frac{db_j}{dc_i} = \frac{db_{i+3}}{dc_{j-3}} \qquad (i=1, 2, 3; j=4, 5, 6).$$

The former give

$$\Sigma_i \left(x_{i+3} \frac{\partial \dot{x}}{\partial c_i} - \dot{x}_{i+3} \frac{\partial x}{\partial c_i} \right) = 1,$$

or

$$\Sigma_i \left(x_{i+3} \frac{d}{dt} x_i - \dot{x}_{i+3} x_i \right) + \Sigma_j \Sigma_i \frac{db_j}{dc_i} x_j x_{i+3} = 1,$$

according as we use the second or first form of $d\dot{x}/dc_i$. It is necessary to use the first form for $i = 1$ on account of the integration with respect to t and it is better to use the second form for $i = 2, 3$. The equation then becomes

$$x_4 \frac{d}{dt} \frac{\partial x}{\partial c_1} - \dot{x}_4 \frac{\partial x}{\partial c_1} = 1 - x_5 \frac{\partial \dot{x}}{\partial c_2} + \dot{x}_5 \frac{\partial x}{\partial c_2} - x_6 \frac{\partial \dot{x}}{\partial c_3} + \dot{x}_6 \frac{\partial x}{\partial c_3} - x_4 \Sigma_j \frac{db_j}{dc_1} x_j$$

$$= Q - x_4^2 \frac{db_4}{dc_1},$$

where the definition of Q is evident.

Dividing by x_4^2, putting $db_5/dc_1 = db_4/dc_2$, $db_6/dc_1 = db_4/dc_3$, and integrating, we obtain

$$\frac{\partial x}{\partial c_1} = x_4 \int \left(\frac{Q}{x_4^2} - \frac{db_4}{dc_1} \right) dt.$$

The only derivative with respect to c_1 in the right hand member is db_4/dc_1. But since $\partial x/\partial c_1$ can not contain any term factored by t, the constant term under the integral sign must vanish. This fact gives

$$\frac{db_4}{dc_1} = \text{const. term in the expansion of } \frac{Q}{x_4^2},$$

and the determination of $\partial x/\partial c_1$ is complete.

10. The objection to leaving the results in this form for practical application is the fact that the formation of the coördinates as functions of c_1, c_2, c_3 is very troublesome, especially as far as c_1 is concerned. But I shall show later that there is no objection to the use of c_2, c_3 instead of e, γ. Hence it will be convenient to develop the results for the *semi-canonical* system $b_4 (= n)$, c_2, c_3, so that c_1 is considered to be a function of the independent constants n, c_2, c_3.

With this semi-canonical system, we find

$$C_{14} = \frac{dc_1}{dn}, \quad C_{24} = \frac{dc_2}{dn}, \quad C_{34} = \frac{dc_3}{dn}, \quad \frac{db_5}{dn} = -\frac{dc_1}{dc_2}, \quad \frac{db_6}{dn} = -\frac{dc_1}{dc_3},$$

$$1 = C_{25} = C_{36} = k_{14}, \qquad 0 = C_{15} = C_{35} = C_{16} = C_{26} = k_{24} = k_{26} = k_{34} = k_{35},$$

$$k_{25} = k_{36} = K = \frac{dc_1}{dn}, \qquad k_{15} = -\frac{dc_1}{dc_2}, \qquad k_{16} = -\frac{dc_1}{dc_3};$$

$$X_1 = x_4 - \frac{dc_1}{dc_2}x_5 - \frac{dc_1}{dc_3}x_6, \qquad X_2 = \frac{dc_1}{dn}x_5, \qquad X_3 = \frac{dc_1}{dn}x_6;$$

$$\frac{dx}{dn} = \frac{\partial x}{\partial n} + t\left(x_4 + \frac{db_5}{dn}x_5 + \frac{db_6}{dn}x_6\right) = \frac{\partial x}{\partial n} + tX_1,$$

$$\frac{d\dot{x}}{dn} = \frac{d}{dt}\frac{\partial x}{\partial n} + X_1 + t\dot{X}_1,$$

and the new form of equation (9) is

$$\frac{\partial x}{\partial n} = X_1 \int \left(\frac{dc_1}{dn}\frac{Q'}{X_1^2} - 1\right) dt \tag{11}$$

where X_1 has the value just given and

$$Q' = 1 - x_5 \frac{\partial \dot{x}}{\partial c_2} + \dot{x}_5 \frac{\partial x}{\partial c_2} - x_6 \frac{\partial \dot{x}}{\partial c_3} + \dot{x}_6 \frac{\partial x}{\partial c_3}.$$

There are no derivatives with respect to n in (11) except dc_1/dn and this is determined by the vanishing of the constant term under the integral sign. It is remarkable that the expression for Q' which arises with the semi-canonical system is shorter than that for Q which arises with the canonical system. The equation (11) might have been obtained from (10) by direct transformation of the set c_1, c_2, c_3 to the set n, c_2, c_3.

