PROJECTIVE COÖRDINATES*

BY

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The definition of coördinates in projective geometry should naturally be projectively stated. There is, I think, a very distinct advantage in taking them as double ratios, a step which I do not find in the literature. †

The matter, restricted to two dimensions for simplicity of statement only, might be presented as follows.

I assume that the theory in one dimension has been explained, and that the notion of the double ratio of two points xy and two lines $\xi\eta$ of a plane is known; on varying the order there are *two* reciprocal double ratios

$$(xy \mid \xi\eta) \qquad \text{and} \qquad (yx \mid \xi\eta).$$

§ 1. *Point coördinates.*

Let now four lines α_0, α_i $(i = 1, 2, 3)$ of a plane be given; call α_0 the *auxiliary line* and the triangle formed by the other 3 lines the triangle of reference. Let its points be a_1, a_2, a_3.

With any point x of the plane, not on α_0, are associated 3 double ratios

$$r_i = (xa_i \mid \alpha_i \alpha_0). \tag{1}$$

These three double ratios are the projective coördinates of the point.

When the auxiliary line is the line at infinity, we have the barycentric case; it appears at once from this metrically canonic case that always

$$r_1 + r_2 + r_3 = 1. \tag{2}$$

That is,

The sum of the projective coördinates is 1.

When one coördinate of x is given, x lies on a line; and any two coördinates determine the point. But for points on the auxiliary line α_0 the coördinates are infinite. See the last paragraph of § 4.

Ordinary Cartesian coördinates require no peculiar treatment; the triangle of reference is formed by the axes and the line joining the unit points on the

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† The nearest approach that I know of is made by KOHN, Wiener Sitzungsberichte, vol. 104 (1895), p. 1167.

axes, the auxiliary line being at infinity. The only difference between barycentric and cartesian coördinates lies in the fact that we usually assume in the latter case an isosceles triangle of reference.

§ 2. *Line coördinates.*

With the same reference triangle take an auxiliary point a_0, and with a line ξ associate the double ratios

$$\rho_i = (\xi\alpha_i \mid a_i a_0). \tag{3}$$

For a metrically canonic case we take a_0 to be the centroid of the reference triangle. Then if δ_j be the distance from a_j to ξ, $(j = 0, \cdots, 3)$,

$$\rho_i = \tfrac{1}{3}\frac{\delta_i}{\delta_0}.$$

We have then *always*,

$$\rho_1 + \rho_2 + \rho_3 = 1. \tag{4}$$

except for lines through a_0, for which ρ_i is infinite.

§ 3. *Point and line.*

In the barycentric case we placed the auxiliary line α_0 at infinity, the auxiliary point a_0 at the centroid. So always we take a_0 as the polar point of α_0 as to the reference triangle. The well-known properties of polar point and line are read off from the barycentric case. Thus:

The polars of a line α_0 as to 2 of 3 points a_i from a triangle perspective with the triangle a, and the perspective point is the polar of α_0. Or:

If a 3-point is incident with a 3-line (i. e., each point on a line) and the two triangles are perspective, then the perspective point and line are polar point and line as to either triangle.

Now in the barycentric case the distance from the point x to the line ξ is

$$\delta = r_1\delta_1 + r_2\delta_2 + r_3\delta_3 = (r\delta).$$

Hence

$$\tfrac{1}{3}\frac{\delta}{\delta_0} = \left(r, \tfrac{1}{3}\frac{\delta}{\delta_0}\right).$$

Hence in general (§ 2) the double ratio $(xa_0 \mid \xi\alpha_0)$ is given by

$$\tfrac{1}{3}(xa_0 \mid \xi\alpha_0) = (r\rho) \tag{5}$$

and in particular *when a point and line are incident*

$$r_1\rho_1 + r_2\rho_2 + r_3\rho_3 = 0. \tag{6}$$

§ 4. *The symmetrical system of point coördinates.*

The double ratio $(xx' \mid \xi\xi')$ of two points xx' and two lines $\xi\xi'$ is (appealing once more to the barycentric case)

$$(xx' \mid \xi\xi') = \frac{(r\rho)(r'\rho')}{(r\rho')(r'\rho)}.$$

And in particular

$$r_1' = (xa_i \mid \alpha_1 \alpha_0) = \frac{(x\alpha_1)\,(a_1 \alpha_0)}{(x\alpha_0)\,(a_1 \alpha_1)}.$$

