THEORY OF LINEAR ASSOCIATIVE ALGEBRA*

BY

JAMES BYRNIE SHAW

Introduction.

The following paper is a development of *Linear Associative Algebra*, as distinguished from linear associative algebras. That is to say, not only are algebraic systems of a definite number of units handled, but the foundation is laid for a treatment of associative numbers in general, irrespective of whether they belong to a system of units of one order rather than of another. From the writer's standpoint, a hypercomplex number and an associative number are extensions of the field of number beyond the limits of ordinary negative or complex numbers. An associative number is a number defined by some equation, or set of equations, whose terms and their component factors are *associative* quantities. Thus, a *quaternion* may be *defined* by the identity

$$q^2 - 2wq + \Sigma t_i^2 \equiv 0,$$

where w and $t_1, t_2 \cdots t_r$ are "scalars," which means that wherever q^2 occurs we might substitute for it

$$2wq - \Sigma t_i^2.$$

From this point of view we should define a radical not by a series, or a "sequence," but by the expression which may be substituted for a term of higher degree, as, in defining θ by the equation (or "identity")

$$\theta^2 \equiv 2.$$

The equation above for q contains *essentially* all the properties of quaternions (at least of real quaternions) and from this point of view the fact that quaternions belong to an algebra of *four* units is of minor importance. It was BENJAMIN PEIRCE who said regarding the classification by "order":

"This artificial division of the algebras is cold and uninstructive like the artificial Linnean system of botany." †

---

* Presented to the Society at the Evanston meeting, September 2–3, 1902, under a slightly different title. Received for publication September 11, 1902.

† PEIRCE, *Linear Associative Algebra*, American Journal of Mathematics, vol. 4 (1884), p. 99.

The first result of this paper is to find an answer to the question:

What are the possible independent numbers defined by an identity

$$q^n - w_1 q^{n-1} + w_2 q^{n-2} - \cdots \pm w_n \equiv 0,$$

where the only limitation on q is associativity, and where $w_1, w_2, \cdots, w_n$ are homogeneous expressions of degrees $1, 2, \cdots, n$ dependent on arbitrary scalars $x_1, x_2, \cdots, x_m$. The explicit answer to this question is to be found in Part 3.

The second result of importance is to establish a *calculus* of linear associative algebra. This calculus depends on the notation for its power and simplicity, and the notation depends on the conception of certain "associative units." It is proved that every associative number is expressible in a form built up by adding together terms, each of which consists of one of these associative units with a coefficient dependent linearly and homogeneously on the arbitrary scalar coefficients of the associative number. These expressions may be built and used without knowing explicitly how many arbitrary scalar coefficients the coefficients of the associative units depend on. There is more than a mere notation involved here, and this is more than an expression of results in a matricular notation. Just as the "quadrate or relative *vids*" of C. S. PEIRCE * enable matrices to be conceived as the sum of n^2 terms, each with definite laws of combination with the others, so these associative units of form λ_{ijk} enable us to conceive all associative numbers as really expressions linear in these units, each with definite and simple laws of combination with the others. And as all matrices are expressible in terms of vids and the theory of matrices is essentially the theory of vids, so the theory of linear associative algebra is the theory of these associative units. From this point of view the matrix becomes a special case of linear associative algebra, and an associative number obeys laws different in some respects from those of matrices. We regard all associative numbers as belonging to an associative algebra of an infinite number of units.

The third result of importance is that these results are independent of the presence or absence of a modulus. Any result based on the group-theory of LIE is dependent on the presence of the "identical transformation," *i. e.*, there must be a *modulus* in the algebra. It is obvious that every such result must needs be shown separately to apply to algebras (by far the more numerous) not including a modulus. It was BENJAMIN PEIRCE who recognized intuitively the great importance of these *nilpotent* algebras.† In fact, the problem of writing all nilpotent algebras and of developing nilpotent numbers, is almost commensurate

* PEIRCE, *Linear Associative Algebra*, American Journal of Mathematics, vol. 4, p. 221.

† PEIRCE, American Journal of Mathematics, vol. 4, p. 118.

with developing the entire field. These numbers satisfy each an identity of the form

$$q^m \equiv 0.$$

They are not determinable from the forms satisfying the complete equation cited above, nor are their properties to be developed from such an equation. This fact is the real basis of Hamilton's refusal to admit "imaginary quaternions" into his system. They belong to a class of number different from the real quaternions. The problem of nilpotents is one on which little has been done, save incidentally in the study of algebras with an idempotent unit. The present paper produces general forms for all nilpotent associative numbers and the combinations of these numbers may be studied aside from their belonging to any limited algebra. A subsequent paper presents them in greater detail.

The fourth result of importance is that the study of algebras of any certain order is made easy. All algebras of order n may be found with little trouble, and by comparatively rapid methods. Furthermore it is not necessary that all algebras of order $n-1$ be known, for the entire list of algebras of order n may be found without knowing those of order $n-1$. The natural transformations and simplifications of the algebra are further evident from the forms of the units, and no multiplication table is needed, as the combination laws are evident from the forms of the units. Any algebra may readily be thrown into this form.

The fifth result is the great simplification of proofs and processes from this standpoint of wider generality (which would be expected). The various systems of algebras that belong together on different bases of classification are readily seen, and many new bases of classification are suggested.

This paper may be looked upon as an introduction to the subject, in which only the fundamental principles are laid down. It is hoped that the field will be developed farther in the near future.

An analytical table of the sections is given to serve as a guide in the reading.

Part 1. *The matrix form of a linear algebra.*

§ 1. Let any n independent numbers of an algebra (associative or otherwise) be represented by

$$\rho_1, \rho_2, \rho_3, \cdots, \rho_n,$$

so that the general expression for any number in this algebra is

$$\rho = x_1\rho_1 + x_2\rho_2 + \cdots + x_n\rho_n.$$

Any other number would be

$$\sigma = y_1\rho_1 + y_2\rho_2 + \cdots + y_n\rho_n.$$

The "product" of ρ into σ is defined by the equation

$$\rho\sigma = \Sigma x_i y_j c_{ijk}\rho_k \qquad (i, j, k = 1, 2, \cdots, n),$$

wherein the c_{ijk} are coefficients (scalar) which belong to the algebra in question.

The algebra is *linear* since each set of coefficients enters homogeneously to the first degree.

§ 2. Let us now annex one more unit ρ_0, an ideal unit which enters only to enable us to build the operators which follow. Let

$$\lambda_{ij} \qquad (i, j = 0, 1, \cdots, n),$$

be a matrix operator subject to the definitions

$$\begin{aligned} \lambda_{i0}\cdot\rho_0 &= \rho_i, \\ \lambda_{i0}\cdot\rho &= 0, \\ \lambda_{ij}\cdot\rho &= x_j\cdot\rho_i, \\ \lambda_{ij}\cdot\lambda_{kl} &= 0 \quad \text{if} \quad j \neq k, \\ &= \lambda_{il} \quad \text{if} \quad j = k. \end{aligned}$$

Then we may represent the *product* defined above by the matrix form

$$\rho\sigma = \phi_\rho \cdot \sigma$$

wherein

$$\begin{aligned}\phi_\rho \equiv x_1\lambda_{10} &+ \lambda_{11}(c_{111}x_1 + c_{211}x_2 + \cdots + c_{n11}x_n) + \lambda_{12}(c_{121}x_1 + \cdots + c_{n21}x_n) \\ &\qquad + \cdots + \lambda_{1n}(c_{1n1}x_1 + \cdots + c_{nn1}x_n) \\ + x_2\lambda_{20} &+ \lambda_{21}(c_{112}x_1 + c_{212}x_2 + \cdots + c_{n12}x_n) + \lambda_{22}(c_{122}x_1 + \cdots + c_{n22}x_n) \\ &\qquad + \cdots + \lambda_{2n}(c_{1n2}x_1 + \cdots + c_{nn2}x_n) \\ + x_3\lambda_{30} &+ \lambda_{31}(c_{113}x_1 + c_{213}x_2 + \cdots + c_{n13}x_n) + \lambda_{32}(c_{123}x_1 + \cdots + c_{n23}x_n) \\ &\qquad + \cdots + \lambda_{3n}(c_{1n3}x_1 + \cdots + c_{nn3}x_n) \\ + \cdot &\cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \\ + x_n\lambda_{n0} &+ \lambda_{n1}(c_{11n}x_1 + c_{21n}x_2 + \cdots + c_{n1n}x_n) \\ &\qquad + \cdots + \lambda_{nn}(c_{1nn}x_1 + \cdots + c_{nnn}x_n).\end{aligned}$$

It is to be noticed that throughout the paper a matrix is viewed as a linear vector operator on a ground of n, or in this case of $n+1$, units.*

We may contract the form above into †

$$\phi_\rho \equiv \Sigma(x_i\lambda_{i0} + x_i c_{ikj}\lambda_{jk}) \qquad (i, j, k = 1, 2, \cdots, n).$$

Let us abbreviate the matrices corresponding to ρ_i and ρ_j into ϕ_i, ϕ_j. Then

$$\phi_i = \lambda_{i0} + \Sigma c_{irs}\lambda_{sr},$$

$$\phi_j = \lambda_{j0} + \Sigma c_{jtr}\lambda_{rt},$$

therefore

$$\phi_i\phi_j = \Sigma c_{ijs}\lambda_{s0} + \Sigma c_{irs}c_{jtr}\lambda_{st} \qquad (r, s, t = 1, \cdots, n).$$

therefore

$$\phi_i\phi_j\sigma = \Sigma c_{irs}c_{jtr}y_t\rho_s.$$

But

$$\rho_j \cdot \sigma = \Sigma y_w c_{jwx}\rho_x,$$

$$\rho_i \cdot \rho_j \cdot \sigma = \Sigma y_w c_{jwx}c_{ixz}\rho_z,$$

or

$$= \Sigma y_t c_{jtr}c_{irs}\rho_s,$$

by changing the notation.

---

* Cf. TABER, *On the Theory of Matrices*, American Journal of Mathematics, vol. 12 (1890), pp. 337–396.

† See also PEIRCE, American Journal of Mathematics, vol. 4, p. 222. The results of this article were given by C. S. PEIRCE in the citation, probably for the first time.

Hence $\phi_i\phi_j$ has the same effect as $\rho_i \cdot \rho_j$. It is to be noticed that

$$\rho_i \cdot \rho_j = \Sigma c_{ijs} \rho_s .$$

Hence

$$\phi_{(\rho_i \cdot \rho_j)} = \Sigma c_{ijs} \lambda_{s0} + \Sigma c_{ijr} c_{rts} \lambda_{st} .$$

In general therefore

$$\phi_{(\rho_i \cdot \rho_j)} \neq \phi_{\rho_i} \cdot \phi_{\rho_j} ,$$

which means that generally the combinations are not associative. It is evident however that a treatment of all combinations, whether associative or not, can be based on these matrix forms.

§ 3. The equation

$$\rho\sigma = \Sigma x_i y_j c_{ijk} \rho_k$$

may also be looked upon as equivalent to the equation

$$\psi_\sigma \cdot \rho = \Sigma x_i y_j c_{ijk} \rho_k ,$$

wherein

$$\psi_\rho \cdot \equiv \Sigma \left(y_j \lambda_{j0} + y_j c_{ijk} \cdot \lambda_{ki} \right) .$$

This matrix is analogous to ϕ_ρ. The use of ϕ_ρ is equivalent to conceiving the facient ρ as operating on the faciend σ to produce the result. The use of ψ_σ is equivalent to conceiving the faciend σ as operating on the facient ρ to produce the result.

§ 4. If the numbers are *associative* we must have for any three of them

$$\rho \cdot (\sigma \cdot \tau) \equiv (\rho \cdot \sigma) \cdot \tau ,$$

which gives the matrix form

$$\phi_\rho \cdot \psi_\tau \sigma \equiv \psi_\tau \cdot \phi_\rho \sigma .$$

As this is true for any three numbers, we have the matrix equation

$$\phi_\rho \psi_\tau \equiv \psi_\tau \phi_\rho ,$$

or simply

$$\phi\psi = \psi\phi .$$

This equation defines associativity completely and upon it is to be based the analysis of associative algebras. It includes the theorem of Poincaré* that *two projective groups are connected with every system of complex numbers ;* the theorem of Study † that *these groups are simply transitive ;* the converse

* *Sur les nombres complexes,* Comptes Rendus, t. 99 (1884), pp. 740–742.