11. There are two determinations to be made. The first is to find the values of c_1, c_2, c_3 from a theory expressed in terms of n, e, γ and the second is to express this theory as well as c_1 in terms of n_1, c_2, c_3.

For the first, I have shown* that

$$c_i = \text{const. term in } \dot{x}x_{i+3} + \dot{y}y_{i+3} + \dot{z}z_{i+3},$$

which permits of the calculation of c_i with ease. We find

$$c_1 = \sqrt{a\mu}P_1(e^2, \gamma^2), \quad c_2 = e^2\sqrt{a\mu}P_2(e^2, \gamma^2), \quad c_3 = \gamma^2\sqrt{a\mu}P_3(e^2, \gamma^2),$$

* *On certain Properties of the Mean Motions*, etc., Proceedings of the London Mathematical Society, vol. 28 (1896), p. 150. The proof involves the supposition $a/a' = 0$. That the result holds when $a/a' \neq 0$ will be shown in a paper to appear later in the Transactions.

where P_1, P_2, P_3 are positive power series in e^2, γ^2, and μ is the sum of the masses of the earth and moon. It is understood that the numerical values of the other constants may have been substituted; it is unnecessary in any case to specify them. From the second and third of these equations, the values of e^2, γ^2 are quickly deduced as power series in $c_2/\sqrt{a\mu}$, $c_3/\sqrt{a\mu}$, particularly as it will probably never be necessary to go beyond the fourth powers of e, γ. Hence c_1 is easily expressible in powers of the same quantities and likewise b_5, b_6.

The coefficients contain odd powers of e, γ in general and therefore will involve powers of the square roots of the new parameters. The final transformation is therefore to put

$$c_2 = \mathrm{E}^2\sqrt{a\mu} = \mathrm{E}^2 a^2 n, \qquad c_3 = \Gamma^2\sqrt{a\mu} = \Gamma^2 a^2 n,$$

so that in Q or Q' we can put

$$\frac{\partial}{\partial c_2} = \frac{1}{2a^2 n\mathrm{E}}\frac{\partial}{\partial \mathrm{E}}, \qquad \frac{\partial}{\partial c_3} = \frac{1}{2a^2 n\Gamma}\frac{\partial}{\partial \Gamma},$$

where the coördinates are transformed by the equations

$$e = \mathrm{E}P_2(\mathrm{E}^2, \Gamma^2), \qquad \nu = \Gamma P_3(\mathrm{E}^2, \Gamma^2).$$

It is to be noticed that the value of z_1 may be obtained without the necessity of an integration from either of the equations

$$\Sigma_i(x_i Z_i - z_i X_i) = 0 = \Sigma_i(y_i Z_i - z_i Y_i), \tag{12}$$

when x_1, y_1 have been obtained.

12. We have next to transform to complex coördinates to avoid the occurrence of vanishing denominators. Put

$$u = x + y\sqrt{-1}, \qquad u_i = x_i + y_i\sqrt{-1}, \qquad U_i = X_i + Y_i\sqrt{-1},$$
$$s = x - y\sqrt{-1}, \qquad s_i = x_i - y_i\sqrt{-1}, \qquad S_i = X_i - Y_i\sqrt{-1}.$$

From the equations (6), (7) we easily obtain

$$0 = \Sigma_i(\dot{u}_i U_i - u_i \dot{U}_i) = \Sigma_i(u_i Z_i - z_i U_i), \tag{13}$$

$$\Sigma_i(\dot{u}_i S_i - s_i \dot{U}_i) = 2K, \qquad \Sigma_i(u_i S_i - s_i U_i) = \Sigma_i(\dot{u}_i Z_i - z_i U_i) = 0, \tag{14}$$

with the corresponding equations formed by changing the sign of $\sqrt{-1}$. The first of (13) determines u_1, with the denominator U_1^2 which does not vanish *; the second equation then gives z_1. The other equations may be used for verification purposes.

* Since X_1, Y_1 never vanish simultaneously.

