

THEORY ON PLANE CURVES IN NON-METRICAL ANALYSIS SITUS*

BY

OSWALD VEBLEN

§ 1. *Introduction.*

JORDAN'S † explicit formulation of the fundamental theorem that a simple closed curve lying wholly in a plane decomposes the plane into an inside and an outside region is justly regarded as a most important step in the direction of a perfectly rigorous mathematics. This may be confidently asserted whether we believe that perfect rigor is attainable or not. His proof, however, is unsatisfactory to many mathematicians. It assumes the theorem without proof in the important special case of a simple polygon ‡ and of the argument from that point on, one must admit at least that all details are not given.

The work of SCHOENFLIES, § especially in formulating a converse theorem has thrown much light on its relation to the theory of point sets and Analysis Situs in general, and elegant proofs under restrictive hypotheses have been given by AMES || and BLISS. ** All these discussions make more or less use of the ideas of analysis, thus implying either an axiom to the effect that a plane is a doubly extended number-manifold or a set of congruence axioms. Either of these hypotheses imposes a restriction upon the formal generality of Analysis Situs as a science independent of the magnitude of the figures treated.

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† C. JORDAN, *Cours d'Analyse*, Paris, 1893, 2d ed., p. 92.

‡ This case was under discussion at the University of Chicago in 1901-02 in connection with Professor MOORE's seminar on Foundations of Geometry. Mr. N. J. LENNES gave a proof in his master's thesis (1903), *Theorems on the simple polygon and polyhedron*. Another proof appears as theorem 28 in the writer's dissertation (for reference, see footnote below). The present paper owes much to the discussions of the subject that have taken place under the leadership of Professor MOORE.

§ A. SCHOENFLIES, *Ueber einen grundlegenden Satz der Analysis Situs*, *Nachrichten der Göttinger Gesellschaft der Wissenschaften*, 1902, p. 185; *Beiträge zur Theorie der Punktmengen*, *Mathematische Annalen*, Vol. 58 (1903), p. 195.

|| L. D. AMES, *On the theorem of Analysis Situs relating to the division of the plane or of space by a closed curve or surface*, *Bulletin of the American Mathematical Society* (2), vol. 10 (1904), p. 301.

** G. A. BLISS, *The exterior and interior of a plane curve*, *Bulletin of the American Mathematical Society* (2), vol. 10 (1904), p. 398.

In the following pages an attempt is made to discuss the theorem of JORDAN and a number of related questions under considerably more general hypotheses than are employed in any of the works referred to above. This undertaking practically results in a statement in logical terms of a body of information that formerly was used without explicit formulation and recently has been cramped by unnecessary restrictions. In other words we are inquiring how wide may be the application of our intuitive notion of a plane curve.*

The arguments and definitions are based upon axioms I–VIII, XI of the system adopted in the writer's dissertation † to which refer all the citations not otherwise indicated. These axioms are sufficient to determine the intersectional properties of straight lines, the ordinal relations ‡ of points on a straight line, and continuity. We accordingly assume nothing about analytic geometry, the parallel axiom, congruence relations, nor the existence of points outside a plane. For example, the theory is as valid in the non-desarguesian geometries of HILBERT and MOULTON § as in the geometries of EUCLID and LOBATCHEWSKY, and is of course as applicable in pure analysis as in geometry. ||

The reader who prefers the JORDAN definition of a simple curve (which according to § 4 is equivalent to ours for purposes of analysis) and does not care about the question of non-metrical hypotheses, may conveniently begin with § 5. A relatively simple proof of the theorem of JORDAN about the decomposition of the plane which applies to any simple closed curve having a straight line inter-

*The general problem of the "mathematics of precision" may be stated in similar terms.

† O. VEBLEN, *A System of Axioms for Geometry*, Transactions of the American Mathematical Society, vol. 5 (1904), pp. 343–384.

‡ The line is open, i. e., between every two points there is a third, and the order ABC excludes BAC and ACB . Single elliptic geometry and projective geometry are therefore excluded unless a properly chosen cut is introduced.

§ F. R. MOULTON, *A Simple non-desarguesian Geometry*, these Transactions, vol. 3 (1902), p. 192.

|| As to the applicability of our results in analysis, it seems desirable to add a remark which, though obvious from the point of view of "foundations of mathematics," may be of service to some readers who are not directly interested in this point of view. Numerical analysis is ordinarily thought of as founded on the concept of the positive integers. In terms of these a proof of existence can be given of a set of elements, or quantities, satisfying the postulates of the system of rational numbers, positive and negative. In terms of the rational numbers, in turn, can be given a proof of the existence of elements satisfying the postulates of the continuous real number system. Finally, the processes of analysis have to do with pairs of real numbers (x, y) . The set of all such number-pairs is a set of objects about which (with proper definition of the term "order") our axioms I–VIII, XI are true theorems. From the axioms of analysis, the line of deduction of our theorems is therefore clear and simple. Not only that, but we may add that any theorem or any definition rigorously based on the assumptions of geometry is *ipso facto* a theorem or definition of analysis. Such considerations as these justify the assertion that while much may be lost in elegance and simplicity, nothing is gained in rigor by the banishment of geometrical language and geometrical styles of exact reasoning from pure analysis. (Of course, under sufficiently strong geometrical axioms, these remarks may be reversed and applied to the rôle of analysis in geometry.)

val is to be obtained by considering theorem 8 applied to a triangular region, theorem 9 applied to a simple polygon, corollary 2 of theorem 9, lemma *B*, theorem 10 and lemma *C*. This "reduced proof" could in turn be slightly modified so as to apply to any curve having at least one non-cuspidal tangent.