† *Complexe Zahlen und Transformationsgruppen,* Leipziger Berichte, 1889, S. 202; *Ueber Systeme Complexer Zahlen und ihre Anwendung in der Theorie der Transformationsgruppen,* Monatsheft f. Math. u. Phys. 1. Jahrg. (1890), S. 332.-

of these theorems; and the theorem that these two groups are *commutative.**† These theorems are *included* in the above equation since the entire set of matrices ϕ or the set ψ may not at all include the matrix 1. The progress of the development of the theory of hyper-complex numbers by means of the group-theory of LIE may be found traced in STUDY's *Ältere und Neuere Untersuchungen über Systeme complexer Zahlen.*‡

§ 5. It is clear that a set of independent ϕ's satisfying the equation

$$\phi\psi = \psi\phi,$$

where ψ is a given matrix, can easily be far more numerous than the order of any field to which ψ may belong. This means that if ψ is a matrix corresponding to some faciend, there are several algebras in which ψ might exist. In fact every set of n independent *matrices* of the set satisfying the equation above, if their products are linearly expressible in terms of these n, gives a linear algebra of n units.

We need now to see what the commutativity of matrices means.

Part 2. *The commutativity of matrices.*

§ 1. Any matrix ψ operating on a field of vectors defined by $\rho_1, \rho_2, \cdots, \rho_n$, may be so expressed, by linear changes in the defining vectors of the field, that if the matrix is written in square form, all coefficients vanish save certain ones on the main diagonal and certain others on the next diagonal to the left. Thus if the equation of lowest degree that ψ satisfies is

$$(\psi - g_1)^{\mu_1}(\psi - g_2)^{\mu_2}\cdots(\psi - g_p)^{\mu_p} \equiv 0,$$

where $g_1 \cdots g_p$ are the p *distinct latent roots*, and where

$$\mu_1 + \mu_2 + \mu_3 + \cdots + \mu_p \overline{\overline{<}} n,$$

then there is a region I_{11} of order μ_1', $\mu_1' \overline{\overline{>}} \mu_1$, for any vector of which we have the equation

$$(\psi - g_1)^{\mu_1} \cdot I_{11} = 0.$$

For no vector with a component outside I_{11} is this equation true, for any power of $(\psi - g_1)$. If we choose the vectors defining this region I_{11} properly, the square array of ψ will start as shown in the diagram

---

* Cf. LIE-ENGEL: *Theorie der Transformationsgruppe*, vol. I, S. 367–429.

† Cf. LIE-SCHEFFERS: *Continuierliche Gruppen*, Kapitel 21.

‡ *Mathematical Papers Read at the International Mathematical Congress Held in Chicago, 1893*, pp. 367–381.

$$\begin{array}{c|cccccccccc}
 & \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} & \rho_{15} & \cdots & \rho_{1\mu_1} & & \cdots & \rho_{1\mu'_1} \\ \hline
\rho_{11} & g_1 \\
\rho_{12} & 1 & g_1 \\
\rho_{13} & & 1 & g_1 \\
\rho_{14} & & & 1 & g_1 \\
\rho_{15} & & & & 1 & g_1 \\
\vdots & & & & & 1 & \ddots \\
\rho_{1\mu_1} & & & & & & 1 & g_1 \\
 & & & & & & & 0 & g_1 \\
\vdots & & & & & & & & 1 & \ddots \\
\rho_{1\mu'_1} & & & & & & & & & & g_1
\end{array}$$

The number of *consecutive* 1's in the second diagonal being $\mu_1 - 1$ at least once and nowhere more than $\mu_1 - 1$. For the other roots there are similar regions, and the whole field may be divided up into mutually exclusive regions corresponding to the distinct roots of ψ. The entire diagonal of the array of ψ will then consist of the roots $g_1, g_2 \cdots g_p$ repeated $\mu'_1, \mu'_2 \cdots \mu'_p$ times; the second diagonal will consist of a series of 1's whose number in the consecutive regions will be, in each region of order μ'_i at least $\mu_i - 1$ once, and not more than $\mu_i - 1$ anywhere in that region.

Let λ_{ii0} be the matrix (linear vector operator) which converts

$$x_1\rho_{i1} + x_2\rho_{i2} + \cdots + x_{\mu'_i}\rho_{i\mu'_i}$$

into itself, and annuls all vectors whose components have the form $x_j\rho_{kj}$, $(i \neq k)$; and further, let ϑ_{ii1} convert

$$x_1\rho_{i1} + x_2\rho_{i2} + \cdots + x_{\mu'_i}\rho_{i\mu'_i}$$

into

$$x_1\rho_{i2} + x_2\rho_{i3} + \cdots + x_{\mu'_i-1}\rho_{i\mu'_i}.$$

Then we may write

$$\psi = g_1\lambda_{110} + g_2\lambda_{220} + \cdots + g_p\lambda_{pp0} + \vartheta_{111} + \vartheta_{221} + \cdots + \vartheta_{pp1}.$$

§ 2. Any other matrix ϕ operating in the same field must be representable in the form

$$\phi = \Sigma a_{ij\,.\,rs}\epsilon_{ij\,.\,rs}$$

where $\epsilon_{ij.rs}$ is a matrix which annuls every vector but ρ_{rs} and converts ρ_{rs} into ρ_{ij}. In square form this means that the constituent of ϕ in the row ij and column rs, is $a_{ij.rs}$. The rows following the vectors $\rho_{i1}, \rho_{i2}, \cdots, \rho_{i\mu_i'}$ form a horizontal strip "belonging" to the region i. Likewise the columns under these vectors, form a vertical strip "belonging" to the region i.

If now we form the equation

$$\phi\psi = \psi\phi,$$

from these more expanded forms, we see

(1) $\psi\phi$ has every row in each region multiplied by the corresponding root g, yielding a matrix ϕ', plus a second matrix ϕ'' which is derived from ϕ by dropping each row of constituents in each region one line lower, the first row of the region being filled with zeros, the last row of the region disappearing entirely.

(2) $\phi\psi$ has every column in each region multiplied by the corresponding root g, yielding a matrix ϕ_1, plus a second matrix ϕ_2, which is derived from ϕ by moving each vertical line in a region to the left one place, the line at the right of each region being filled with zeros, and the line at the left of each region disappearing altogether.

§ 3. Suppose now that

$$\phi\psi = \psi\phi.$$

Let us consider first the rectangular block on each side of this equation in the square array of the product and lying in the strip i horizontally and j vertically, $i \neq j$. The matrices ϕ', ϕ'', ϕ_1, ϕ_2 appear thus:

$$\begin{array}{c|cccc} & \rho_{j1} & \rho_{j2} & \rho_{j3} & \cdot \\ \hline \rho_{i1} & g_i a_{i1j1}, & g_i a_{i1j2}, & g_i a_{i1j3}, & \cdot \\ \rho_{i2} & g_i a_{i2j1}, & g_i a_{i2j2}, & g_i a_{i2j3}, & \cdot \\ \rho_{i3} & g_i a_{i3j1}, & g_i a_{i3j2}, & g_i a_{i3j3}, & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array} + \begin{array}{c|cccc} & \rho_{j1} & \rho_{j2} & \rho_{j3} & \cdot \\ \hline \rho_{i1} & 0 & 0 & 0 & \cdot \\ \rho_{i2} & a_{i1j1}, & a_{i1j2}, & a_{i1j3}, & \cdot \\ \rho_{i3} & a_{i2j1}, & a_{i2j2}, & a_{i2j3}, & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

$$\begin{array}{c|cccc} & \rho_{j1} & \rho_{j2} & \rho_{j3} & \cdot \\ \hline \rho_{i1} & g_j a_{i1j1}, & g_j a_{i1j2}, & g_j a_{i1j3}, & \cdot \\ \rho_{i2} & g_j a_{i2j1}, & g_j a_{i2j2}, & g_j a_{i2j3}, & \cdot \\ \rho_{i3} & g_j a_{i3j1}, & g_j a_{i3j2}, & g_j a_{i3j3}, & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array} + \begin{array}{c|cccc} & \rho_{j1} & \rho_{j2} & \rho_{j3} & \cdot \\ \hline \rho_{i1} & a_{i1j2}, & a_{i1j3}, & a_{i1j4}, & \cdot \\ \rho_{i2} & a_{i2j2}, & a_{i2j3}, & a_{i2j4}, & \cdot \\ \rho_{i3} & a_{i3j2}, & a_{i3j3}, & a_{i3j4}, & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

For the equality to hold we must have then

(1)
$$g_i a_{i1j1} = g_j a_{i1j1} + a_{i1j2},$$
$$g_i a_{i1j2} = g_j a_{i1j2} + a_{i1j3},$$
$$\cdots\cdots$$
$$g_i a_{i1j\mu'_j} = g_j a_{i1j\mu'_j},$$

therefore
$$a_{i1j2} = (g_i - g_j)\, a_{i1j1},$$
$$a_{i1j3} = (g_i - g_j)^2 a_{i1j1},$$
$$\cdots\cdots$$
$$a_{i1j\mu'_j} = (g_i - g_j)^{\mu'_j - 1} a_{i1j1},$$
$$0 = (g_i - g_j)^{\mu'_j} a_{i1j1}.$$

By hypothesis
$$g_i \neq g_j.$$

Hence
$$0 = a_{i1j1} = a_{i1j2} = a_{i1j3} = \cdots = a_{i1j\mu'_j}.$$

(2) Each line yields a similar set of equations, hence all a's of the form a_{irjs} $(i \neq j)$ must be zero.

This means that ϕ reduces to a sum of matrices each of which acts only in one of the regions of ψ mentioned above.

(3) We may suppose then that ψ consists of one such region only, and that

$$g_1 = g_2 = g_3 = \cdots = g_p.$$

The "regions" may now be taken to be those produced by the 1's in the second diagonal and the 0's that separate them, each set of consecutive 1's lying in a separate region. The equation of the matrix ψ (a matrix still of order n) is

$$(\psi - g)^{\mu} = 0 \qquad (\mu \leqq n),$$

the "widths" of the regions being $\mu_1, \mu_2, \cdots, \mu_p$, where $\mu_1 + \mu_2 + \cdots + \mu_p = n$. Since we may take $g = 1$, with no loss of generality, we may write

$$\psi = \lambda_{110} + \lambda_{220} + \lambda_{330} + \cdots + \lambda_{pp0} + c_1\lambda_{111} + c_2\lambda_{221} + c_3\lambda_{331} + \cdots + c_p\lambda_{pp1},$$

wherein
$$c_i = 0 \quad \text{if} \quad \mu_i = 1,$$
$$c_i = 1 \quad \text{if} \quad \mu_i > 1.$$

In this case the matrices above called ϕ' and ϕ_1 are identical and may be omitted. Hence we have to consider the equality:

$$
\begin{array}{c|ccccc}
 & \rho_{j1} & \rho_{j2} & \rho_{j3} & \cdots & \rho_{j\mu_j} \\ \hline
\rho_{i1} & 0 & 0 & 0 & \cdots & 0 \\
\rho_{i2} & a_{i1j1} & a_{i1j2} & a_{i1j3} & \cdots & a_{i1j\mu_j} \\
\rho_{i3} & a_{i2j1} & a_{i2j2} & a_{i2j3} & \cdots & a_{i2j\mu_j} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\rho_{i\mu_i} & a_{i\mu_i-1j1} & a_{i\mu_i-1j2} & a_{i\mu_i-1j3} & \cdots & a_{i\mu_i-1j\mu_j}
\end{array}
=
\begin{array}{c|ccccc}
 & \rho_{j1} & \rho_{j2} & \rho_{j3} & \cdots & \rho_{j\mu_j} \\ \hline
\rho_{i1} & a_{i1j2} & a_{i1j3} & a_{i1j4} & \cdots & 0 \\
\rho_{i2} & a_{i2j2} & a_{i2j3} & a_{i2j4} & \cdots & 0 \\
\rho_{i3} & a_{i3j2} & a_{i3j3} & a_{i3j4} & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\rho_{i\mu_i} & a_{i\mu_ij2} & a_{i\mu_ij3} & a_{i\mu_ij4} & \cdots & 0.
\end{array}
$$

It is plain that

$$0 = a_{i1j2} = a_{i1j3} = a_{i1j4} = \cdots = a_{i1j\mu_j} = a_{i2j\mu_j} = \cdots = a_{i\mu_i-1\,j\mu_j}.$$

That is, all the coefficients vanish from the upper row and the right hand line of the block in the strips i and j, except the one in the upper left hand corner and the one in the lower right hand corner.