It is no longer convenient to think of the triangle a_i or α_i as the triangle of reference—rather it is any triangle, the reference triangle being placed in the background. The point a_1, the join of the lines α_2 and α_3, is given by

$$(a_1 \alpha_2) = 0, \qquad (a_1 \alpha_3) = 0.$$

Hence

$$r_1 = \frac{(x\alpha_1) \mid \alpha_2 \alpha_3 \alpha_0 \mid}{(x\alpha_0) \mid \alpha_2 \alpha_3 \alpha_1 \mid},$$

and the relation

$$r_1 + r_2 + r_3 = 1$$

becomes

$$(x\alpha_1) \mid \alpha_2 \alpha_3 \alpha_0 \mid + (x\alpha_2) \mid \alpha_3 \alpha_1 \alpha_0 \mid + (x\alpha_3) \mid \alpha_1 \alpha_2 \alpha_0 \mid = (x\alpha_0) \mid \alpha_2 \alpha_3 \alpha_1 \mid,$$

or say

(7) $$x_1 + x_2 + x_3 + x_0 = 0.$$

These "supernumerary homogeneous coördinates" x_j, or as I would call them *symmetrical coördinates*, are used only in connection with homogeneous equations; whereas with the double ratios it is mainly as a matter of convenience in polarizing that we prefer homogeneous equations.

It will be noticed that the symmetrical system is not intuitively a projective system at all; that is, we do not have (at all events at present) a direct projective significance for x_j expressed in terms of the 4 lines and the point alone, but only for a ratio of two x_j's.

In passing from 3 dimensions where $r_i = (xa_i \mid \alpha_i \alpha_0)$ $(i = 1, 2, 3, 4)$, to a plane of reference α_i we pass at once to the projective system in that plane, but in passing to the auxiliary plane α_0 we pass at once to the symmetrical system in that plane. And so in general.

§ 5. *The symmetrical system of line coördinates.*

In the same way, given 4 points a_j $(j = 0, 1, 2, 3)$ and a line ξ, we have

$$\rho_1 = \frac{(\xi a_1) \mid a_2 a_3 a_0 \mid}{(\xi a_0) \mid a_2 a_3 a_1 \mid},$$

and we take the 4 numbers

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(8) \qquad
\begin{aligned}
\xi_1 &= (\xi a_1)\,|\,a_2 a_3 a_0\,|, \\
\xi_2 &= (\xi a_2)\,|\,a_3 a_1 a_0\,|, \\
\xi_3 &= (\xi a_3)\,|\,a_1 a_2 a_0\,|, \\
\xi_0 &= -\,(\xi a_0)\,|\,a_1 a_2 a_3\,|,
\end{aligned}
$$

whose sum is zero, as symmetrical coördinates of the line with reference to the four points.

But the 4 points have to be connected with the 4 lines of § 4, as otherwise the geometrical use of symmetrical systems would be extremely restricted.

Any rational equation in r_i gives a curve which is of course a covariant of the 4 lines α_j; the curve simply carries its equation with it when projected. And so any homogeneous equation in x_j defines a covariant curve of the 4 lines; but of special interest are the symmetrical equations, defining curves not altered by interchange of lines.* And of these the simplest is

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0.$$

The polars of the 4 lines as to this nullipartite conic are the 4 points to be associated with the 4 lines. Let us call the two tetrads *counter-tetrads;* their combination a *frame of reference.*

§ 6. *The metrically canonic frame.*

Denote by S_n a flat space of n dimensions. Let the plane be the plane at infinity in S_3; and replace the points and lines by lines and planes through an origin O. Let the nullipartite absolute of this (elliptic) geometry be given by $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$.

In the plane $x_j = 0$ we have a geometry with the intersection of that plane and the absolute cone as absolute; hence from the theory in S_1 the planes $x_1 x_2 x_3$ cut x_0 in equispaced lines; whence the four planes are parallel to the faces of a regular tetrahedron, or, say, are *equispaced.* With these planes associate their normals through O; these are the diagonals of a cube. The combination is the metrically canonic frame, and the mutuality of the tetrads is evident.