§ 2. *Non-metrical definition of limit point.*

For the definition of the terms, triangle, polygon, broken line, triangular region, separate, decompose, the reader is referred to § 4, chapter II. Of the theorems there proved we assume for the present purpose only that a triangle decomposes a plane in which it lies into two regions, an interior and an exterior.

DEFINITION 1. A *triangular region* is the interior of a triangle. A *geometrical limit point* of a set of points, $[X]$,* in a plane is a point P such that every triangular region including P includes a point X , distinct from P . A *triangular region* including a point is called a *neighborhood* of the point.

The continuity axiom was assumed for only one segment of a straight line and proved by projection for all lines. In like manner by projection it can be proved that for every point, P , of any line there exists a numerably infinite set of segments $[\sigma_\nu]$ ($\nu = 1, 2, \dots$) such that σ_ν contains $\sigma_{\nu+1}$ and such that P is the only point that lies on every σ_ν . It is an easy consequence of this that for every point in a plane there exists a set of triangular regions $[t_\nu]$ with a similar property. We also prove without difficulty the theorem that a limit point of a set of limit points of a set of points, $[X]$, is itself a limit point of $[X]$.

DEFINITION 2. A *region* is a set of points, any two of which are points of at least one broken line composed entirely of points of the set. An *interior point* of a region, R , is one that can be surrounded by a triangle containing only points of R . Consequently, an interior point of R is a geometrical limit point of no set of points that does not contain points of R . A *frontier point* of a region R is a point or geometrical limit point of R not an interior point, i. e., it is a limit point both of R points and of not R points. An *exterior point* of R or a point exterior to R is any point neither an interior nor a frontier point of R . The *frontier* or *boundary* of a region is a set of all frontier points. An *open region* contains no frontier points. A *closed region* contains all its frontier points.

One of the most familiar examples of an open region is obtained by letting $[C]$ stand for a closed set of points and $[P]$ for the set of all points that can be joined with a point P_0 not of $[C]$ by broken lines not meeting $[C]$; $[P]$ is an open region.—It is to be noted that the points exterior to a region $[P]$, if such exist, need not constitute only a single region.

*The notation $[X]$ denotes a set of elements any one of which is denoted by X alone or with suffixes. If we wish to indicate that the set is *ordered* we use $\{X\}$ instead of $[X]$.

§ 3. Definition of simple curve.

Simple curves, closed and unclosed, are composed of sets of points subject to certain conditions which we arrange in the following groups:

A. LINEAR ORDER. Among the points of a set of points $\{P\}$ there exists a relation, \odot , which we may read *precedes*, such that:

1. $\{P\}$ contains at least two points.
2. If P_1 and P_2 are any two distinct points of $\{P\}$, then either $P_1 \odot P_2$ or $P_2 \odot P_1$.

3.* If $P_1 \odot P_2$, then not $P_2 \odot P_1$.

4. If $P_1 \odot P_2$ and $P_2 \odot P_3$, then $P_1 \odot P_3$.

B. ORDINAL CONTINUITY.

1. If P_1 and P_3 are any two points of $\{P\}$, such that $P_1 \odot P_3$, then there is a point P_2 of $\{P\}$ such that $P_1 \odot P_2$ and $P_2 \odot P_3$.

2. If every point of $\{P\}$ belongs to $[P_1]$ or $[P_2]$, two infinite subsets of $\{P\}$ such that for every P_1 and P_2 , $P_1 \odot P_2$, then there is a point P' such that for every P_1 and P_2 distinct from P' , $P_1 \odot P'$ and $P' \odot P_2$.

C. GEOMETRICAL CONTINUITY.

1. Let P_0 be any point of $\{P\}$ for which there is an infinity of points P' such that $P' \odot P_0$. Denote the set of all such points by $[P']$; then for every triangular region, t , including P_0 , there is a point of $[P']$, P'_t such that t includes all points of $[P']$ for which $P'_t \odot P'$.

2. Let P_0 be any point of $\{P\}$ for which there is an infinity of points P'' such that $P_0 \odot P''$. Denote the set of all such points by $[P'']$; then for every triangular region, t , including P_0 there is a point of $[P'']$, P''_t such that t includes all points of $[P'']$ for which $P'' \odot P''_t$.

DEFINITION 3. By the term *arc* or *arc of curve* is meant a set of points $\{P\}$ satisfying conditions A, B, C and including two points P_1, P_2 such that every point P , distinct from P_1 and P_2 , satisfies the further conditions that $P_1 \odot P$ and $P \odot P_2$. The arc is said to *join* P_1 and P_2 which are called its *end-points*.

DEFINITION 4. A *simple closed curve*, j , is a set of points, $\{J\}$, consisting of two arcs joining two points J_1 and J_2 but having in common no points other than J_1, J_2 .