But we have also

$$a_{irjs} = a_{ir+1\,js+1} \quad (r = 1, \cdots, \mu_i - 1\,;\ s = 1, \cdots, \mu_j - 1).$$

This means that all coefficients in the block on a diagonal parallel to the main diagonal are equal. Hence only those coefficients do not vanish which may be formed by filling the diagonals successively with coefficients, all coefficients in one diagonal being equal, and all diagonals which do not run from the left hand line to the lowest line and *vice versa* being zero. For example we may have blocks as follows for values of μ_i, μ_j:

$$
\begin{array}{ccc}
\mu_i = 4,\ \ \mu_j = 3. & \mu_i = 4,\ \ \mu_j = 4. & \mu_i = 3,\ \ \mu_j = 4. \\
\begin{array}{ccc} 0 & 0 & 0 \\ a_1 & 0 & 0 \\ a_2 & a_1 & 0 \\ a_3 & a_2 & a_1 \end{array}
&
\begin{array}{cccc} a_1 & 0 & 0 & 0 \\ a_2 & a_1 & 0 & 0 \\ a_3 & a_2 & a_1 & 0 \\ a_4 & a_3 & a_2 & a_1 \end{array}
&
\begin{array}{cccc} a_1 & 0 & 0 & 0 \\ a_2 & a_1 & 0 & 0 \\ a_3 & a_2 & a_1 & 0 \end{array}
\end{array}
$$

Let the matrix operator λ_{ijk} convert

$$y_{k+1}\rho_{j1} + y_{k+2}\rho_{j2} + y_{k+3}\rho_{j3} + \cdots y_{k+r}\rho_{jr}$$

into

$$y_{k+1}\rho_{ik+1} + y_{k+2}\rho_{ik+2} + y_{k+3}\rho_{ik+3} + \cdots + y_{k+r}\rho_{ik+r},$$

and be subject to the conditions

$$r \leqq \mu_j, \qquad k + r \leqq \mu_i,$$

or

$$k \geqq \mu_i - \mu_j \qquad \text{if} \qquad \mu_i > \mu_j,$$

$$k \geqq \mu_i - 1 \qquad \text{if} \qquad \mu_j \geqq \mu_i.$$

Then the block used above becomes (changing the a's to conform to the λ notation):

$$a_{ij0}\lambda_{ij0} + a_{ij1}\lambda_{ij1} + \cdots + a_{ij\mu_i-1}\lambda_{ij\mu_i-1} \qquad (\mu_i = \mu_j),$$

$$a_{ij0}\lambda_{ij0} + a_{ij1}\lambda_{ij1} + \cdots + a_{ij\mu_i-1}\lambda_{ij\mu_i-1} \qquad (\mu_i < \mu_j),$$

$$a_{ij\mu_i-\mu_j+1}\lambda_{ij\mu_i-\mu_j+1} + a_{ij\mu_i-\mu_j+2}\lambda_{ij\mu_i-\mu_j+2} + \cdots + a_{ij\mu_i-1}\lambda_{ij\mu_i-1} \qquad (\mu_i > \mu_j).$$

This is obviously the sole condition that

$$\phi\psi = \psi\phi.$$

As no restriction was placed upon i and j these results apply to every block formed by the crossing of any horizontal strip and any vertical strip of ϕ.

To illustrate, we may determine the matrix form (general) commutative with

$$\psi = \lambda_{110} + \lambda_{111} + \lambda_{220} + \lambda_{221} + \lambda_{330} + \lambda_{331} + \lambda_{440} + \lambda_{441} + \lambda_{550} + \lambda_{551},$$

whence

$$\mu_1 = 4, \qquad \mu_2 = 4, \qquad \mu_3 = 3, \qquad \mu_4 = 2, \qquad \mu_5 = 2.$$

The square array for ϕ would appear thus:

	ρ_{11}	ρ_{12}	ρ_{13}	ρ_{14}	ρ_{21}	ρ_{22}	ρ_{23}	ρ_{24}	ρ_{31}	ρ_{32}	ρ_{33}	ρ_{41}	ρ_{42}	ρ_{51}	ρ_{52}
ρ_{11}	a_{110}				a_{120}										
ρ_{12}	a_{111}	a_{110}			a_{121}	a_{120}			a_{131}						
ρ_{13}	a_{112}	a_{111}	a_{110}		a_{122}	a_{121}	a_{120}		a_{132}	a_{131}		a_{142}		a_{152}	
ρ_{14}	a_{113}	a_{112}	a_{111}	a_{110}	a_{123}	a_{122}	a_{121}	a_{120}	a_{133}	a_{132}	a_{131}	a_{143}	a_{142}	a_{153}	a_{152}
ρ_{21}	a_{210}				a_{220}										
ρ_{22}	a_{211}	a_{210}			a_{221}	a_{220}			a_{231}						
ρ_{23}	a_{212}	a_{211}	a_{210}		a_{222}	a_{221}	a_{220}		a_{232}	a_{231}		a_{242}		a_{252}	
ρ_{24}	a_{213}	a_{212}	a_{211}	a_{210}	a_{223}	a_{222}	a_{221}	a_{220}	a_{233}	a_{232}	a_{231}	a_{243}	a_{242}	a_{253}	a_{252}
ρ_{31}	a_{310}				a_{320}				a_{330}						
ρ_{32}	a_{311}	a_{310}			a_{321}	a_{320}			a_{331}	a_{330}		a_{341}		a_{351}	
ρ_{33}	a_{312}	a_{311}	a_{310}	0	a_{322}	a_{321}	a_{320}	0	a_{332}	a_{331}	a_{330}	a_{342}	a_{341}	a_{352}	a_{351}
ρ_{41}	a_{410}				a_{420}				a_{430}			a_{440}		a_{450}	
ρ_{42}	a_{411}	a_{410}	0	0	a_{421}	a_{420}	0	0	a_{431}	a_{430}	0	a_{441}	a_{440}	a_{451}	a_{450}
ρ_{51}	a_{510}				a_{520}				a_{530}			a_{540}		a_{550}	
ρ_{52}	a_{511}	a_{510}	0	0	a_{521}	a_{520}	0	0	a_{531}	a_{530}	0	a_{541}	a_{540}	a_{551}	a_{550}

That is

$$\begin{aligned}
\phi = {} & a_{110}\lambda_{110} + a_{120}\lambda_{120} \\
& + a_{111}\lambda_{111} + a_{121}\lambda_{121} + a_{131}\lambda_{131} \\
& + a_{112}\lambda_{112} + a_{122}\lambda_{122} + a_{132}\lambda_{132} + a_{142}\lambda_{142} + a_{152}\lambda_{152} \\
& + a_{113}\lambda_{113} + a_{123}\lambda_{123} + a_{133}\lambda_{133} + a_{143}\lambda_{143} + a_{153}\lambda_{153} \\
& + a_{210}\lambda_{210} + a_{220}\lambda_{220} \\
& + a_{211}\lambda_{211} + a_{221}\lambda_{221} + a_{231}\lambda_{231} \\
& + a_{212}\lambda_{212} + a_{222}\lambda_{222} + a_{232}\lambda_{232} + a_{242}\lambda_{242} + a_{252}\lambda_{252} \\
& + a_{213}\lambda_{213} + a_{223}\lambda_{223} + a_{233}\lambda_{233} + a_{243}\lambda_{243} + a_{253}\lambda_{253} \\
& + a_{310}\lambda_{310} + a_{320}\lambda_{320} + a_{330}\lambda_{330} \\
& + a_{311}\lambda_{311} + a_{321}\lambda_{321} + a_{331}\lambda_{331} + a_{341}\lambda_{341} + a_{351}\lambda_{351} \\
& + a_{312}\lambda_{312} + a_{322}\lambda_{322} + a_{332}\lambda_{332} + a_{342}\lambda_{342} + a_{352}\lambda_{352} \\
& + a_{410}\lambda_{410} + a_{420}\lambda_{420} + a_{430}\lambda_{430} + a_{440}\lambda_{440} + a_{450}\lambda_{450} \\
& + a_{411}\lambda_{411} + a_{421}\lambda_{421} + a_{431}\lambda_{431} + a_{441}\lambda_{441} + a_{451}\lambda_{451} \\
& + a_{510}\lambda_{510} + a_{520}\lambda_{520} + a_{530}\lambda_{530} + a_{540}\lambda_{540} + a_{550}\lambda_{550} \\
& + a_{511}\lambda_{511} + a_{521}\lambda_{521} + a_{531}\lambda_{531} + a_{541}\lambda_{541} + a_{551}\lambda_{551}.
\end{aligned}$$

Generally we shall write the a's only, or even occasionally omit both a and λ, writing the subscripts only. Thus

$$\begin{array}{lccccc}
\phi = & 110 & 120 & & & \\
& 111 & 121 & 131 & & \\
& 112 & 122 & 132 & 142 & 152 \\
& 113 & 123 & 133 & 143 & 153 \\
& & \text{etc.} & & \text{etc.} &
\end{array}$$

With a further object in view we may write this matrix also in the form

	ρ_{11}	ρ_{21}	ρ_{31}	ρ_{41}	ρ_{51}	ρ_{12}	ρ_{22}	ρ_{32}	ρ_{42}	ρ_{52}	ρ_{13}	ρ_{23}	ρ_{33}	ρ_{14}	ρ_{24}
ρ_{11}	a_{110}	a_{120}													
ρ_{21}	a_{210}	a_{220}													
ρ_{31}	a_{310}	a_{320}	a_{330}												
ρ_{41}	a_{410}	a_{420}	a_{430}	a_{440}	a_{450}										
ρ_{51}	a_{510}	a_{520}	a_{530}	a_{540}	a_{550}										
ρ_{12}	a_{111}	a_{121}	a_{131}			a_{110}	a_{120}								
ρ_{22}	a_{211}	a_{221}	a_{231}			a_{210}	a_{220}								
ρ_{32}	a_{311}	a_{321}	a_{331}	a_{341}	a_{351}	a_{310}	a_{320}	a_{330}							
ρ_{42}	a_{411}	a_{421}	a_{431}	a_{441}	a_{451}	a_{410}	a_{420}	a_{430}	a_{440}	a_{450}					
ρ_{52}	a_{511}	a_{521}	a_{531}	a_{541}	a_{551}	a_{510}	a_{520}	a_{530}	a_{540}	a_{550}					
ρ_{13}	a_{112}	a_{122}	a_{132}	a_{142}	a_{152}	a_{111}	a_{121}	a_{131}			a_{110}	a_{120}			
ρ_{23}	a_{212}	a_{222}	a_{232}	a_{242}	a_{252}	a_{211}	a_{221}	a_{231}			a_{210}	a_{220}			
ρ_{33}	a_{312}	a_{322}	a_{332}	a_{342}	a_{352}	a_{311}	a_{321}	a_{331}	a_{341}	a_{351}	a_{310}	a_{320}	a_{330}		
ρ_{14}	a_{113}	a_{123}	a_{133}	a_{143}	a_{153}	a_{112}	a_{122}	a_{132}	a_{242}	a_{152}	a_{111}	a_{121}	a_{131}	a_{110}	a_{120}
ρ_{24}	a_{213}	a_{223}	a_{233}	a_{243}	a_{253}	a_{212}	a_{222}	a_{232}	a_{242}	a_{252}	a_{211}	a_{221}	a_{231}	a_{210}	a_{220}

That the general form may always be so arranged is easily seen, for the coefficient in any arrangement opposite ρ_{ir} and under ρ_{js} is

$$\begin{array}{lll} a_{ij\,r-s} & \text{if} & r \geqq s, \\ 0 & \text{if} & r < s, \\ 0 & \text{if} & r - s < \mu_i - \mu_j. \end{array}$$

In this arrangement no block to the right of the main diagonal has any coefficient other than zero.*

This form we shall use frequently, writing only the a's or their subscripts, thus

$$\begin{array}{lll} \phi = a_{110} & a_{120} & \\ a_{210} & a_{220} & \\ a_{310} & a_{320} & a_{330} \end{array}$$

* Cf. DICKSON: *Linear Groups*, p. 229 et seq.

$$\begin{array}{ccccc} a_{410} & a_{420} & a_{430} & a_{440} & a_{450} \\ a_{510} & a_{520} & a_{530} & a_{540} & a_{550} \\ a_{111} & a_{121} & a_{131} & & \\ a_{211} & a_{221} & a_{231} & & \\ a_{311} & a_{321} & a_{331} & a_{341} & a_{351} \\ & \text{etc.} & \text{etc.} & & \end{array}$$

This form is of the highest importance, and the λ's entering it will be called its "*frame.*" It will be referred to as *the form* of ϕ when it is commutative with ψ.

§ 4. It is profitable to arrive at these results from another starting point, not using the square array.

(1) If ψ is any matrix operating on a field of n dimensions, the field may be divided up into p mutually exclusive regions corresponding to the p distinct roots. Let these regions be

$$I_{11},\ I_{21},\ \cdots,\ I_{p1},$$

of dimensions $\mu'_1, \mu'_2, \cdots, \mu'_p$. These regions are found by operating on the whole field by the expressions

$$(g_1-\psi)^{\mu_1}(g_2-\psi)^{\mu_2}\cdots(g_{i-1}-\psi)^{\mu_{i-1}}(g_{i+1}-\psi)^{\mu_{i+1}}\cdots(g_p-\psi)^{\mu_p},$$

where $i=1, 2, \cdots, p$ and $\mu_r \gtreqless \mu'_r$. This expression reduces the whole field to I_{i1}.