It is quite evident that, in order to obtain the equations corresponding to those of arts. 7–11, all that is necessary is to substitute u for x, U for X, and omit the term unity in Q and Q'. To find $\partial z/\partial u$ we have

$$(15)\qquad \frac{\partial z}{\partial u} = \frac{u_1}{U_1} Z_1 - \frac{1}{U_1}\frac{dc_1}{dn}\left(\frac{\partial u}{\partial c_2} z_5 - \frac{\partial z}{\partial c_2} u_5 + \frac{\partial u}{\partial c_3} z_6 - \frac{\partial z}{\partial c_3} u_6\right),$$

where

$$U_1 = u_4 - \frac{dc_1}{dc_2} u_5 - \frac{dc_1}{dc_3} u_6, \qquad Z_1 = z_4 - \frac{dc_1}{dc_2} z_5 - \frac{dc_1}{dc_3} z_6.$$

13. As a rule it is more convenient to have the coördinates referred to axes of which one is directed to the mean place of the sun or moon. There is no trouble in either case in making the change. We substitute

$$u \text{ exp. } + \sqrt{-1}\,(n't + \epsilon') \quad \text{or} \quad u \text{ exp. } + \sqrt{-1}\,(nt + \epsilon),$$

in the respective cases, for u. The various small charges of this nature which experience has shown to be best for actual calculation are made in the summary which follows. It is scarcely useful to give all the details of the algebraical work which leads up to the results. The formulæ are somewhat long and necessarily unsymmetrical, as is generally the case when they are best adapted for numerical computation. It is to be noted also that the calculations are of the same nature as those which constitute the bulk of the work in my lunar theory, namely, multiplications of series. The great majority of these multiplications are, however, already done, having been previously needed, so that the new work is much less extensive than would appear from the formulæ. The order of accuracy with respect to e, γ to which they give the values of the derivatives with respect to n is one less than that of the coördinates, but in any case it can be found to a sufficient degree of accuracy for practical purposes.

§ (iii) *Summary of the results adapted for calculation.*

14. The notation adopted in this section is that of my theory.* It is therefore necessary to substitute for the symbols denoted above by u, s, the values

$$u \text{ exp. } \sqrt{-1}\,(n't + \epsilon'), \qquad s \text{ exp. } - \sqrt{-1}\,(n't + \epsilon')$$

to reduce to axes moving with the sun's mean motion. We shall suppose that this substitution has been made.

Now u is expansible in powers of ζ, ζ^{c}, ζ^{g}, ζ^{m}. Let

$$D_1 = \zeta^1 \frac{\partial}{\partial \zeta^1}, \qquad D_c = \zeta^c \frac{\partial}{\partial \zeta^c}, \qquad D_g = \zeta^g \frac{\partial}{\partial \zeta^g}, \qquad D_m = \zeta^m \frac{\partial}{\partial \zeta^m},$$

* T. M. M., part I.

a notation which can be adopted since c, g, m, are incommensurable with unity and their numerical values are not substituted in the index of ζ. Then

$$D = D_1 + cD_c + gD_g + mD_m.$$

Put *

$$au' = u\zeta^{-1}. \qquad as' = s\zeta, \qquad az' = z;$$

then $u'a$ is the complex coördinate referred to the mean place of the moon as axis of x.

15. I shall now indicate briefly the various steps required to find the derivatives with respect to six arbitraries.

(a) The derivatives with respect to ϵ, π, θ are given by

$$(A) \qquad \frac{du'}{d\epsilon} = (D_1 + D_c + D_g)u' \sqrt{-1},$$

$$\frac{du'}{d\pi} = -D_c u' \sqrt{-1}, \qquad \frac{du'}{d\theta} = -D_g u' \sqrt{-1}.$$

(b) The canonical constants c_1, c_2, c_3 are given by the method referred to in art. 11. The equations for them and for certain new constants (not quite the same as those of art. 11) introduced for convenience, are

$$(n - n')a^2 A = c_1 = -\tfrac{1}{2}(n - n')a^2 [(D + 1 + m)u' \cdot (D_1 + D_c + D_g - 1)s'$$
$$+ (D - 1 - m)s' \cdot (D_1 + D_c + D_g + 1)u'$$
$$+ 2Dz' \cdot (D_1 + D_c + D_g)z']^\circ,$$