THEOREM 1. Any two points of j may be taken as the points, J_1, J_2 in the above definition.

The proof of this theorem is here omitted as it involves no difficulty. The existence of sets of points satisfying the conditions of our definition is proved by the examples of an interval of a straight line, which is an arc, and the boundary of a simple polygon, which is a simple closed curve. We shall use the letter j ,

* From this it follows that if $P_1 \odot P_2$, then $P_1 \neq P_2$.

to denote a simple closed curve, in honor of CAMILLE JORDAN. The term *arc* of course does not cover the most general case of an unclosed curve. On the other hand the conditions A , B , C are too general to define an unclosed curve since they are satisfied, for example, by the boundary of a triangle exclusive of one vertex. We therefore set down the following condition which is evidently satisfied by an arc.

DEFINITION 5. A *simple unclosed curve* is a set of points $\{C\} = c$ that satisfies conditions A , B , C and also the following:

D . If C is any point of the curve, no point except C is a limit point (in the geometrical sense of definition 1) both of the set of all points C' such that $C' \odot C$ and of the set of all points C'' such that $C \odot C''$.

Any simple closed or simple unclosed curve is called a *simple curve*. For a set of points satisfying conditions A , B and C , it is evident that there hold all the propositions usually proved in the theory of linear point-sets with the exception of those that involve the length of intervals. We may mention particularly the propositions of section 5, chapter II, including the HEINE-BOREL theorem and the definition of ordinal limit point, the properties of point-free intervals in connection with closed sets, and the proposition that a line cannot be separated into two subsets each of which includes all its limit points. We do not stop here to prove these propositions though we make use of the last one in the following theorem.

DEFINITION 6. A relation satisfying conditions A , B , C is called a *sense*. A sense in which $P_1 \odot P_2$ is said to be *from* P_1 *to* P_2 .

THEOREM 2. *From one point to another upon a simple unclosed curve there is one and but one sense, while upon a simple closed curve there are two and but two senses.*

Proof. We have to show that if ρ is any relation satisfying the conditions A , B , C and D imposed on \odot , then if $P_1 \rho P_2$ implies for one pair $P_1 P_2$ of $\{P\}$ that $P_1 \odot P_2$, $P_1 \rho P_2$ implies that $P_1 \odot P_2$ for every pair $P_1 \rho P_2$ of $\{P\}$.

If P_1 is any point of $\{P\}$, let $[P']$ be the set of all points such that simultaneously $P_1 \rho P'$ and $P_1 \odot P'$. Every limit point \bar{P} ($\bar{P} \neq P_1$) of $[P']$ with respect to the sense ρ must by conditions A , B and C be such that $P_1 \rho \bar{P}$. Moreover \bar{P} , by condition C , is a geometrical limit point of $[P']$. But in view of condition D , \bar{P} being a point of $\{P\}$ and a geometrical limit point of points P' such that $P_1 \odot P'$ must be such that $P_1 \odot \bar{P}$; otherwise \bar{P} ($\bar{P} \neq P_1$) would be a geometrical limit point both of points \bar{P} such that $\bar{P} \odot P_1$ and of points P' such that $P_1 \odot P'$.

Therefore the set $[P']$, if existent, contains all its limit points with respect to the sense ρ , except the point P_1 . Similarly the set of all points P'' , such that simultaneously $P_1 \rho P''$ and $P'' \odot P_1$ must, if existent, contain all its limit points with respect to the sense ρ . Therefore, since the set of points $\{P_2\}$

such that $P_1 < P_2$ cannot consist of two subsets, each closed with the exception of P_1 , every point P_2 must either be such that $P_1 \odot P_2$ or every point P_2 must be such that $P_2 \odot P_1$. From this result the conclusion of our theorem follows at once.

DEFINITION 7. If with respect to any sense on a curve, $P_1 \odot P_2$ and $P_2 \odot P_3$, P_2 is *between* P_1 and P_3 in that sense. The set of all points between P_1 and P_3 in the given sense is called a *segment* $P_1 P_2 P_3$ whose *end-points* are P_1, P_3 . The segment and its end-points together constitute an *arc* or *interval* of the curve. On a simple unclosed curve, if $P_1 \odot P_2 \odot P_3$, P_2 is said to *separate* P_1 and P_3 . On any simple curve if $P_1 \odot P_2 \odot P_3 \odot P_4$, P_1 and P_3 are said to separate and be separated by P_2 and P_4 . If a set $[P_\nu]$ ($\nu = 1, 2, 3, \dots$) is such that $P_\nu \odot P_{\nu+1}$, the points P_ν are said to be in *the order along the curve*, $P_1 P_2 P_3 \dots P_\nu P_{\nu+1} \dots$. A point P_0 is the *first* of a set $[P]$ if $P_0 \odot P$ for every $P \neq P_0$; P_1 is the *last* of the set $[P]$ if $P \odot P_1$ for every $P \neq P_1$.

Either of the relations of "betweenness" or "separation" which are here defined in terms of "precedence" could have been used as fundamental* and a definition of a simple curve equivalent to the above would have resulted. The deduction of the properties of these relations will be omitted.

§ 4. *Remarks on the definition of a simple closed curve.*

While the definitions of the preceding section are stated so as to apply only to plane curves, it is obvious that if one replaces triangles by tetrahedrons or the corresponding figures in space of more dimensions, the conditions A, B, C , etc., give a definition of a simple curve in space of any number of dimensions.