(2) We have

$$(\psi-g_i)I_{j1}=I_{j1} \qquad (i \neq j).$$

But

$$(\psi-g_i)I_{i1}=I_{i2},$$

I_{i2} being a region included in I_{i1} and of dimensions at least one less than μ'_i. If $\mu_i \neq \mu'_i$ the dimensionality may be still smaller than μ'_i-1. Let it be $\mu'_1-\delta_{i1}$.

Likewise

$$(\psi-g_i)I_{i2}=I_{i3},$$

I_{i3} is included in I_{i2} and is of dimensionality $\mu'_1-\delta_{i1}-\delta_{i2}$, where $\delta_{i2} \leqq \delta_{i1}$.

By repetition of this operation we arrive at $I_{i\mu_i}$ for which

$$(\psi-g_i)I_{i\mu_i}=0.$$

Of course

$$I_{i3}=(\psi-g_i)^2 I_{i1},$$

$$\cdots\cdots\cdots$$

$$I_{i\mu_i}=(\psi-g_i)^{\mu_i-1}I_{i1},$$

$$0=(\psi-g_i)^{\mu_i}I_{i1}.$$

(3) It is to be noticed that if $\delta_{ik+1} < \delta_{ik}$, say $\delta_{ik} = \delta_{1k+1} + h_{ik}$, then there is a region of dimensionality h_{ik} included in I_{ik}, say I_{ikh}, for which

$$(\psi - g_i) I_{ikh} = 0.$$

The region I_{i1} has a subregion I_{i1e_1} for which

$$(\psi - g_i) I_{i1e_1} = 0.$$

This is made up of parts (generally) of the regions $I_{i1}, I_{i2}, \cdots$. Also, I_{i1} has a subregion I_{i1e_2} for which

$$(\psi - g_i)^2 I_{i1e_2} = 0.$$

This includes I_{i1e_1} of course, and so on.

(4) We call I_{i1e_1} the *first invariant region* of ψ for the root g_i; I_{i1e_2} the *second invariant region*. I_{i1} is the μ_1 *invariant region*.

We call I_{i1} the *first projective region* of ψ for the root g_i; I_{i2} the *second projective region*, etc.

For example, suppose I_{i1} is a region of fifteen dimensions, acted upon by $\psi - g_i$ according to the scheme below, $\psi - g_i$ converting each vector into the one below it, and each projective region being defined by all the vectors below the corresponding bar:

$$\begin{array}{ccccc|l}
\hline & & & & & I_{i1} \\
\rho_{11} & \rho_{21} & \rho_{31} & \rho_{41} & \rho_{51} & \\
\hline & & & & & I_{i2} \\
\rho_{12} & \rho_{22} & \rho_{32} & \rho_{42} & \rho_{52} & \\
\hline & & & & & I_{i3} \\
\rho_{13} & \rho_{23} & \rho_{33} & & & \\
\hline & & & & & I_{i4}. \\
\rho_{14} & \rho_{24} & & & &
\end{array}$$

Then the invariant regions are found by dropping each column until its lowest vector is on the bottom line, each invariant region consisting of all vectors below the corresponding bar:

$$\begin{array}{ccccc|l}
\hline & & & & & I_{i1e_4} \\
\rho_{11} & \rho_{21} & & & & \\
\hline & & & & & I_{i1e_3} \\
\rho_{12} & \rho_{22} & \rho_{31} & & & \\
\hline & & & & & I_{i1e_2} \\
\rho_{13} & \rho_{23} & \rho_{32} & \rho_{41} & \rho_{51} & \\
\hline & & & & & I_{i1e_1}. \\
\rho_{14} & \rho_{24} & \rho_{33} & \rho_{42} & \rho_{52} &
\end{array}$$

That is to say

$$(\psi - g_i)\{\rho_{14}, \rho_{24}, \rho_{33}, \rho_{42}, \rho_{52}\} = 0,$$

and $\psi - g_i$ does not reduce *any other vectors* to 0. So

$$(\psi - g_i)^2\{\rho_{14}, \rho_{24} \cdots \rho_{52}, \rho_{13}, \rho_{23} \cdots \rho_{51}\} = 0,$$

$$(\psi - g_i)^3\{\rho_{14} \cdots \rho_{52}, \rho_{13} \cdots \rho_{51}, \rho_{12} \cdots \rho_{31}\} = 0.$$

Also if ρ is any vector of the region

$$(\psi - g_i)^3 \rho = I_{i4} = \{ \rho_{14}, \rho_{24} \},$$

$$(\psi - g_i)^2 \rho = I_{i3} = \{ \rho_{13}, \rho_{23}, \rho_{33}, \rho_{14}, \rho_{24} \},$$

$$(\psi - g_i) \rho = I_{i2} = \{ \rho_{12} \cdots \rho_{52}, \rho_{13} \cdots \rho_{33}, \rho_{14}, \rho_{24} \}.$$

(5) Common to the first projective and last invariant regions are $\{ \rho_{11}, \rho_{21} \}$; and $(\psi - g_i)$ successively applied converts this region into $\{ \rho_{12}, \rho_{22} \}$, $\{ \rho_{13}, \rho_{23} \}$, $\{ \rho_{14}, \rho_{24} \}$. These regions together constitute the *shear region* of I_{i1}, of degree 4, or in general μ_i. Again, common to the first projective and third invariant regions is $\{ \rho_{31} \}$, giving the *shear region* $\{ \rho_{31}, \rho_{32}, \rho_{33} \}$. Common to the first projective and the second invariant regions are $\{ \rho_{41}, \rho_{51} \}$ giving the *shear* $\{ \rho_{41}, \rho_{51}, \rho_{42}, \rho_{52} \}$.

Suppose now we have two matrices satisfying the relation

$$\phi\psi = \psi\phi.$$

Then

$$(\psi - g_i)\phi = \phi(\psi - g_i),$$

and conversely. If also

$$\psi \{ R \} = \{ R \};$$

then

$$\phi\psi \{ R \} = \phi \{ R \} = \psi\phi \{ R \},$$

hence if $\{ R \}$ is an invariant region of ψ, so is $\phi \{ R \}$; also, if

$$(\psi - g_i) \{ I \} = \{ I' \},$$

then

$$(\psi - g_i)\phi \{ I \} = \phi \{ I' \};$$

and if

$$(\psi - g_i) I = 0,$$

$$(\psi - g_i)\phi I = 0,$$

$$(\psi - g_i)^2 I = 0,$$

$$(\psi - g_i)^2 \phi I = 0,$$

etc.,

so that if the successive projective regions of ψ are

$$I_{i1}, \quad I_{i2}, \quad I_{i3}, \quad I_{i4},$$

the same projective regions are represented by

$$\phi I_{i1}, \quad \phi I_{i2}, \quad \phi I_{i3}, \quad \phi I_{i4}.$$

In the example used above,

$$(\psi - g_i)^3 \cdot \rho = I_{i4},$$

therefore

$$(\psi - g_i)^3 \cdot \phi\rho = \phi I_{i4} = I_{i4},$$

hence ϕ leaves I_{i4} unchanged, or at least adds nothing to it. So

$$(\psi - g_i)^2 \cdot \rho = I_{i3},$$

therefore

$$(\psi - g_i)^2 \cdot \phi I_{i3} = I_{i3}.$$

It follows then that ϕ *does not change the contents of any projective region of* ψ, *but only its internal structure.*

Hence ϕ does not project a vector from one region I_{i1} into another I_{j1}, so that the regions of the roots of ϕ lie in the regions of the roots of ψ, and do not overlap them. Therefore as ϕ must have n dimensions, or the entire field for its operation, we may consider it as made up of matrices each acting only on the ground that the submatrices of ψ act upon, viz. I_{i1}, I_{i2}, etc. In other words we may suppose that all the roots of ψ were equal, and that the example above is of a matrix of order 15, with all its roots equal.

Since

$$(\psi - g)\{\rho_{14}, \rho_{24}, \cdots, \rho_{52}\} = 0,$$

and only for this region is this true, then

$$\phi(\psi - g)\{\rho_{14} \cdots \rho_{52}\} = 0 = (\psi - g)\phi\{\rho_{14} \ldots \rho_{52}\}.$$

Therefore

$$\phi\{\rho_{14} \cdots \rho_{52}\} = \{\rho_{14} \cdots \rho_{52}\}.$$

Likewise

$$\phi\{\rho_{14} \cdots \rho_{52}, \rho_{13} \cdots \rho_{51}\} = \{\rho_{14} \cdots \rho_{51}\},$$

$$\phi\{\rho_{14} \cdots \rho_{51}, \rho_{12}, \rho_{22}, \rho_{31}\} = \{\rho_{14} \cdots \rho_{31}\}.$$

That is, ϕ *does not change the invariant regions of* ψ, *as to their content, but only as to their structure.*

Finally, in the particular case given, since ρ_{14}, ρ_{24} are common to both the fourth projective region and the first invariant region, and since neither changes content, therefore

$$\phi(\rho_{14}, \rho_{24}) = (\rho_{14}, \rho_{24}).$$

Likewise,

$$\phi\rho_{33} = (\rho_{14}, \rho_{24}, \rho_{33}),$$

$$\phi(\rho_{42}, \rho_{52}) = (\rho_{14}, \rho_{24}, \rho_{33}, \rho_{42}, \rho_{52}),$$

$$\phi(\rho_{13}, \rho_{23}) = (\rho_{13}, \rho_{23}, \rho_{33}, \rho_{14}, \rho_{24}),$$

$$\phi\rho_{32} = (\rho_{32}, \rho_{42}, \rho_{52}, \rho_{13}, \rho_{23}, \rho_{33}, \rho_{14}, \rho_{24}),$$

$$\phi(\rho_{41}, \rho_{51}) = (\rho_{41}, \rho_{51}, \rho_{32} \cdots \rho_{52}, \rho_{13} \cdots \rho_{33}, \rho_{14}, \rho_{24}),$$

$$\phi(\rho_{12}, \rho_{22}) = (\rho_{12} \cdots \rho_{24}),$$

$$\phi \cdot \rho_{31} \qquad = (\rho_{31} \cdots \rho_{24}),$$

$$\phi(\rho_{11}, \rho_{21}) = (\rho_{11} \cdots \rho_{24}).$$

If we let

$$\phi\rho_{13} = x_{13}\rho_{13} + x_{23}\rho_{23} + x_{33}\rho_{33} + x_{14}\rho_{14} + x_{24}\rho_{24},$$

then

$$(\psi - g)\phi\rho_{13} = \phi(\psi - g)\rho_{13} = \phi\rho_{14} = x_{13}\rho_{14} + x_{23}\rho_{24}.$$

If also

$$\phi\rho_{23} = y_{13}\rho_{13} + \cdots + y_{24}\rho_{24},$$

then

$$(\psi - g)\phi\rho_{23} = \phi(\psi - g)\rho_{23} = \phi\rho_{24} = y_{13}\rho_{14} + y_{23}\rho_{24}.$$

Hence if we put

$$\phi \cdot \rho_{24} = a_{120}\rho_{14} + a_{220}\rho_{24},$$

$$\phi \cdot \rho_{14} = a_{110}\rho_{14} + a_{210}\rho_{24},$$

$$\phi \cdot \rho_{33} = a_{330}\rho_{33} + a_{131}\rho_{14} + a_{231}\rho_{24},$$

then

$$\phi \cdot \rho_{23} = a_{120}\rho_{13} + a_{220}\rho_{23} + a_{320}\rho_{33} + a_{121}\rho_{14} + a_{221}\rho_{24},$$

$$\phi\rho_{13} = a_{110}\rho_{13} + a_{210}\rho_{23} + a_{310}\rho_{33} + a_{111}\rho_{14} + a_{211}\rho_{24}.$$

A full set of equations which amount to the table already given for ϕ, may easily be deduced.

The first arrangement of the constituents of ϕ, it will be observed, is according to the shear regions, the second according to the projective region.

§ 5. This also gives the entire group of matrices commutative with a given matrix whose roots are all equal. With regard to this group we observe the following:

(1) Let the symbol $S_a^{(b)}$ designate any part of any shear region common to the projective region of order a and the invariant region of order b. If the shear order is μ,

$$a + b = \mu + 1.$$

Then the group leaves invariant the regions made up of $S_a^{(b)}$ and all subregions $S_{a'}^{(b')}$ such that

$$b' \leqq b_1, \qquad a' \geqq a_1.$$

For example in the above let

$$(\rho_{41}, \rho_{51}) = S_1^{(2)},$$

then the group leaves invariant

$$(S_1^{(2)}, S_2^{(2)}, S_4^{(1)}, S_3^{(1)}, S_2^{(1)}),$$

that is,

$$(\rho_{41}, \rho_{51}, \rho_{32}, \rho_{14}, \rho_{24}, \rho_{13}, \rho_{23}, \rho_{33}).$$

If the shears are all of *width* unity, it is obvious that there is for the group a set of invariant regions, of orders one, two, three, $\cdots$ to n, each region including the preceding.* The theorem above is to the effect that each group of matrices commutative with a given matrix whose roots are all equal leaves invariant a system of regions, which in the main are each included in the succeeding. The exceptions arise from the character of the subregions.