$$(B) \qquad -(n - n')a^2 E^2 = c_2 = \tfrac{1}{2}(n - n')a^2 [(D + 1 + m)u' \cdot D_c s'$$
$$+ (D - 1 - m)s' \cdot D_c u' + 2Dz' \cdot D_c z']^\circ,$$

$$-(n - n')a^2 \Gamma^2 = c_3 = \tfrac{1}{2}(n - n')a^2 [(D + 1 + m)u' \quad D_g s'$$
$$+ (D - 1 - m)s' \cdot D_g u' + 2Dz' \cdot D_g z']^\circ,$$

where [ ]° denotes the constant term in the development of the enclosed function and μ is put equal to unity. The actual calculation from these rather long formulæ is really quite brief. The operators D_1, D_c, D_g introduce only integers and the necessary multiplications of series are nearly all at hand; with these, a few hours' work will suffice to determine c_1, c_2, c_3 to the fourth order.†

We thus find A, E^2, Γ^2 expanded in powers of e^2, k^2 (the constants of my

* The symbol u' is not used in T. M. M.

† The relations of c_i to DELAUNAY's final L, G, H, are given by

$$c_1 = L, \qquad c_2 = (G - L), \qquad c_3 = (H - G),$$

when $\mu = 1$.

theory) with numerical coefficients if the numerical values of e', α have been substituted, and in powers of e'^2, α^2 if their numerical values have not been substituted.

(c) Revert the series for E^2, Γ^2 so as to find e^2, k^2 expressed in powers of E^2, Γ^2. Then find A, c, g, in terms of the latter constants. Take the positive square roots to obtain e, k.

(d) Substitute the values of e, k in the expressions for u', z', so as to obtain the coördinates in terms of E, Γ.

(e) Put

$$U' = (D_1 + D_c + D_g + 1)\,u' - \frac{d\mathrm{A}}{d\mathrm{E}^2}\,D_c u' - \frac{d\mathrm{A}}{d\Gamma^2}\,D_g u',$$

$$Q = \frac{D_c u'}{\mathrm{E}}\,\frac{\partial}{\partial \mathrm{E}}\,Du' - \frac{D_c Du'}{\mathrm{E}}\,\frac{\partial u'}{\partial \mathrm{E}} + 2D_g u'\,\frac{\partial}{\partial \Gamma^2}\,D_g u' - 2D_g Du'\,\frac{\partial u'}{\partial \Gamma^2},$$

$$\frac{1}{\beta} = \left[\frac{Q}{2U'^2}\right]^{\circ}, \qquad \frac{dc_1}{dn} = -\,\mathrm{a}^2\beta,$$

and compute U', Q and the constant β. Then we find the derivative with respect to n from the formula

$$(C) \qquad \frac{n-n'}{\mathrm{a}}\,\frac{\partial(u'\mathrm{a})}{\partial n} = U'\,D^{-1}\left(\frac{\beta Q}{2U'^2} - 1\right),$$

no added constant being necessary on performing the operation D^{-1}. The operations $\partial/\partial\mathrm{E}$, $\partial/\partial\Gamma$ are performed only on the *coefficients* of the various powers of ζ in u', Du'. The final result is the value of $\partial(u'\mathrm{a})/\partial n$ when E, Γ have been replaced by their values in terms of c_2, c_3: the change was a temporary one for convenience only.

The actual calculation of Q/U'^2 is performed in the following way. The principal terms in u' (those depending only on m and therefore of the form $\Sigma_i a_i \zeta^{2i}$ where the a_i are numerical coefficients and $a_0 = 1$) being denoted by u'_0, the principal terms in U' are given by

$$U'_0 = (D_1 + 1)\,u'_0 = (D + 1)\,u'_0.$$

Hence

$$\frac{1}{U'^2} = \frac{1}{U_0'^2}\left\{1 - 2\,\frac{U' - U'_0}{U'_0} + 3\left(\frac{U' - U'_0}{U'_0}\right)^2 - \cdots\right\}.$$

The functions $1/U_0'^p$, being power series in ζ^2 are best calculated by the method of special values. It is to be noted that all terms on which D_c has operated contain the factor E, and those on which D_g has operated, the factor Γ^2.

In the expansion of $1/U'^2$, the value of u' is used as far as the order actually required; in that of Q, the value of u' to one order higher in E, Γ.

The values of $\partial u'/\partial \mathrm{E}$, $\partial u'/\partial \Gamma$ have been obtained in the course of the investigation and we therefore have the derivatives of with respect to all six constants.