It may be of interest to note that when one passes from the realm of plane geometry, the distinction between metric and non-metric theory loses much of its importance. For if we add to our assumptions (axioms I–VIII, XI) the assumption (axiom IX) that there exists a point outside a plane, then it is possible to define the ideal elements of projective geometry (cf. chapter III) and by choosing among these ideal elements an "absolute" plane† and polar system to establish a projective theory of congruence. We are thus enabled to operate in the most general case by ordinary analytic geometry as if dealing with the whole or a limited region of euclidean space.

As to the relation of the above definition to the current definition‡ in terms of a numerical parameter, it has not yet been determined whether, in the presence of axioms I–VIII, XI alone, the two definitions are or are not equivalent.

* Cf. B. RUSSELL, *The Principles of Mathematics*, Cambridge, 1903, chapters 24, 25. On the definition by postulate of "separation," see G. VAILATI, *Sulle relazioni di posizione tra punti d'una linea chiusa*, Rivista di Matematica, vol. 5 (1895), p. 75; and also *ibid.*, p. 183.

† That there always is an ideal plane depends in particular on axioms III and XI which determine that straight lines shall be open.

‡ See JORDAN, *loc. cit.* p. 90.

If however we introduce axiom IX as indicated above or bring in a set of congruence axioms like HILBERT's group IV, then the two definitions can be shown to be equivalent by reference to a theorem of CANTOR.*

Let a be any arc of a simple closed curve j . The points of a , excluding the end points, evidently constitute what CANTOR calls a *perfect* set, i. e., with respect to the sense \odot . Let $\{t_n\}$ denote a set of equilateral triangles concentric and similarly placed such that the lengths of the side of t_n is $1/n$. For every point J of a there is such a set of triangles $\{t_n\}_J$, having J as a common center. Each triangle t_n determines an arc i_n of j which lies wholly within t_n (cf. condition C) and includes the central point J of t_n . Among the arcs i_n , there is by the HEINE-BOREL theorem applied to a , a finite subset such that every point of a is interior to one of the arcs i_n . The end points of these arcs that lie on a , excluding the end points of a , we denote by

$$A'_n, A_n^2, \dots, A_n^{m_n}.$$

The set of points $\{A_n^k\}$ is evidently numerable, is ordered according to one of the senses of a , and moreover is *everywhere dense on a*. For if it were not everywhere dense on a there would be some interval i of a which for *every* n lies wholly within *some* i_n and therefore within some t_n ; whereas two of its points are a certain distance apart greater than $1/n$ for n sufficiently great.

Now by the theorem of CANTOR cited above, any perfect set which possesses a numerable subset everywhere dense can be set in one-to-one reciprocal continuous correspondence, with the real numbers between 0 and 1. Thus we have a continuous one-to-one correspondence of the points of any arc, and hence of any simple closed curve, with a numerical parameter, t . If a system of coördinates (x, y) has been introduced, the simple closed curve may be expressed in parameter form by defining $x(t)$ as the abscissa of the point of j that corresponds to t and $y(t)$ as the ordinate of the same point. The continuity of $x(t)$ and $y(t)$ is evident.

Regarding the conditions of definition 4 as a set of postulates for the determination of the notion, simple curve, the proposition just proved is in effect that in the presence of axioms I-VIII, XI, together with IX or a set of congruence axioms, the system of postulates is "categorical."† The conditions are also independent; i. e., each item of the definition is indispensable to the full definition. To prove this we give a list of point-sets each of which satisfies all the conditions except one. Our independence proofs apply to conditions A , B , C , D since the closed simple curve is defined in terms of the unclosed arc.

A_1 . $\{P\}$ consists of one point.

*G. CANTOR *Zur Begründung der transfiniten Mengenlehre* I, *Mathematische Annalen*, vol. 46 (1895), p. 510.

† See vol. 5, p. 346, of these Transactions.

A_2 . $\{P\}$ consists of a straight segment P_1P_2 and a point P_3 not on the straight line P_1P_2 , the relation \odot referring to a fixed sense on the straight line P_1P_2 .

A_3 . $\{P\}$ consists of two points P_1, P_2 with the conventions $P_1 \odot P_2, P_1 \odot P_1, P_2 \odot P_2$.

A_4 . $\{P\}$ consists of seven points $P_1 \dots P_7$, the relation \odot being defined by the following table

\odot	1	2	3	4	5	6	7
1		•		•			•
2			•		•		•
3	•					•	•
4		•	•			•	
5	•		•	•			
6	•	•			•		
7				•	•	•	

B_1 . $\{P\}$ consists of two points P_1 and P_2 with the convention $P_1 \odot P_2$.

B_2 . $\{P\}$ consists of all the points of a straight line with one exception, \odot being one of the two senses along the line.

C_1 . $\{P\}$ consists of all the points of a straight line P_1P_2 with the exception of the point P_1 and the segment P_1P_2 , \odot being the sense from P_1 to P_2 .

C_2 . $\{P\}$ consists of all the points of a straight line P_1P_2 with the exception of the point P_2 and the segment P_1P_2 , \odot being the sense from P_1 to P_2 .