For example, ϕ_a above leaves invariant

$$(\rho_{14}, \rho_{24}), \quad (\rho_{33}, \rho_{14}, \rho_{24}), \quad (\rho_{13}, \rho_{23}, \rho_{33}, \rho_{14}, \rho_{24}), \quad (\rho_{42}, \rho_{52}, \rho_{33}, \rho_{14}, \rho_{24}), \text{ etc.}$$

The fourth does not include the third.

(2) It is obvious that the characteristic equation of any matrix of the group is the product of the determinants along its diagonal, with $-\phi$ inserted after every diagonal coefficient. For the special case above the equation is

$$\begin{vmatrix} a_{110}-\phi & a_{120} \\ a_{210} & a_{220}-\phi \end{vmatrix}^4 [a_{330}-\phi]^3 \begin{vmatrix} a_{440}-\phi & a_{450} \\ a_{540} & a_{550}-\phi \end{vmatrix}^2 \equiv 0.$$

This equation obviously may degenerate for some of the matrices of the group.

Let the determinant factors of the general characteristic equation for any frame be represented by $\Phi_1, \Phi_2, \cdots, \Phi_e$, of multiplicities $m_1, m_2, \cdots, m_e$, and widths $w_1, w_2, \cdots, w_e$. These factors we shall call *shear factors*. The general equation is then

$$\Phi_1^{m_1} \Phi_2^{m_2} \cdots \Phi_e^{m_e} \equiv 0.$$

Thus, in the example above

$$\Phi_1 = \begin{vmatrix} a_{110}-\phi & a_{120} \\ a_{210} & a_{220}-\phi \end{vmatrix} \qquad (m_1 = 4,\ w_1 = 2),$$

$$\Phi_2 = a_{330} - \phi \qquad (m_2 = 3,\ w_2 = 1),$$

$$\Phi_3 = \begin{vmatrix} a_{440}-\phi & a_{450} \\ a_{540} & a_{550}-\phi \end{vmatrix} \qquad (m_3 = 2,\ w_3 = 2).$$

In any case

$$n = w_1 m_1 + w_2 m_2 + \cdots + w_e m_e.$$

In a degenerate equation we have two cases, either

(*a*) all the shear factors are present,

(*b*) some shear factors are absent.

If we consider the shear factors we notice that any one of them as Φ_c reduces a certain region, which we may call the *shear region* c, by the width w_c and

* Cf. LIE–ENGEL: *Theorie der Transformationsgruppen*, vol. I, p. 589, Satz 4.

also projects it into other shear regions (the internal change in the shear region c is immaterial.) If this factor is

$$\Phi_c = \begin{vmatrix} a_{r_c r_c 0} - \phi & a_{r_c r_c+1\,0} & \cdots & a_{r_c r_c+s_c\,0} \\ a_{r_c+1\,r_c 0} & a_{r_c+1 r_c+1 0} - \phi & \cdots & a_{r_c+1 r_c+s_c\,0} \\ \cdot & \cdot & \cdot & \cdot \\ a_{r_c+s_c\,r_c 0} & a_{r_c+s_c\,r_c+1\,0} & \cdots & a_{r_c+s_c\,r_c+s_c\,0} - \phi \end{vmatrix},$$

since it has the effect pointed out, it cannot contain in its expanded form in terms of the λ's any of the block:

$$\begin{matrix} \lambda_{r_c,\,r_c,\,0} & \lambda_{r_c,\,r_c+1,\,0} & \cdots & \lambda_{r_c,\,r_c+s_c,\,0} \\ \lambda_{r_c+1,\,r_c,\,0} & \lambda_{r_c+1,\,r_c+1,\,0} & \cdots & \lambda_{r_c+1,\,r_c+s_c,\,0} \\ \cdot & \cdot & \cdot & \cdot \\ \lambda_{r_c+s_c,\,r_c,\,0} & & \cdots & \lambda_{r_c+s_c,\,r_c+s_c,\,0}. \end{matrix}$$

The presence of any one of these would prevent the reduction that must take place.

Thus in the above, Φ_1 cannot contain $\lambda_{110}, \lambda_{120}, \lambda_{210}, \lambda_{220}$; Φ_2 cannot contain λ_{330}; Φ_3 cannot contain $\lambda_{440}, \lambda_{450}, \lambda_{540}, \lambda_{550}$.

It is obvious that $\Phi_1 \Phi_2$ operating on $(\rho_{11}, \rho_{21}, \rho_{31})$ projects this region outside itself, hence $\Phi_1 \Phi_2$ cannot contain $\lambda_{110}, \lambda_{120}, \lambda_{210}, \lambda_{220}, \lambda_{310}, \lambda_{320}, \lambda_{330}$. Operating on $(\rho_{33}, \rho_{14}, \rho_{24})$ it annuls the region completely, hence it cannot contain λ_{131}, λ_{231}. For a similar reason, $\Phi_1 \Phi_2 \Phi_3$ cannot contain $\lambda_{110}, \cdots, \lambda_{210}, \cdots, \lambda_{310}, \cdots, \lambda_{410}, \cdots, \lambda_{510}, \cdots, \lambda_{550}$. Likewise $\Phi_1^2 \Phi_2$ cannot contain $\lambda_{110}, \cdots, \lambda_{220}, \lambda_{310}, \cdots, \lambda_{330}, \lambda_{111}, \lambda_{121}, \lambda_{131}, \lambda_{211}, \lambda_{221}, \lambda_{231}$. Further products and the λ's that can be determined to be missing from their expression can easily be traced from the diagram in Part 2, § 3 (3). The generalization for any frame is obvious.

(3) If we have an equation of order lower than the most general one belonging to the frame, this equation will include *every* shear factor to some power, or only *part* of the shear factors. In the first case let it be

$$\Phi_1^{m_1'} \Phi_2^{m_2'} \Phi_3^{m_3'} \cdots \Phi_e^{m_e'} = 0 \qquad (m_i' \gtreqless m_i,\ m_1 \gtreqless m_2 \gtreqless m_3 \cdots \gtreqless m_e)$$

The coefficients in the determinantal form of the shear factors we suppose to be arbitrary (the only case interesting us here). If we write the expression

$$\Psi = \Phi_1^{m_1'} \Phi_2^{m_2'} \cdots \Phi_e^{m_e'-1},$$

this expression, since it does not vanish by hypothesis, *must* contain some of the terms, and can contain only the following:

$$\left.\begin{array}{l}\lambda_{r_e+t,\, r_e+u,\, 0}\\ \lambda_{r_e+t,\, r_e+u,\, 1}\\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\ \lambda_{r_e+t,\, r_e+u,\, m_e-1}\end{array}\right\} \qquad (t,\, u=0,\, 1\cdots s_e).$$

In Ψ let all coefficients become zero, which have the form

$$a_{r_e+t,\, r_e+u,\, 0};$$

let Ψ' represent this value of Ψ; then since by hypothesis Ψ' cannot vanish until multiplied by the corresponding value of Φ_e, or $(-\phi)^{w_e}$, it must follow that Ψ' contains terms of the form

$$\lambda_{r_e+t,\, r_e+u,\, i} \qquad (0<i<m_c).$$

If we let the aggregate of terms of the form

$$a_{r_e+t,\, r_e+u,\, 0}\,\lambda_{r_e+t,\, r_e+u,\, 0}$$

be represented by Ψ_0, the remaining ones being Ψ', then $\Psi=\Psi_0+\Psi'$.

If we raise Ψ to a high enough power, since no power of it can vanish, and since the coefficients of Ψ_0 are arbitrary, and since the powers of Ψ_0 contain the same λ's, it follows that we arrive at a series of non-vanishing expressions or matrices, which may be so added and subtracted as to yield each of the matrices

$$\lambda_{r_e+t,\, r_e+u,\, 0}.$$

These elementary matrices belong then to the subgroup consisting of all matrices having this equation. That being the case, their products into Ψ' must yield matrices of the form

$$\lambda_{r_e+t,\, r_e+u,\, i} \qquad (b<i<m_e)$$

as members of the same subgroup. The lower limit of i, namely b, must be determined from the fact that the lowest of these raised to the m'_e power must vanish; that is

$$\lambda^{m'_e}_{r_e+t,\, r_e+t,\, b+1}=0,$$

whence

$$m'_e(b+1)>m_e,$$

giving

$$b>\frac{m_e}{m'_e}-1.$$

Obviously $\lambda_{r_e+t,\, r_e+u,\, k}$ is produced by the product $\lambda_{r_e+t,\, r_e+t,\, k}\cdot\lambda_{r_e+t,\, r_e+u,\, 0}$. Hence $m_e>i>m_e/m'_e-1$.

These λ's being removed from the general expression of ϕ, we proceed to find those for the other factors.

The subgroup then of all matrices whose equation is of the form

$$\Phi_1^{m_1'} \Phi_2^{m_2'} \cdots \Phi_e^{m_e'} \equiv 0 \qquad (m_i' \leqq m_i),$$

contains all the elementary matrices

$$\lambda_{r_i+t,\, r_i+u,\, 0} \qquad (i=1, 2, \cdots, e;\ t, u=0, 1, \cdots, s_i),$$

$$\lambda_{r_i+t,\, r_i+u,\, k_i} \qquad \left(m_i > k_i > \frac{m_i}{m_i'} - 1\right).$$

Any other matrix of the subgroup must then depend on forms not included in these, and *not determinable from the equation.*

If the equation lacks some of the shear factors, then some of the coefficients in the wider shear factor are equal to those in the narrower which may be considered as similarly placed. In this case the problem is more complicated, though to be handled on these same general lines. For example, in the special case used above, if

$$\begin{vmatrix} a_{110} - \phi & a_{120} \\ a_{210} & a_{220} - \phi \end{vmatrix}^4 \begin{vmatrix} a_{440} - \phi & a_{450} \\ a_{540} & a_{550} - \phi \end{vmatrix}^2 \equiv 0,$$

Then $a_{330} = a_{110}$ or a_{220}, it is immaterial which. The forms corresponding to Φ_1 become $\lambda_{110} + \lambda_{330}$, λ_{120}, λ_{210}, λ_{220}, λ_{111}, λ_{331}, λ_{121}, λ_{211}, λ_{221}, λ_{112}, λ_{332}, λ_{122}, λ_{212}, λ_{222}, λ_{113}, λ_{123}, λ_{213}, λ_{223}.

§ 6. The most striking feature of the form which any number takes, is that it is expressible linearly in terms of the elementary matrices λ_{rst}. In fact these forms may be considered to be the elementary units, called *associative units*, of a general algebra, which is of necessity linear and associative. They have from this point of view the definitions

$$\lambda_{rst}\lambda_{r's't'} = c\vartheta_{sr'}\lambda_{rs'\,t+t'} \qquad (\mu_r - 1 \geqq t + t' \geqq \mu_r - \mu_{s'}),$$

where $\vartheta_{sr'} = 1$ if $s = r'$, 0 if $s \neq r'$, $c = 1$ if $t + t'$ satisfies the condition, 0 if $t + t'$ does not satisfy the condition.

The group of commutative matrices is then representable by $\Sigma\, a_{rst}\lambda_{rst}$, which is commutative with

$$\lambda_{110} + \lambda_{111} + \lambda_{220} + \lambda_{221} + \cdots + \lambda_{pp0} + \lambda_{pp1}.$$

It is worth noticing that if

$$i \neq j, \quad \mu_i = \mu_j,$$

then

$$\lambda_{ij0}\lambda_{ji0} = \lambda_{ii0},$$

$$\lambda_{ji0}\lambda_{ij0} = \lambda_{jj0}.$$

If

$$\lambda' = \lambda_{ij0}\lambda_{ji0} - \lambda_{ji0}\lambda_{ij0} = \lambda_{ii0} - \lambda_{jj0},$$

then

$$\lambda'\lambda_{ij0} - \lambda_{ij0}\lambda' = 2\lambda_{ij0},$$

$$\lambda'\lambda_{ji0} - \lambda_{ji0}\lambda' = -2\lambda_{ji0}.$$

The four forms λ_{ii0}, λ_{jj0}, λ_{ij0}, λ_{ji0} form a quaternion matrix group, i. e. the general forms

$$a_{ii0}\lambda_{ii0} + a_{ij0}\lambda_{ij0} + a_{ji0}\lambda_{ji0} + a_{jj0}\lambda_{jj0}$$

obey the laws of quaternions. In fact these four units are practically the four canonical quaternion units.