To obtain a canonic frame in a plane we may cut the frame just mentioned across a diagonal of the cube, thus getting for the four points the vertices and center of an equilateral triangle, and for one of the lines the line at infinity;

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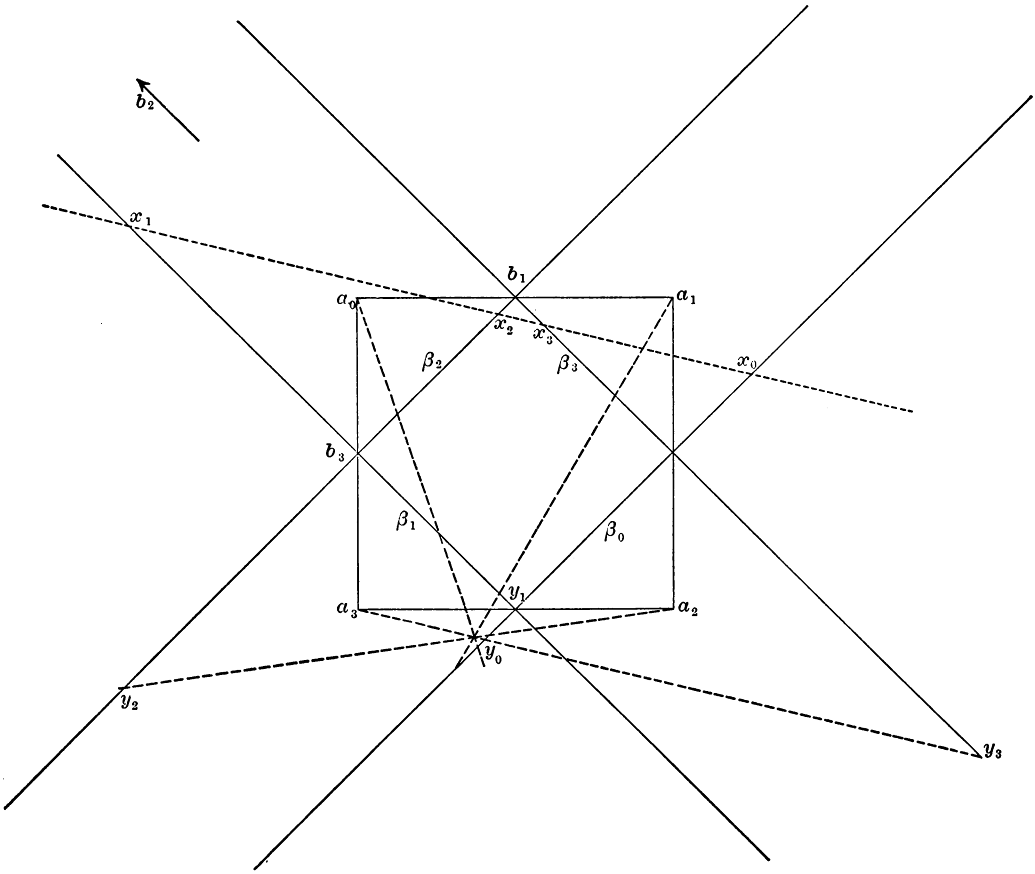
* That is, invariant under KLEIN's group of 24 collineations which leave the 4-line unaltered. KLEIN, Mathematische Annalen, vol. 4 (1871), p. 346, or *Icosaeder*, p. 166; MOORE, American Journal of Mathematics, vol. 22 (1900).

whence *each line is the polar line of one of the points as to the 3-line formed by the other points*, and reciprocally *each point is the polar point of a line as to the 3-point formed by the other lines.* Or, again, we may cut the frame by a face of the cube; we thus get the most convenient form (see figure), but it is not one that can be generalized for S_n, whereas the regular $(n+1)$-point and its center are always available in S_n.