D . P consists of the points of a broken line $P_1P_2P_3P_4$, where P_4 is a point of the segment P_1P_2 , \odot being the sense P_1P_2 , and P_4 being counted as a point of P_1P_2 . This case shows the necessity of condition D in theorem 2 since \odot may also be the sense along the broken line $P_1P_4P_3P_2$.

§ 5. A simple curve as a planar point set.

DEFINITION 8. A geometrically closed set of points is a set that includes all its geometrical limit points.

THEOREM 3. If $[P]$ is any geometrically closed set of points and a any arc that does not have any point in common with $[P]$, then (1) there exists a finite set of triangles $\{t_n\}$ such that every point of a is interior to at least one t_n and every point of $[P]$ is exterior to every t_n , and (2) the two end points A_1A_2 of a can be joined by a broken line not meeting $[P]$.

Proof. (1) If A is any point of a there must be a triangle, t , including A and not including any point of $[P]$; otherwise A would be a limit point of $[P]$. By condition c , each of these triangles, t , determines an arc, i , of a which lies entirely within t and includes the point A to which t belongs. By

the HEINE-BOREL theorem applied to the arc a , there is a finite subset $[i_n]$ of the arcs i such that every point of a belongs to one arc i_n . The finite set of triangles, t_n , that determined these arcs i_n is the set required by conclusion (1) of the theorem.

(2) The end points of the arcs i_n constitute a finite set of points which we take as ordered by the sense of a from A_1 to A_2 . The broken line * joining these points taken in order is such that each side lies within a triangle t and therefore cannot meet $[P]$.

COROLLARY. If $[P]$ is any geometrically closed set of points and Q_0 a point not of $[P]$, then Q_0 and the set of points, Q , that can be joined to Q_0 by arcs not meeting $[P]$ constitute an open region.

The following theorem is a direct consequence of definitions 4 and 5 and its proof as well as that of theorem 5 is omitted.

THEOREM 4. *About any point of a segment of a simple curve there is a triangle which includes no points of the curve not on the segment.*

In the sense of definition 8, a straight line is a geometrically closed set. A straight line, however, lacks a property possessed by any one of its intervals, namely that every infinite subset has a limit point. For this kind of set we introduce the phrase "finitely closed" because any such set can be enclosed by a finite set of triangles. This property, however, is not used and not proved in the present paper.

DEFINITION 9. A *finitely closed* set of points is a geometrically closed set of which every infinite subset possesses a geometrical limit point. A finitely closed set, every point of which is a geometrical limit point, is a *finitely perfect* set. A finitely perfect set of points which cannot consist entirely of two closed subsets is called a *coherent* set of points.†

THEOREM 5. *A closed curve or an arc of curve is a finitely perfect set of points which cannot consist entirely of two subsets, each of which includes all its limit points. In other words a closed curve or an arc of curve is a coherent set of points.*

THEOREM 6. *If every point of a coherent set of points $[A]$ is on a simple curve c , closed or unclosed, then $[A]$ is an interval of c .*

Proof. If $[A]$ were not an arc of c there must in case c is unclosed be one, and in case c is closed, two points, C_1, C_2 , of C not on a which separate the points of c into two sets, c', c'' each containing points of $[A]$. Let $[A']$ denote the points common to $[A]$ and c' and $[A'']$ denote the points common to $[A]$ and c'' . Every geometrical limit point of $[A']$ would be a

* This broken line of course need not be simple. A broken line with multiple points has a sense independent of the definition of sense on a simple curve. See chapter II, § 4.

† This is the "Begriff des Zusammenhangs" of JORDAN and SCHOENFLIES. Cf. SCHOENFLIES, *Mathematische Annalen*, vol. 58 (1903), p. 208.

geometrical limit point of $[A]$, therefore a point of $[A]$, and hence a point of c . Being a point of c and a geometrical limit point of c' , by theorem 4, it would be a point of c' and hence a point of $[A']$. $[A']$ would therefore be a closed set and by parity of reasoning $[A'']$ also would be closed and thus the definition of $[A]$ would be contradicted.

Corollary. If every point of an arc, a , is on a simple curve, c , then a is an interval of c .

THEOREM 7. *If c is any simple curve, any triangle, t , of the plane includes points not on c .*

Proof. Let a be a straight line interval lying wholly within t . By theorem 6, a either contains points not on c , in which case our conclusion holds, or a is an arc of c . In the latter case, by theorem 4, a triangle, t' , exists about any interior point of a including no points of c not on a . Points of the boundary of this triangle within t and not on a are not on c .

§ 6. *The approach to and crossing of a boundary.*

DEFINITION 10. Let P be an interior point of a region, R , and B a point of the boundary b of R . An arc of a curve, a , whose end points are P and B *approaches B from P through R* if every interval of a , one of whose end points is B , contains interior points of R . The approach is *one-sided* if, besides the above condition, the arc, a , contains no points exterior to R . The approach is *simple* if all the points of a , except B , are interior points of R .

An arc a' *departs from a point B' of b to a point Q exterior to R* if every interval of a' with B' as an end point contains points exterior to R . The departure is *one-sided* if, besides the above condition, the arc a' contains no points interior to R . The departure is *simple* if all the points of a' except B' are exterior to R .