(f) Finally separate into real and imaginary parts and obtain the derivatives of the two coördinates referred to moving axes, that of x pointing to the mean place of the moon. For the transformation we have

$$(D) \qquad \begin{aligned} \zeta^{i} &= \cos iD + \sqrt{-1}\sin iD, & \zeta^{ic} &= \cos il + \sqrt{-1}\sin il, \\ \zeta^{ig} &= \cos iF + \sqrt{-1}\sin iF, & \zeta^{im} &= \cos il' + \sqrt{-1}\sin il', \end{aligned}$$

where D, l, l', F are the usual angles in DELAUNAY's notation.

(g) Put

$$Z' = (D_1 + D_c + D_g)z' - \frac{d\mathrm{A}}{d\mathrm{E}^2}D_c z' - \frac{d\mathrm{A}}{d\Gamma^2}D_g z',$$

and determine $\partial z/\partial n$ from

$$(E) \qquad \frac{n-n'}{\mathrm{a}}\frac{\partial z}{\partial n} = Z'D^{-1}\left(\frac{\beta Q}{2U'^2} - 1\right) + \frac{\beta}{2U'}Q',$$

where

$$Q' = \frac{\partial u'}{\partial \mathrm{E}}\cdot\frac{D_c z'}{\mathrm{E}} - \frac{\partial z'}{\partial \mathrm{E}}\cdot\frac{D_c u'}{\mathrm{E}} + 2\frac{\partial u'}{\partial \Gamma^2}D_g z' - \frac{\partial z'}{\partial \Gamma}\frac{D_g u'}{\Gamma}.$$

The expansion of $1/U'$ is made as before and the calculation of the various functions follows a quite similar plan.

16. The value of $\partial(u'_0\mathrm{a})/\partial n$ has already been obtained. If the suffix zero denotes the parts depending only on m, the connection between the results here and those in T. M. M., pt. II, chap. V, section (ii), is given by

$$U'_0 = \mathrm{u}'_3, \qquad n\frac{\partial(u'_0\mathrm{a})}{\partial n} = 2\mathrm{u}_4\mathrm{a}\beta_0,$$

$$-\beta_0 = \frac{1}{\mathrm{a}^2}\frac{d}{dn}(c_1)_0 = -\frac{1}{2\mathrm{q}} = -.33022, \qquad \frac{d}{dn}(c_1)_0 = -\frac{\mathrm{a}^2}{2\mathrm{q}} = -.32962a^2.$$

I have found elsewhere* by a totally different method,

$$\frac{d}{dn}(c_1)_0 = \tfrac{3}{2}\frac{d^2}{dn^2}\left[\frac{\mu}{r}\right]_0^{\circ}, \qquad \frac{d^2}{dn^2}\left[\frac{\mu}{r}\right]_0^{\circ} = -.21975a^2,$$

which gives the same value for $d(c_1)_0/dn$.

*Page 153 of the paper referred to in art. 11 and page 348 of the first paper referred to in art. 6.

18. If it be intended actually to use the canonical equations for the variation of arbitrary constants, namely,

$$\frac{dc_i}{dt}=\frac{\partial R}{\partial w_{i+3}}, \qquad \frac{dw_j}{dt}=-\frac{\partial R}{\partial c_{j-3}}+b_j \qquad (i=1, 2, 3; j=4, 5, 6),$$

it is easy to make the necessary changes. Denote by parentheses the derivatives when c_1, c_2, c_3 are considered to be the arbitraries. Then if ϕ be any one of the functions used, the derivatives of ϕ with regard to c_1, c_2, c_3 are found by solving the equations

$$\frac{\partial \phi}{\partial n}=\left(\frac{\partial \phi}{\partial c_1}\right)\frac{dc_1}{dn}, \quad \frac{\partial \phi}{\partial c_2}=\left(\frac{\partial \phi}{\partial c_1}\right)\frac{dc_1}{dc_2}+\left(\frac{\partial \phi}{\partial c_2}\right), \quad \frac{\partial \phi}{\partial c_3}=\left(\frac{\partial \phi}{\partial c_1}\right)\frac{dc_1}{dc_3}+\left(\frac{\partial \phi}{\partial c_3}\right),$$

with respect to $(\partial\phi/\partial c_1)$, $(\partial\phi/\partial c_2)$, $(\partial\phi/\partial c_3)$; $dc_1/dn=-\beta a^2$ is given by art. 15 (e) and dc_1/dc_2, dc_1/dc_3 will have been found.

HAVERFORD COLLEGE,
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