§ 7. *Point and line in the symmetrical system.*

It is now to be shown that when a point x is referred to 4 lines and a line ξ to the counter-tetrad of 4 points, the condition of incidence is

$$(x\xi) \equiv x_0\xi_0 + x_1\xi_1 + x_2\xi_2 + x_3\xi_3 = 0.$$

Let the lines be β_j, the points a_j. When referred to the triangle $\beta_1\beta_2\beta_3$ or $b_1b_2b_3$, for which the auxiliary point and line are a_0 and β_0, the projective coördinates of x are simply

$$r_i = -x_i/x_0.$$

Those of ξ are $\rho_i = (\xi\beta_i \mid b_i a_0)$, while $-\xi_i/\xi_0 = (\xi\alpha_i \mid a_2 a_0)$. Estimating along the diagonal $\overline{a_0 a_2}$ of the figure we have

$$\rho_2 = \frac{\beta_2 - a_0}{\xi - a_0},$$

$$-\xi_2/\xi_0 = \frac{x - a_2}{x - a_0} \cdot \frac{\alpha_2 - a_0}{\alpha_2 - a_2}$$

$$= -\frac{x - a_2}{x - a_0}$$

$$= -1 + 4\frac{\beta_2 - a_0}{x - a_0}$$

$$= -1 + 4\rho_2.$$

That is,

$$1 - \xi_i/\xi_0 = 4\rho_i.$$

Hence

$$4(r\rho) = 1 + \sum^3 \frac{x_i \xi_i}{x_0 \xi_0} = \frac{(x\xi)}{x_0 \xi_0}.$$

If point and line unite, then

$$(r\rho) = 0,$$

and therefore

(10) $$(x\xi) = 0.$$

If not, then

(11) $$\tfrac{1}{3}(xa_0 \mid \xi\beta_0) = \tfrac{1}{4}\frac{(x\xi)}{x_0 \xi_0},$$

or say

$$\tfrac{1}{3}\lambda_j = \tfrac{1}{4}\frac{(x\xi)}{x_j \xi_j},$$

whence the relation between the double ratios $\lambda_j \equiv (xa_j \mid \xi\beta_j)$ is

(12) $$\sum \frac{1}{\lambda_j} = \tfrac{4}{3}.$$

[And so for the double ratios formed by a point and an S_{n-1} in S_n, when referred to a frame, defined by $n+2$ points a_j and the counter-scheme of S_{n-1}'s β_j, if

$$\lambda_j = (xa_j \mid \xi\beta_j),$$

$$\sum \frac{1}{\lambda_j} = \frac{n+2}{n+1}.$$

For $n = 1$ we may regard a_j and β_j as complex numbers and the canonic frame is then the vertices of a regular hexagon.]

§ 8. *Like-named point and line.*

It is worth while to show how to construct the polar system

$$\xi_0/x_0 = \xi_1/x_1 = \xi_2/x_2 = \xi_3/x_3$$

which is that of the covariant conic

$$\sum x_j^2 = 0.$$

Referring to the cube-diagonals this is merely to construct the plane normal to a given line; but we wish a ruler-construction in the plane.

We suppose the problem solved for S_1; that is that the partner of a point, or its polar as to the Hessian pair of three points, has been constructed. The quickest way (though of course not a projective way) is, when the three points a_j on a line are given to draw arcs containing the angle $\pi/3$ through $a_0 a_1$ and $a_1 a_2$ on the same side of the line; a T-square whose vertex is placed when the arcs meet marks off on the line the required involution.

Construct then the partner of the point where a line ξ meets β_0, as to the triad of points where $\beta_1 \beta_2 \beta_3$ meet β_0. Join this partner y_0 to a_0. We get thus 4 lines $y_j a_j$ which meet in the point required.

The polar system being now constructed for S_2 can be extended in the same way for S_3 and so on. That is, *the polar system of the invariant quadric is constructed.*

The figure (drawn by Mr. Carver) shows the present case $n = 2$.

§ 9. *Like-named points for two frames.*

A word on the beginning of the treatment of collineations may be added. The distinction between two views must be made—call them *alias* and *alibi*. In the former we consider the relations of the coördinates of a point x referred to different frames F, F'; in the second F' is projected into F and thereby x into x'. The like-named points of *alias* are the fixed points of *alibi*.