So if

$$i \neq j \neq k, \qquad \mu_i = \mu_j = \mu_k,$$

we have the nonion group

$$\begin{matrix} \lambda_{ii0} & \lambda_{ij0} & \lambda_{ik0} \\ \lambda_{ji0} & \lambda_{jj0} & \lambda_{jk0} \\ \lambda_{ki0} & \lambda_{kj0} & \lambda_{kk0}. \end{matrix}$$

For these we have

$$\lambda_{ij0}\lambda_{jk0}\lambda_{ki0} = \lambda_{ii0}, \text{ etc.,}$$

If $\omega^3 = 1$, and

$$\lambda' = \lambda_{ij0}\lambda_{jk0}\lambda_{ki0} + \omega\lambda_{jk0}\lambda_{ki0}\lambda_{ij0} + \omega^2\lambda_{ki0}\lambda_{ij0}\lambda_{jk0} = \lambda_{ii0} + \omega\lambda_{jj0} + \omega^2\lambda_{kk0},$$

then

$$\lambda'^2 = \lambda_{ij0}\lambda_{jk0}\lambda_{ki0} + \omega^2\lambda_{jk0}\lambda_{ki0}\lambda_{ij0} + \omega\lambda_{ki0}\lambda_{ij0}\lambda_{jk0} = \lambda_{ii0} + \omega^2\lambda_{jj0} + \omega\lambda_{kk0},$$

$$\lambda'\lambda_{ij0}\lambda_{jk0} + \omega\lambda_{ij0}\lambda'\lambda_{jk0} + \omega^2\lambda_{ij0}\lambda_{jk0}\lambda' = \lambda_{ik0}(1 + \omega^2 + \omega) = 0,$$

$$\lambda'\lambda_{ij0}\lambda_{jk0} + \omega^2\lambda_{ij0}\lambda'\lambda_{jk0} + \omega\lambda_{ij0}\lambda_{jk0}\lambda' = 3\lambda_{ik0}, \text{ etc.}$$

Obviously any matrix may be put into a similar form.

There are other sets of these forms, as for example $\lambda_{110} + \lambda_{220}$, λ_{111}, λ_{221}, λ_{210} λ_{310}, λ_{330}, which involve the compound idempotents $\lambda_{110} + \lambda_{220}$, etc.

§ 7. The matrix form given yields the equations

$$\phi \cdot \rho_{11} = a_{110}\rho_{11} + a_{210}\rho_{21} + \cdots + a_{113}\rho_{14} + a_{213}\rho_{24},$$

$$\phi \cdot \rho_{12} = a_{120}\rho_{11} + a_{220}\rho_{22} + \cdots + a_{123}\rho_{14} + a_{223}\rho_{24},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\phi \cdot \rho_{14} = a_{110}\rho_{14} + a_{210}\rho_{24},$$

$$\phi \cdot \rho_{24} = a_{120}\rho_{14} + a_{220}\rho_{24}.$$

Inasmuch as these correspond to matrix equations of the algebra, we must have, if ϕ_{11} corresponds to ρ_{11}, etc.,

$$\phi \cdot \phi_{11} = a_{110}\phi_{11} + a_{210}\phi_{21} + \cdots + a_{113}\phi_{14} + a_{213}\phi_{24},$$

$$\phi \cdot \phi_{12} = a_{120}\phi_{11} + a_{220}\phi_{22} + \cdots + a_{123}\phi_{14} + a_{223}\phi_{24},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\phi \cdot \phi_{14} = a_{110}\phi_{14} + a_{210}\phi_{24},$$

$$\phi \cdot \phi_{24} = a_{120}\phi_{14} + a_{220}\phi_{24}.$$

From the last two equations

$$\begin{vmatrix} a^{(24)}_{110} - \phi_{24} & a^{(24)}_{210} \\ a^{(24)}_{120} & a^{(24)}_{220} - \phi_{24} \end{vmatrix} \phi_{24} = 0,$$

$$\begin{vmatrix} a^{(14)}_{110} - \phi_{14} & a^{(14)}_{210} \\ a^{(14)}_{120} & a^{(14)}_{220} - \phi_{14} \end{vmatrix} \phi_{14} = 0.$$

We have further

$$\begin{vmatrix} a^{(24)}_{110} - \phi_{24} & a^{(24)}_{210} \\ a^{(24)}_{120} & a^{(24)}_{220} - \phi_{24} \end{vmatrix} \phi_{14} = 0, \qquad \begin{vmatrix} a^{(14)}_{110} - \phi_{14} & a^{(14)}_{210} \\ a^{(14)}_{120} & a^{(14)}_{220} - \phi_{14} \end{vmatrix} \phi_{24} = 0.$$

There are other equations easily written down from the equations above. These are useful in determining the laws of algebras as well as structure. The consequences of these equations are of great importance, but are deferred for further consideration.

Part 3. *Linear associative algebras.*

§ 1. We may now enunciate as the result of the preceding analysis the following proposition:

Any number of any linear associative algebra of n units is of the form

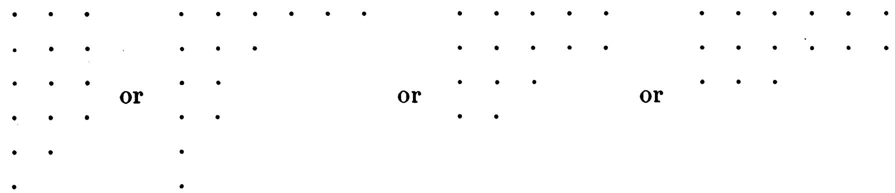
$$\phi = \Sigma \cdot a_{rst}\lambda_{rst} \qquad (r, s = 1, \cdots, p),$$

and for each r or s exists a multiplicity number μ_r or μ_s; these numbers satisfy the conditions $\mu_1 + \mu_2 + \cdots + \mu_p = n + 1$; $\mu_r \sim \mu_s \gtreqless t < \mu_r$.

We have $n + 1$ and not n in this equation just above, because of the ideal unit that we added to the list of units (Part 1, § 2). This ideal unit plays no part in the theory. The form of ϕ may be shown in diagrams similar to those of Part 2, § 3.

The number ϕ satisfies an equation of degree $n + 1$ or less, and the factors of this equation are certain determinants, the shear factors mentioned before. The constituents of these determinants are arbitrary, also the degree of this *equation being less than* $n+1$, *there are two distinct types of equation:* (1) *when the factors belonging to the frame are all represented*, (2) *when some of the factors are not represented.*

If the order is n, we may have in the equation such factors and multiplicities as can occur in the partitions of $n+1$. Thus if $n=14$, we may have among other forms



corresponding to the cases:

$\mu_1=6,\ w_1=1,$	$\mu_1=6,\ w_1=1,$	$\mu_1=4,\ w_1=2,$	$\mu_1=3,\ w_1=3,$
$\mu_2=5,\ w_2=1,$	$\mu_2=4,\ w_2=1,$	$\mu_2=3,\ w_2=1,$	$\mu_2=2,\ w_1=3.$
$\mu_3=4,\ w_3=1;$	$\mu_3=2,\ w_3=1,$	$\mu_3=2,\ w_2=2;$	
	$\mu_4=1,\ w_4=1,$		
	$\mu_5=1,\ w_5=1,$		
	$\mu_6=1,\ w_6=1;$		

The general equations of the corresponding frames would be respectively:

$$(a_{110}-\phi)^6(a_{220}-\phi)^5(a_{330}-\phi)^4=0,$$

$$(a_{110}-\phi)^6(a_{220}-\phi)^4(a_{330}-\phi)^2(a_{440}-\phi)(a_{550}-\phi)(a_{660}-\phi)=0,$$

$$\begin{vmatrix} a_{110}-\phi & a_{120} \\ a_{210} & a_{220}-\phi \end{vmatrix}^4 [a_{330}-\phi]^3 \begin{vmatrix} a_{440}-\phi & a_{450} \\ a_{540} & a_{550}-\phi \end{vmatrix}^2 = 0,$$

$$\begin{vmatrix} a_{110}-\phi & a_{120} & a_{130} \\ a_{210} & a_{220}-\phi & a_{230} \\ a_{310} & a_{320} & a_{330}-\phi \end{vmatrix}^3 \begin{vmatrix} a_{440}-\phi & a_{450} & a_{460} \\ a_{540} & a_{550}-\phi & a_{560} \\ a_{640} & a_{650} & a_{660}-\phi \end{vmatrix}^2 = 0.$$

These equations might degenerate for algebras of each class. The characteristic equation, if the degree is less than $n+1$, might thus arise in different ways. For example, if an algebra of the first frame above had the equation

$$(a_{110}-\phi)^6\,(a_{220}-\phi)^4=0,$$

the units might include $\lambda_{110}+\lambda_{330}$ or $\lambda_{220}+\lambda_{330}$, according to whether $a_{330}\equiv a_{110}$ or $a_{330}\equiv a_{220}$.

The frames that are possible determine the possible associative numbers. There may be for a given frame, obviously, more numbers expressible than the order of the frame, e. g. for the frame

$$\begin{array}{lllll} 110 & & & & \\ 210 & 220 & & & \\ 111 & 121 & 110 & & \\ 211 & 221 & 210 & 220 & \\ 112 & 122 & 111 & 121 & 110 \end{array} \qquad \text{or} \qquad \begin{array}{ll} 110 & \\ 210 & 220 \\ 111 & 121 \\ 211 & 221 \\ 112 & 122 \end{array}$$

which is of order *five*, we may have as distinct numbers the *nine*

$$\lambda_{110},\ \lambda_{220},\ \lambda_{210},\ \lambda_{111},\ \lambda_{121},\ \lambda_{211},\ \lambda_{221},\ \lambda_{112},\ \lambda_{122}.$$

We might look upon any algebra built upon this frame as a sub-algebra of this one of order nine. The general equation of the algebra is

$$(a_{110}-\phi)^3(a_{220}-\phi)^2=0.$$

But this is equally the equation of the sub-algebra

$$\lambda_{110},\ \lambda_{111},\ \lambda_{112},\ \lambda_{220},\ \lambda_{221},\ \lambda_{210},\ \lambda_{211}.$$

On the other hand the sub-algebra of five units

$$\lambda_{110}+\lambda_{220}, \qquad \lambda_{210}+a\lambda_{121}, \qquad \lambda_{111}+\lambda_{221}, \qquad \lambda_{211}+a\lambda_{122}, \qquad \lambda_{112},$$

has the equation

$$(a_{110}-\phi)^5=0,$$

and is isomorphic with λ_{110}, λ_{111}, λ_{112}, λ_{113}, λ_{114} of a different frame. Evidently the equation of a frame cannot have a higher degree than the frame, and if all species of the general equation are considered we have corresponding species of sub-algebras, which may each consist of many individual algebras. The equation just given might however have arisen from some other general equation, belonging to a different frame.

§ 2. *Every algebra or group of numbers whose characteristic equation is*

$$\Phi_1^{m'_1}\Phi_2^{m'_2}\cdots\Phi_e^{m'_e}=0,$$

(*the factors* Φ_i *being of width* w_i) *and belonging to a frame whose general equation is*

$$\Phi_1^{m_1}\Phi_2^{m_2}\cdots\Phi_e^{m_e}=0 \qquad (m_i \gtreqless m'_i),$$

must contain the associative units

$$\lambda_{r_i+t,\, r_i+u,\, 0} \qquad (i=1,\,2,\,\cdots,\,e),$$

$$\lambda_{r_i+t,\, r_i+u,\, k_i} \quad \left(t,\, u=0,\,1,\,\cdots,\,s_i;\, m_i > k_i > \frac{m_i}{m_i'}-1\right),$$

by Part 2, § *5 (3)*.