A curve c *crosses the boundary in a point B* if, with respect to a fixed sense, B is between two points C_1, C_2 of c , C_1 interior and C_2 exterior to R , in such a way that the arc C_1B approaches B through R and BC_2 departs from B to C_2 .

A curve c *crosses the boundary b in a pair of points BB'* if, with respect to a certain sense, one arc BB' of c is composed entirely of boundary points and if there are two points C_1C_2 of c such that C_1 is interior to R and an arc C_1B of c approaches B from C_1 while C_2 is exterior to R and an arc $B'C_2$ departs from B' to C_2 .

The crossing of a boundary is *simple* if both the approach and departure at the point B or point pair BB' are simple.

The crossing of a straight line by a curve is a special case of the definition just given. A curve is said to cross a segment AB if the curve crosses the line AB in a point or a pair of points.

THEOREM 8. *Any simple curve joining an interior point of a region to an exterior point crosses the boundary in a point or a pair of points.*

Proof. Let I be the interior point, O the exterior point, and a any arc of the curve from I to O . Let $\{A\}$ be the set of all points, A , of the arc a such that every point following I and preceding A is an interior or boundary point of the region. There are such points because of condition C of the definition in § 3. By the ordinal continuity of a , the set $\{A\}$ has a first forward bound B , i. e., a first point in the sense from I to O that follows every point of $\{A\}$ except possibly B itself.

The arc BO of a departs from B to O as otherwise every arc BB'' of BO would contain only interior or boundary points of the region and thus B would not be a bound of $\{A\}$. Two cases can now occur. Either B is approached from I by the arc IB of a in which case our conclusion follows, or there are points A' of $\{A\}$ such that the arcs $A'B$ include only boundary points. In the last case the set of all points, A' , must have a first forward bound B' in the sense from B to I . The point B' is evidently a boundary point and is approached from I by the arc of a , IB' . Thus in the second case, the boundary is crossed in the pair of points $B'B$.

THEOREM 9. *If a simple closed curve crosses a side of a polygon (simple or not) in one point or point-pair, it must pass through a vertex or cross the same or another side in another point or point-pair.*

Proof. Let the polygon be $P_1P_2 \dots P_n$ and let the curve, j , cross it in a point of P_1P_2 . If there is another crossing on the segment P_1P_2 or if j passes through a vertex, $P_1 \dots P_n$, the theorem is verified. These cases disposed of, $P_1P_2P_3$ may either be collinear or non-collinear. In the first case the original crossing may have been on P_2P_3 in which case the theorem is verified or it may have been on P_1P_3 in which case we pass to the paragraph below. In case $P_1P_2P_3$ are non-collinear there must be a point J_1 of j and a point O_1 common to j and P_1P_2 such that in a certain sense on j the arc J_1O_1 of j approaches O_1 through the region on one side of P_1P_2 ; likewise there must be a point J_2 of j on the opposite side of P_1P_2 from J_1 and a point O_2 common to P_1J_1 and j such that in the same sense the arc O_2J_2 departs from O_2 to J_2 . Moreover the points J_1 and J_2 may be so chosen that one and only one of them lies within the triangle $P_1P_2P_3$. Since j crosses P_1P_2 only once, O_1 and O_2 are on the same arc of j with end points J_1J_2 . The other arc, α , of j with end points J_1J_2 must, by theorem 8, cross the boundary of the triangle $P_1P_2P_3$ and since it does not pass through a vertex, must either cross P_2P_3 verifying the theorem or cross P_1P_3 . In the latter case, let O'_1 be the first point in the sense from J_1 to J_2 in which α meets P_1P_3 and O'_2 the last such point. Upon the arcs $J_1O'_1$ and O'_2J_2 there must be two points of α , J'_1 and J'_2 on opposite sides of P_1P_3 such that in opposite senses along j the arcs $J'_1O'_1$ and $J'_2O'_2$ approach O'_1 and O'_2

from opposite sides of P_1P_3 . In case $P_1P_3P_4$ are non-collinear J'_1 and J'_2 may be so chosen that one is interior and the other exterior to the triangle $P_1P_3P_4$.

Thus whether $P_1P_3P_4$ are or are not collinear we proceed as with $P_1P_2P_3$, either verifying the theorem or arriving at the case $P_1P_4P_5$. Continuing this process, by a finite number of steps we come to $P_1P_{n-1}P_n$ and verify the theorem if it is not fulfilled at one of the intermediate steps.

COROLLARY 1. If j_1 is a simple closed curve having an arc which is a linear interval J_1J_2 , and if the segment J_1J_2 is crossed by a simple closed curve j_2 in one point or point pair, then either J_1J_2 is crossed in another point or point pair or the non-linear arc J_1J_2 of j_1 has a point in common with j_2 .

Proof. In case J_1J_2 were not crossed more than once and the other arc J_1J_2 of j_1 did not meet j_2 , by theorem 3 J_1 and J_2 could be joined by a broken line not meeting j_2 and we should thus have a contradiction with theorem 9.

COROLLARY 2. Any simple closed curve j_1 having a linear arc J_1J_2 decomposes its plane into at least two regions.