We ask now for the points which are like-named as to two frames F, F'. That is, the points for which

$$r_i = r'_i,$$

or

$$(xa_i \mid \alpha_i \alpha_0) = (xa'_i \mid \alpha'_i \alpha'_0),$$

or, in the symmetrical system,

$$x_i = \kappa x'_i.$$

They are the points common to all the conics

(13) $$\frac{x_0}{x_0'} = \frac{x_1}{x_1'} = \frac{x_2}{x_2'} = \frac{x_3}{x_3'}.$$

The general case is that of three distinct intersections. If these like-named points are taken as points of both frames, then the "change of coördinates" is merely a change of auxiliary point and therefore of auxiliary line. Particularly important is the case when the new auxiliary point lies on the old auxiliary line; for then if

$$r_i = \kappa_i r_i'$$

be the transformation we have $\kappa_i/\sum\kappa_i$ on the line for which r_i is ∞, so that

$$\sum\kappa_i = 0,$$

and the collineation is *normal*.

§ 10. *Hun's theorem.*

It was shown by PASCH* that, given a normal collineation N of a plane which sends any figure F into NF, there are triangles T inscribed in NT. In fact if two points ab become Na, Nb and the joins of a and Nb, b and Na determine c, then Nc is on ab.

There are, for given fixed points of N, ∞^5 such PASCH triangles. Mr. HUN has proved that *the relation of a Pasch triangle and the triangle of the fixed points is mutual.* I wish to prove this fact in such form that the restriction to two dimensions falls away. The fixed points being reference points, N is of the form

(14) $$\kappa_1 x_1 \xi_1 + \kappa_2 x_2 \xi_2 + \kappa_3 x_3 \xi_3 = 0,$$

where

$$\kappa_1 + \kappa_2 + \kappa_3 = 0.$$

And

$$x_1\xi_1 = (x\alpha_1 \,|\, \alpha_1\alpha_0)(\xi a_1 \,|\, a_1 a_0)$$

$$= \frac{(x\alpha_1)(a_1\alpha_0)}{(x\alpha_0)(a_1\alpha_1)}\,\frac{(\xi a_1)(\alpha_1 a_0)}{(\xi a_0)(\alpha_1 a_1)}.$$

Hence (14) takes the form

$$\Sigma\lambda_i(x\alpha_1)(\xi a_i) = 0,$$

so that if there be a triangle x_i, ξ_i of the PASCH kind (in which case the collineation is normal) then

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*Mathematische Annalen, vol 23 (1884).

$$\begin{vmatrix} (x_1\alpha_1)(\xi_1 a_1) & (x_1\alpha_2)(\xi_1 a_2) & (x_1\alpha_3)(\xi_1 a_3) \\ (x_2\alpha_1)(\xi_2 a_1) & (x_2\alpha_2)(\xi_2 a_2) & (x_2\alpha_3)(\xi_2 a_3) \\ (x_3\alpha_1)(\xi_3 a_1) & (x_3\alpha_2)(\xi_3 a_2) & (x_3\alpha_3)(\xi_3 a_3) \end{vmatrix} = 0,$$

or better, dividing rows by $(x_i \xi_i)$ and columns by $(\alpha_i a_i)$,

(15) $$|(x_i a_k \,|\, \alpha_k \xi_i)| = 0 \qquad (i, k = 1, 2, 3).$$

The mutuality is now evident; and the extension to n dimensions is also evident, we have merely $i, k = 1, 2, \cdots, n+1$. The determinant contains an extraneous factor which vanishes if the $n+1$ points x_i or a_i lie in an S_{n-1} or the $n+1$ S_{n-1}'s lie in a point. In fact the sum of the elements of the determinants in any row or column is 1.

Thus, the determinant being written $|r_{ik}|$, the condition is, for $n = 1$,

$$\begin{vmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{vmatrix} = 0, \quad \text{or} \quad \begin{vmatrix} r_{11} & 1 \\ 1 & 2 \end{vmatrix} = 0;$$

and, for $n = 2$,

$$\begin{vmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{vmatrix} = 0, \quad \text{or} \quad \begin{vmatrix} r_{11} & r_{12} & 1 \\ r_{21} & r_{22} & 1 \\ 1 & 1 & 3 \end{vmatrix} = 0.$$

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