For example, in SCHEFFER's V_8,* with equation

$$(x-x_5e)(x-x_4e)(x-x_5e)=0$$

we have

$$e_5=\lambda_{330}, \qquad e_4=\lambda_{220}, \qquad e_3=\lambda_{110}, \qquad e_2=\lambda_{310}, \qquad e_1=\lambda_{320}.$$

The frame is of order 5, general equation

$$(x-x_5e)^2(x-x_4e)^2(x-x_5e)=0.$$

V_9 is from the same frame, but

$$e_5=\lambda_{330}, \qquad e_4=\lambda_{220}, \qquad e_3=\lambda_{110}, \qquad e_2=\lambda_{210}, \qquad e_1=\lambda_{320}.$$

V_{10} has the equation $(x-x_4e)(x-x_5e)^2=0$, and this equation may come from either

$$(x-x_4e)^3(x-x_5e)^2=0 \qquad \text{or} \qquad (x-x_4e)^2(x-x_5e)^3=0.$$

The first corresponds to V_{10}, the second to V_{11}, which has the same equation, giving

$$V_{10}, \qquad e_5=\lambda_{220}, \qquad e_4=\lambda_{110}, \qquad e_3=\lambda_{221}, \qquad e_2=\lambda_{211}, \qquad e_1=\lambda_{210}.$$

$$V_{11}, \qquad e_5=\lambda_{110}, \qquad e_4=\lambda_{220}, \qquad e_3=\lambda_{112}, \qquad e_2=\lambda_{121}, \qquad e_1=\lambda_{122}.$$

All algebras of this type for the order 5 may be arrived at by considering the general equations of order 5 which can reduce without losing any shear factor. These are evidently, $(x-x_0)^5$, $(x-x_0)^4(x-x_1)$, $(x-x_0)^3(x-x_1)(x-x_2)$, $(x-x_0)^3(x-x_1)^2$, $(x-x_0)^2(x-x_1)(x-x_2)(x-x_3)$, $(x-x_0)^2(x-x_1)^2(x-x_2)$, giving respectively $(x-x_0)^4$, $(x-x_0)^3$, $(x-x_0)^2$,

$$(x-x_0)^3(x-x_1),\ (x-x_0)^2(x-x_1),\ (x-x_0)(x-x_1),$$

$$(x-x_0)^2(x-x_1)(x-x_2),\ (x-x_0)(x-x_1)(x-x_2),$$

$$(x-x_0)^2(x-x_1)^2,\ (x-x_0)(x-x_1)^2,\ (x-x_0)^3(x-x_1),$$

$$(x-x_0)^2(x-x_1),\ (x-x_0)(x-x_1),$$

$$(x-x_0)(x-x_1)(x-x_2)(x-x_3),$$

$$(x-x_0)^2(x-x_1)(x-x_2),\ (x-x_0)(x-x_1)(x-x_2).$$

---

* Mathematische Annalen, vol. 39 (1891), S. 356.

The equations which are alike but derived from different equations, would generally belong to algebras with different multiplication tables, which shows again that the characteristic equation alone does not determine the algebra, though it conditions it.

We have to consider next the case in which the shear factors coalesce, giving an equation of the form

$$\Phi_1^{m_1'}\Phi_2^{m_2'}\cdots\Phi_{c'}^{m_{c'}'}=0 \qquad (c<e).$$

The only way in which this can happen is by certain identities between the otherwise arbitrary constituents being added to the general equation, e. g., let the general equation be

$$\begin{vmatrix} a_{110}-\phi & a_{120} \\ a_{210} & a_{220}-\phi \end{vmatrix}^4 (a_{330}-\phi)^3 \begin{vmatrix} a_{440}-\phi & a_{450} \\ a_{540} & a_{550}-\phi \end{vmatrix}^2 = 0,$$

the special equation being

$$\begin{vmatrix} a_{110}-\phi & a_{120} \\ a_{210} & a_{220}-\phi \end{vmatrix}^3 = 0.$$

In this case the factors Φ_2, Φ_3, have been absorbed into Φ_1, that is, the operation of Φ_1^3 on the field annuls it altogether. This must mean that the operation of Φ_1 is equivalent to the simultaneous operation of $\Phi_1\Phi_2\Phi_3$, or that

$$a_{110}\equiv a_{330}\equiv a_{440},$$
$$a_{120}\equiv a_{450},$$
$$a_{210}\equiv a_{540},$$
$$a_{220}\equiv a_{550}.$$

Therefore there must be involved in ϕ the compound units

$$\lambda_{110}+\lambda_{330}+\lambda_{440},$$
$$\lambda_{220}+\lambda_{550},$$
$$\lambda_{120}+\lambda_{450},$$
$$\lambda_{210}+\lambda_{540},$$

so that

$$\phi = a_{110}(\lambda_{110}+\lambda_{330}+\lambda_{440})+a_{220}(\lambda_{220}+\lambda_{550})+a_{120}(\lambda_{120}+\lambda_{450}) + a_{210}(\lambda_{210}+\lambda_{540})+\cdots.$$

The effect of Φ_1 on the field is to reduce the field by all the vectors which belong to the region ρ_{11}, ρ_{21}, ρ_{31}, ρ_{41}, ρ_{51}. In general the coalesced shear factor will reduce the corresponding shear regions into the second invariant region. A further application will reduce the same into the third invariant region. If the factor Φ_1^2 or $\Phi_1^3 \cdots$ occurs in the equation of the group of algebras under discussion, there must be terms of the form

$$\Sigma a_{rs4} \lambda_{rs4}$$

or

$$\Sigma a_{rs3} \lambda_{rs3} + \Sigma a_{rs4} \lambda_{rs4},$$

and so on, according to the character of the coalescence. The further discussion of this case must be reserved for another paper. It is to be observed that a form arising thus may be isomorphic with a simpler form. There is in fact a sort of resemblance between the relation of these associative units to the algebras they represent and the relation of substitutions to abstract groups.

The determination of all forms of units for the equation

$$\Phi_1^{m'_1} = 0$$

which has only one shear factor, is obviously the basal problem of the subject. The determination of these units enables us to determine corresponding units for any equation of the form

$$\Phi_1^{m'_1} \Phi_2^{m'_2} \cdots = 0 .$$

These may be called *direct units.* Units of the form

$$\Sigma a_{rst} \lambda_{rst}$$

where r and s belong to different shear regions are called *oblique units.* The square of any such obviously vanishes. The determination for a given frame and given direct units is a very simple matter.

§ 3. The frames which have shears whose width is *unity* only, include SCHEFFERS' *nonquaternionic* * algebras. In none of his cases can any diagonal coefficient a_{rr0} vanish, since all his algebras imply a modulus. The η's of this treatment are the forms

$$\lambda_{110}, \ \lambda_{220},$$

or

$$\lambda_{110} + \lambda_{220}, \text{ etc.}$$

Obviously the e's fall into the two classes

$$\Sigma a_{rrt} \lambda_{rrt},$$

$$\Sigma a_{rst} \lambda_{rst} \qquad (r \neq s).$$

The equation is of the form

$$(x - x_1)^{\mu_1} (x - x_2)^{\mu_2} \cdots (x - x_s)^{\mu_s} = 0 .$$

* *Complexe Zahlensysteme,* Mathematische Annalen, vol. 39 (1891), pp. 293–390.

A table of the expression of SCHEFFERS' nonquaternionic forms up to order 5 follows in a later section.

The frames which have shears whose width is *two*, include for every such shear a *quaternionic* system. To such a factor correspond as necessary members of the algebra

$$\begin{matrix} \lambda_{rr0} & \lambda_{rr+1\,0} \\ \lambda_{r+1\,r0} & \lambda_{r+1\,r+1\,0} \end{matrix}$$

and these are essentially quaternionic, in the canonical form of quaternions. The corresponding factor of the equation is

$$\begin{vmatrix} x_1 - x & x_2 \\ x_3 & x_4 - x \end{vmatrix}^m,$$

or equally

$$[x^2 - 2x_0 x + x_0^2 + x_1^2 + x_2^2 + x_3^2]^m.$$

If a system is of order more than four, and its equation contains the factor

$$\begin{vmatrix} x_1 - x & x_2 \\ x_3 & x_4 - x \end{vmatrix}^m$$

it *must* contain the quaternion system and also forms of the type

$$\begin{matrix} \lambda_{rr1} & \lambda_{r,\,r+1,\,1} \\ \lambda_{r+1,\,r,\,1} & \lambda_{r+1,\,r+1,\,1} \end{matrix}$$

etc.

This extends the theory given by SCHEFFERS.*

It is obvious that if the frame of any algebra is of the form

110	120				
210	220				
111	121	110	120		
211	221	210	220		
112	122	111	121	110	120
212	222	211	221	210	220
. . .		. . .		. . .	
. . .		. . .		. . .	

* Loc. cit., S. 364.

then it contains the units

$$\{\lambda_{110}, \lambda_{120}, \lambda_{210}, \lambda_{220}\} = \{Q\},$$

$$\{\lambda_{111}, \lambda_{121}, \lambda_{211}, \lambda_{221}\} = \{Q\}(\lambda_{111} + \lambda_{221}) = (\lambda_{111} + \lambda_{221})\{Q\},$$

$$\{\lambda_{112}, \lambda_{122}, \lambda_{212}, \lambda_{222}\} = \{Q\}(\lambda_{112} + \lambda_{222}) = (\lambda_{112} + \lambda_{222})\{Q\}.$$

If we have the frame below,

110	120				
210	220				
310	320	330	340		
410	420	430	440		
111	121	131	141	110	120
211	221	231	241	210	220

and if also

$$a_{110} = a_{330}, \qquad a_{120} = a_{340}, \qquad a_{210} = a_{430}, \qquad a_{220} = a_{440},$$

then we must have the units

$$\{\lambda_{110} + \lambda_{330}, \lambda_{120} + \lambda_{340}, \lambda_{210} + \lambda_{430}, \lambda_{220} + \lambda_{440}\} = \{Q\},$$

$$\{\lambda_{111}, \lambda_{121}, \lambda_{211}, \lambda_{221}\} = \{Q\}(\lambda_{111} + \lambda_{221}) = (\lambda_{111} + \lambda_{221})\{Q\},$$

and we may have either or both of the sets

$$\{\lambda_{310}, \lambda_{320}, \lambda_{410}, \lambda_{420}\} = \{Q\}(\lambda_{310} + \lambda_{420}) = (\lambda_{310} + \lambda_{420})\{Q\},$$

$$\{\lambda_{131}, \lambda_{141}, \lambda_{231}, \lambda_{241}\} = \{Q\}(\lambda_{131} + \lambda_{241}) = (\lambda_{131} + \lambda_{241})\{Q\}.$$

These examples show that the algebras are equivalent to the "product" of $\{Q\}$, the quaternion system, into the systems

110		110		
111	or	111	110	121
112		210		220
. . .				

respectively.

A theorem given by SCHEFFERS* follows easily and as a matter of course. As may be seen at once it is easily extended to the classes of algebras that are higher in character, viz. *nonionic*, and *matrical*, in general. This is a very

* Loc. cit., S. 374.

remarkable generalization, stated, it is believed, for the first time. It may be stated thus:

Let there be given a certain frame corresponding to a set of numbers Q; let there be also given a frame corresponding to a set of numbers P; let there be constructed a frame S by substituting for each λ_{ii0} in P, the frame (square matrix) for Q, writing this matrix with different subscripts for λ_{110}, λ_{220}, $\lambda_{330}, \cdots, \lambda_{pp0}$; also let square matrices be substituted for every λ_{ijk}, each similar in structure to Q, and each deriving its first subscripts from those in λ_{ii0}, its second subscripts from those in λ_{jj0}, its third subscripts by the addition of k to corresponding third subscripts in λ_{ii0} or λ_{jj0}; finally, let the shear factors of the diagonal of S coalesce, then is

$$\{S\} = \{P\} \cdot \{Q\},$$

and conversely.

Thus, the first case above is derived from

$$Q = \left\{ \begin{matrix} 110 & 120 \\ 210 & 220 \end{matrix} \right\}, \qquad P = \left\{ \begin{matrix} 110 & & \\ 111 & 110 & \\ 112 & 111 & 110 \end{matrix} \right\};$$

from the second case

$$Q = \left\{ \begin{matrix} 110 & 120 \\ 210 & 220 \end{matrix} \right\}, \qquad P = \left\{ \begin{matrix} 110 & & \\ 111 & 110 & 121 \\ 211 & & 220 \end{matrix} \right\},$$

This theorem, however, applies to frames P and Q of *any* character.

§ 4. Algebras whose frames have factors of width three, might be called *nonionic*, and if the factors are *any form of matrix*, *matrical*; thus we may have algebras with the equation

$$\begin{vmatrix} a_{110} - \phi & a_{120} & a_{130} \\ a_{210} & a_{220} - \phi & a_{230} \\ a_{310} & a_{320} & a_{330} - \phi \end{vmatrix}^2 (a_{440} - \phi)^3 = 0,$$

or with

$$\begin{vmatrix} a_{110} - \phi & a_{120} & 0 & 0 \\ a_{210} & a_{220} - \phi & a_{230} & 0 \\ a_{310} & a_{320} & -\phi & a_{340} \\ 0 & 0 & a_{430} & a_{440} - \phi \end{vmatrix}^2 = 0.$$

§ 5. The associative units arising from the main diagonal squares, or from the shear factors, in other words, form sub-algebras. These are MOLIEN's "begleitende" and "ursprüngliche" * number systems. MOLIEN's theorems do not touch the other units corresponding to factors of multiplicity more than unity.