Proof. Let PQ be a linear segment crossing J_1J_2 in a point O . The region composed of all points that can be joined to P by broken lines not meeting j_1 is by theorem 9 separated from the region similarly connected with Q .

LEMMA A. Any simple closed curve j decomposes the plane in which it lies into at least two regions.

Proof. Let J_1 and J_2 be two points of j such that the linear segment J_1J_2 has no point in common with j . Such points J_1J_2 exist, for if a is any line joining two points of j , it either has an interval free of j points and whose end-points are the required points J_1J_2 or its points in common with j constitute a single arc of j (theorem 6, corollary). In the latter case any line a' joining a point of j on a to a point of j not on a evidently has the required points J_1J_2 .

Let t be a triangle about J_1 such that one of its sides meets the linear segment J_1J_2 in a point O . Let Q' and Q'' be two points of this side separated by O and such that the linear interval $Q'Q''$ contains no point of j . The existence of these points depends on the theorem that j is a geometrically perfect set.

J_1 and J_2 decompose j into two segments which with the linear interval J_1J_2 constitute two closed curves j' and j'' . Assign the notation so that the first point, J'_1 , after Q'' in the sense $Q'OQ''$ in which the boundary of t meets j shall be a point of j' . It follows that the first point J''_1 after Q' in the sense $Q'OQ'$ in which the boundary of t meets j is a point of j'' . For if it were a point of j' , the closed curve composed of the boundary of t from J'_1 to J'_1 in the sense $Q'OQ''$ and the arc common to j and j' between J'_1 and J''_1 would cross the linear segment J_1J_2 of j'' simply in O and would meet j'' in no other point. This would contradict corollary 1, theorem 9.

Thus J''_1 is a point of j'' . Let J''_0 be the first point after J'_1 in the sense $Q'OQ''$ in which the boundary of t meets j'' . By the continuity of j , there

exists a segment of the boundary of t just preceding J''_0 in the sense $Q'OQ''$ and containing no point of j' or j'' . Let X be any point of this segment. The broken line b'' composed of the boundary of t in the sense $Q'OQ''$ from Q'' to X does not meet j'' . Likewise X is joined to Q' by a simple curve c' composed of the linear segment XJ''_0 , the common part of j'' and j from J''_0 to J''_1 * and the part of the boundary of t from J''_1 to Q' in the sense $Q'OQ''$. Thus c' cannot meet j' and, applying theorem 3, c' can be replaced by a broken line b' joining X to Q' without meeting j' . We are now ready to complete the proof of our lemma by showing that X cannot be joined to O by a broken line not meeting j .

In the sense from X to O any such broken line would meet the linear segments J_1J_2 and $Q'Q''$ in some first point O_1 . If O_1 were on J_1J_2 , some point S preceding O_1 in the sense from X to O could be joined to a point B of $Q'Q''$ by a segment not meeting j' or j'' . Call b the resulting broken line from X to B . In case O_1 were not on J_1J_2 it would be on $Q'Q''$ and different from O , and b would be the broken line from X to $O_1 = B$.

If B were on the same side of the line J_1J_2 with Q'' then the polygon composed of b and b' and BQ' would be crossed by j' in O and would meet j' in no other point, contradicting theorem 9. If B were on the opposite side of the line J_1J_2 from Q'' the polygon composed of b and b'' and BQ'' would be crossed by j'' in O and would meet j'' in no other point. X and O are therefore two points that cannot be joined by a broken line not meeting j .

§ 7. *Finite accessibility.*

DEFINITION 11. A point C of a curve c is *finitely accessible* from a point P not on c if there is a broken line from C to P not meeting c except in C .

LEMMA B. If P is a point not on a simple closed curve j , and J_1 and J_3 are any two points of j finitely accessible from P or limit points† of the points finitely accessible from P , then there exists a pair of points J_2 and J_4 finitely accessible from P that separate J_1 and J_3 .

Proof. Let t_1 be a triangle about J_1 not including J_3 , and t_3 a triangle about J_3 not including any point of t_1 . By condition C of the definition of j , there is a segment of j including J_1 and lying wholly within t_1 ; by theorem 4 there is a triangle t'_1 about J_1 within t_1 and including no point of j not on this segment. Thus every segment of j with end points on t_1 which meet t'_1 must include J_1 . Similarly there is within t_3 a triangle t'_3 such that every segment of j with end points on t_3 which meets t'_3 must include J_3 .

* Of course it may happen that $J''_0 = J''_1$. In this case c' is a broken line.

† On a simple closed curve the notions of ordinal and geometrical limit points are interchangeable: therefore we drop the distinction.