It was C. S. PEIRCE who first demonstrated the matrix character of linear associative algebra.† The work of BENJAMIN PEIRCE was first put forth in 1870. It was the first to prove the existence of the different idempotent units, and the consequent separation of an algebra into sets of units on the basis of these idempotent units. The idempotents are the η's of SCHEFFERS, and the "gerade" units are the units of PEIRCE's first or fourth, the "schiefe" of the second or third groups, respectively. C. S. PEIRCE seems to have made little mathematical use of his theorem that all algebras are modified forms of "relative" algebras.

§ 6. We close with a general theorem in extension of one given by PEIRCE.

By differentiating the equation of an algebra and remembering that $d\phi$ is any number, we arrive at equations which the numbers of the algebra must satisfy.

For instance, let the equation be

$$(a_{110} - \phi)^2 = 0,$$

then

$$(a_{110} - \phi)(a'_{110} - \phi') + (a'_{110} - \phi')(a_{110} - \phi) = 0$$

is true of any two numbers ϕ, ϕ' of the algebra.

Let the equation be

$$(a_{110} - \phi)^2 (a_{220} - \phi) = 0,$$

then

$$(a_{110} - \phi)(a'_{110} - \phi')(a_{220} - \phi) + (a'_{110} - \phi')(a_{110} - \phi)(a_{220} - \phi) + (a_{110} - \phi)^2 (a'_{220} - \phi') = 0,$$

and

$$(a''_{110} - \phi'')(a'_{110} - \phi')(a_{220} - \phi) + (a_{110} - \phi)(a'_{110} - \phi')(a''_{220} - \phi'') + (a'_{110} - \phi')(a''_{110} - \phi'')(a_{220} - \phi) + (a'_{110} - \phi')(a_{110} - \phi)(a''_{220} - \phi'') + (a''_{110} - \phi'')(a_{110} - \phi)(a'_{220} - \phi') + (a_{110} - \phi)(a''_{110} - \phi'')(a'_{220} - \phi') = 0.$$

Corollary. In the case

$$\phi^n = 0,$$

$$\phi_1 \phi_2 \cdots \phi_n + \phi_2 \phi_1 \cdots \phi_n + \cdots + \phi_1 \phi_2 \cdots \phi_n = 0,$$

all permutations of the numbers appearing.

This corollary was first stated by BENJAMIN PEIRCE.

---

* *Ueber Systeme höherer complexer Zahlen*, Mathematische Annalen, vol. 41 (1892), p. 83–156.

† Proceedings American Academy of Arts and Sciences, May 11, 1875. See also *Linear Associative Algebra*, American Journal of Mathematics, vol. 4, p. 97.

In a manner quite similar to the development of quaternions, combinations of the numbers may be named and studied, further developing the calculus side of the theory, as

$$\phi_2\phi_1 - \phi_1\phi_2, \text{ etc.}$$

§ 7. As examples, the following cases of SCHEFFERS are expressed in the forms given here.

II_1.	$\lambda_{110}, \lambda_{111},$	$[(a_{110}-\phi)^2=0]$.
III_1.	$\lambda_{110}, \lambda_{111}, \lambda_{112},$	$[(a_{110}-\phi)^3=0]$.
III_2.	$\lambda_{110}, \lambda_{220}, \lambda_{210},$	$[(a_{110}-\phi)(a_{220}-\phi)=0]$.
III_3.	$\lambda_{110}+\lambda_{220}, \lambda_{111}, \lambda_{210},$	$[(a_{110}-\phi)^2=0]$.
IV_1.	$\lambda_{110}, \lambda_{111}, \lambda_{112}, \lambda_{113},$	$[(a_{110}-\phi)^2=0]$.
IV_2.	$\lambda_{110}, \lambda_{220}, \lambda_{111}, \lambda_{121},$	$[(a_{110}-\phi)^2(a_{220}-\phi)=0]$.
IV_3.	$\lambda_{110}+\lambda_{220}, \frac{1}{2}(\lambda+1)\lambda_{210}+2\cdot\lambda_{122}, \lambda_{111}+\lambda_{210}, \lambda_{112},$	$[(a_{110}-\phi)^3=0]$.
IV_4.	$\lambda_{110}+\lambda_{220}, \lambda_{111}, b\lambda_{210}+\frac{1}{b}\lambda_{122}, \lambda_{112},$	$[(a_{110}-\phi)^3=0]$.
IV_5.	$\lambda_{110}+\lambda_{220}, \lambda_{111}, \lambda_{210}, \lambda_{112},$	$[(a_{110}-\phi)^3=0]$.
IV_6.	$\lambda_{110}, \lambda_{220}, \lambda_{210}, \lambda_{211},$	$[(a_{110}-\phi)(a_{220}-\phi)=0]$.
IV_7.	$\lambda_{110}, \lambda_{220}, \lambda_{211}, \lambda_{121},$	$[(a_{110}-\phi)(a_{220}-\phi)=0]$.
IV_8.	$\lambda_{110}+\lambda_{220}, \lambda_{111}-\lambda_{221}, -\lambda_{210}, \lambda_{211},$	$[(a_{110}-\phi)^2=0]$.
IV_9.	$\lambda_{110}+\lambda_{220}, \lambda_{111}+\lambda_{221}, \lambda_{211}, \lambda_{121},$	$[(a_{110}-\phi)^2=0]$.
IV_{10}.	$\lambda_{110}, \lambda_{120}, \lambda_{210}, \lambda_{220},$	$\left[\begin{vmatrix} a_{110}-\phi & a_{120} \\ a_{210} & a_{220}-\phi \end{vmatrix}=0\right]$.
V_1.	$\lambda_{110}, \lambda_{111}, \lambda_{112}, \lambda_{113}, \lambda_{114},$	$[(a_{110}-\phi)^5=0]$.
V_2.	$\lambda_{110}, \lambda_{220}, \lambda_{111}, \lambda_{112}, \lambda_{122},$	$[(a_{110}-\phi)^3(a_{220}-\phi)=0]$.
V_3.	$\lambda_{110}, \lambda_{220}, \lambda_{111}, \lambda_{221}, \lambda_{210},$	$[(a_{110}-\phi)^2(a_{220}-\phi)^2=0]$.
V_4.	$\lambda_{110}+\lambda_{220}, \lambda_{210}+\lambda_{123}-\lambda_{112}, \lambda_{111}+2\cdot\lambda_{123}, \lambda_{112}, \lambda_{113},$	$[(a_{110}-\phi)^4=0]$.
V_5.	$\lambda_{110}+\lambda_{220}, \lambda_{111}, \lambda_{112}, \lambda_{113}, \lambda_{210}+\lambda_{123},$	$[(a_{110}-\phi)^4=0]$.
V_6.	$\lambda_{110}+\lambda_{220}, \lambda_{111}+2\lambda_{123}, \lambda_{112}, \lambda_{113}, \lambda_{211},$	$[(a_{110}-\phi)^4=0]$.
V_7.	$\lambda_{110}+\lambda_{220}, \lambda_{111}, \lambda_{112}, \lambda_{113}, \lambda_{210},$	$[(a_{110}-\phi)^4=0]$.
V_8.	$\lambda_{110}, \lambda_{220}, \lambda_{330}, \lambda_{310}, \lambda_{320},$	$[(a_{110}-\phi)(a_{220}-\phi)(a_{330}-\phi)=0]$.
V_9.	$\lambda_{110}, \lambda_{220}, \lambda_{330}, \lambda_{210}, \lambda_{320},$	$[(a_{110}-\phi)(a_{220}-\phi)(a_{330}-\phi)=0]$.

V_{10}.	$\lambda_{110}, \lambda_{220}, \lambda_{221}, \lambda_{211}, \lambda_{210},$	$[(a_{110}-\phi)(a_{220}-\phi)^2=0]$.
V_{11}.	$\lambda_{110}, \lambda_{112}, \lambda_{220}, \lambda_{121}, \lambda_{122},$	$[(a_{110}-\phi)^2(a_{220}-\phi)=0]$.
V_{12}.	$\lambda_{110}, \lambda_{220}, \lambda_{210}, \lambda_{122}, \lambda_{112},$	$[(a_{110}-\phi)^2(a_{220}-\phi)=0]$.
V_{13}.	$\lambda_{110}, \lambda_{220}, \lambda_{211}, \lambda_{122}, \lambda_{112},$	$[(a_{110}-\phi)^2(a_{220}-\phi)=0]$.

§ 8. As further examples, the following of PEIRCE's cases are expressed in the present form:

(a_1)	$\lambda_{110},$	$[\phi-xi=0]$.
(b_1)	$\lambda_{111},$	$[\phi=0]$.
(a_2)	$\lambda_{110}, \lambda_{111},$	$[(\phi-xi)^2=0]$.
(b_2)	$\lambda_{110}, \lambda_{120},$	$[(\phi-xi)\phi=0]$.
(c_2)	$\lambda_{111}, \lambda_{112},$	$[\phi^3=0]$.
(a_3)	$\lambda_{110}, \lambda_{111}, \lambda_{112},$	$[(\phi-xi)^3=0]$.
(b_3)	$\lambda_{111}, \lambda_{112}, \lambda_{113},$	$[\phi^3=0]$.
(c_3)	$a\lambda_{210}+\lambda_{111}, \lambda_{112}, \lambda_{210}+\lambda_{122},$	$[\phi^3=0]$.
(d_3)	$\lambda_{210}+\lambda_{111}, \lambda_{112}, \lambda_{122},$	$[\phi^3=0]$.
(e_3)	$\lambda_{211}, \lambda_{210}, \lambda_{111}-\lambda_{121},$	$[\phi^2=0]$.
(a_4)	$\lambda_{110}, \lambda_{111}, \lambda_{112}, \lambda_{113},$	$[(\phi-xi)^4=0]$.
(b_4)	$\lambda_{110}+\lambda_{220}, a\lambda_{210}+\lambda_{111}, \lambda_{112}, \lambda_{210}+\lambda_{122},$	$[(\phi-xi^3)=0]$.
(c_4)	$\lambda_{110}+\lambda_{220}, \lambda_{210}+\lambda_{111}, \lambda_{112}, \lambda_{122},$	$[(\phi-xi)^3=0]$.
(d_4)	$\lambda_{110}+\lambda_{220}, \lambda_{211}, \lambda_{210}, \lambda_{111}-\lambda_{121},$	$[(\phi-xi)^2=0]$.
(e_4)	$\lambda_{220}, \lambda_{221}, \lambda_{210}, \lambda_{211},$	$[(\phi-xi)^2\phi=0]$.
(f_4)	$\lambda_{220}, \lambda_{222}, \lambda_{212}, \lambda_{120},$	$[(\phi-xi)^3=0]$.
(g_4)	$\lambda_{110}, \lambda_{120}, \lambda_{210}, \lambda_{220},$	$\left[\begin{vmatrix} x_1 i-\phi & x_2 \\ x_3 & x_4 l-\phi \end{vmatrix}=0\right]$.
(h_4)	$\lambda_{110}, \lambda_{120}, \lambda_{212}, \lambda_{222},$	$[(\phi-xi)\phi^3=0]$.
(i_4)	$\lambda_{111}, \lambda_{112}, \lambda_{113}, \lambda_{114},$	$[\phi^4=0]$.

The completion of these tables is a simple matter. It is proposed in a following paper to extend them and systematize them.

§ 9. A natural classification of algebras seems to be that according to the number and character of distinct factors. Thus we have algebras of class:

linear matrix, all factors linear;

quadratic matrix, some factors determinants of second order, quadratic-linear, quadratic-quadratic, etc.;

cubic matrix, etc.;

of degree:

one, *two*, *three*, according to the number of factors;

of multiplicity:

one, *two*, *three*, according to the powers of factors;

A natural treatment seems to be first to consider those of equation

$$(a-\phi)^m = 0,$$

second, of equation

$$(a_1-\phi)^{m_1}(a_2-\phi)^{m_2}\cdots = 0,$$

third, of equation

$$\begin{vmatrix} a_{11}-\phi & a_{12} \\ a_{21} & a_{22}-\phi \end{vmatrix}^m = 0,$$

etc.

This method we might call the *statistical classification.*

A second classification is according to the order of the framework necessary. Thus for a framework of order 2, we may have

(1) (2)

(1) $\lambda_{110}, \quad \lambda_{120}, \quad \lambda_{210}, \quad \lambda_{220},$

with sub-cases

(11) $\lambda_{110}, \quad \lambda_{220}, \quad \lambda_{120},$

(12) $\lambda_{110}, \quad \lambda_{220}, \quad \lambda_{210},$

(13) $\lambda_{110}, \quad \lambda_{220},$

(14) $\lambda_{110}, \quad \lambda_{120},$

(15) $\lambda_{110},$

(16) $\lambda_{120},$

(2) $\lambda_{110}+\lambda_{220}, \quad \lambda_{210},$

(3) $\lambda_{110}, \quad \lambda_{111}.$

In such cases we should frequently find algebras of units whose substitution effect on the *framework* would be different, yet whose combination tables would be the same.

September 1, 1902.