Let J'_1 be a point finitely accessible from P within t'_1 and J'_3 a point finitely accessible from P within t'_3 . The points J'_1 and J'_3 are thus joined by a broken line b , meeting j only in J'_1 and J'_3 , which without loss of generality may be supposed simple. On this broken line let P_1 be the first point in the sense from J'_3 to J'_1 in which it meets the boundary of t_1 . P_1 lies on an interval i_1 of the boundary of t_1 containing no points of j but such that its end points are points of j . Let P_2 be the last point in the sense from J'_3 to J'_1 in which b meets the interval i_1 . In case P_2 is distinct from P_1 replace the portion of the broken line from P_1 to P_2 by the portion of i_1 from P_1 to P_2 , calling the new broken line b_1 . If b_1 crosses the interval i_1 , J_2 and J_4 are the end points of the interval. If b_1 does not cross i_1 there must be some point P_3 beyond P_2 in the sense from J'_3 to J'_1 in which b_1 meets the boundary of t . The point P_3 must lie on an interval i_3 of the boundary of t_1 analogous to i_1 . Proceed with i_3 as with i_1 . Since J'_1 is inside t_1 and J'_3 outside t_1 and since b has but a finite number of sides, we must by repeating the process above come to a *first* interval i_k , in which the boundary of t_1 is crossed by a reduced broken line b_k from J'_3 to J'_1 in a point P_k or a point pair $P_k P_{k+1}$. The end points of the j -point free interval i_k of the boundary of t_1 are now to be shown to be the required J_2 and J_4 .

We prove first that J_2 and J_4 separate J'_3 and J'_1 . If this were not so, let the simple closed curve formed by b_k and the arc $J'_1 J'_3$ of j not including J_2 and J_4 be denoted by j_b . Also let j_i denote the simple closed curve formed by i_k and the arc $J_2 J_4$ of j not including J'_3 and J'_1 . The simple closed curve j_b would cross the arc i_k of j_i in the point P_k or point pair $P_k P_{k+1}$ and would meet j_i in no other point, contrary to corollary 1, theorem 9.

Hence J'_1 and J'_3 are on different arcs of j with end points J_2 and J_4 . But by the construction of the triangle t'_1 , J_1 must be on the same arc with J'_1 and by the construction of t'_3 , J_3 must be on the same arc with J'_3 . Hence $J_2 J_4$ separate $J_1 J_3$.

THEOREM 10. *The set of points of a simple curve j finitely accessible from a point P not on j is everywhere dense on j .*

Proof. Denote by $[J]$ the set of points of j which are either finitely accessible from P or are limit points of the set of finitely accessible points. The theorem amounts to showing that $[J']$ is identical with j . But if any point J_0 of j should not belong to $[J']$ it would lie on an arc of j free of points J' and having two points of $[J']$ as end points. This would contradict lemma *B*.

§ 8. *Decomposition of a plane by a simple closed curve.*

LEMMA C. Any simple closed curve of which one arc is a linear interval decomposes its plane into two open regions.

Proof. It has been shown (corollary 2, theorem 9) that j decomposes the plane into at least two regions. The regions are open because a supposed frontier point of the set of points $[P]$ that can be joined to a point P_0 not on j could, if not itself a point of j , be surrounded by a triangle not meeting j and containing points of $[P]$; it would therefore be an interior point of $[P]$, contrary to hypothesis.

By theorem 10 every point O of the straight arc of j is finitely accessible from any point of the plane. Thus if there were three distinct regions there would be three segments meeting in O and one lying in each of the three regions. But as two of these must lie on the same side of the straight segment of j they could be joined by a straight segment not meeting j , contrary to the hypothesis that the three regions are separated from one another by j . Hence j decomposes the plane into two and only two open regions.

THEOREM 11. *Every simple closed curve, j , decomposes its plane into two open regions.*

Proof. By lemma A the curve decomposes the plane into at least two regions which by the reasoning of the first paragraph of the proof of lemma C are open regions. Let P be any point not on j and let PJ_1 and PJ_2 be two linear intervals meeting j only in J_1 and J_2 . J_1 and J_2 exist because j is a perfect set of points. Let Q be any point not on j and not in the same region with P and let J_3 be a point on j such that the linear segment QJ_3 does not meet j and such that J_3 is distinct from J_1 and J_2 . Then QJ_3 does not meet PJ_1 or PJ_2 and Q can by theorem 10 be joined by a broken line not meeting PJ_1 , PJ_2 , QJ_3 , or j except in J_4 to a point J_4 of j in the order $J_1J_2J_3J_4$. The broken line J_1PJ_2 , the points between J_2 and J_3 in the sense $J_1J_2J_3$, the broken line J_3QJ_4 and the points between J_4 and J_1 in the sense $J_1J_2J_3$ constitute a simple closed curve j' of the type which we have proved to decompose the plane into two and only two regions. The points of the segments J_1J_2 and J_3J_4 in the sense $J_1J_2J_3$, are not points of j' and must lie both in the same region or in opposite regions with respect to j' . If they were in the same region a point in the region not containing the segments J_1J_2 and J_3J_4 could by theorem 10 be joined by broken lines not meeting j to P and Q , thus contradicting the hypothesis that P and Q are in different regions.

Having shown that the arcs J_1J_2 and J_3J_4 (in the fixed sense $J_1J_2J_3$) are in opposite regions with respect to j' we are ready to complete the proof that j does not decompose the plane into more than two regions. A point R in a sup-

posed third region could be joined because of theorem 10 by a broken line not meeting j except in its end-point to a point J_5 of J_1J_2 and by a similar broken line to a point J_6 of J_3J_4 . Since R would not be in the same region with P or Q these broken lines would not meet the broken line part of j' . Thus we should have two points J_5 and J_6 in opposite regions with respect to j' joined by a broken line not meeting j' contrary to lemma C' . Hence j decomposes the plane into not more than two, and therefore into exactly two, open regions.
