ON QUADRATIC, HERMITIAN AND BILINEAR FORMS*

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Introduction.

Part I is concerned with the reduction of quadratic forms in an arbitrary field to canonical types, a problem hitherto treated only for finite fields and for the field of all real or all complex numbers.

Part II treats of the reduction of hermitian forms in a field Q obtained by the adjunction to an arbitrary field F of a root of a quadratic equation belonging to and irreducible in F. The problem is completely solved when F is any finite field, the field of all real numbers, or the field of all rational numbers.

Part III deals with the bilinear forms in an arbitrary field F which are invariant under a given substitution S with coefficients in F. The necessary and sufficient conditions on S for the existence of such invariants are obtained, and the reduction of the invariants to a single normal form is effected by a transformation commutative with S. Here and in Part IV use is made of the writer's determination of the canonical form of a linear transformation in an arbitrary field.†

Part IV treats the analogous questions on quadratic forms and gives the generalization to an arbitrary field F of JORDAN's recent results for the case of a field of order p, a prime.‡ The explicit form of the general invariant is determined with the same ease, but the difficult problem of its reduction to canonical forms by substitutions commutative with S becomes much more troublesome for an arbitrary field than for a finite field. When F does not have modulus 2, the general character of the result may be indicated as follows. As semi-normal forms of the invariant we obtain

$$\sum B_i + \sum a_i H_i + \sum a_i' H_i' + \cdots + \sum A_i + \sum b_i Q_i + \sum b_i' Q_i' + \cdots,$$

where the invariants B_i, H_i, A_i and Q_i have respectively the character of bilinear, hermitian, alternate bilinear and quadratic forms, each with completely

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† American Journal of Mathematics, vol. 24 (1902), pp. 101–108.

‡ *Mémoire sur les formes quadratiques, suivant un module premier p, invariantes par une substitution linéaire donnée,* Journal de Mathématiques, ser. 6, vol. 1 (1905), pp. 217–284.

fixed coefficients, while b_i, b'_i, $\cdots$ are any non-vanishing elements of F and a_i, a'_i, $\cdots$, any non-vanishing elements of certain fields $F(\rho)$. The question of ultimate normal forms is the question of the extent to which these parameters can be specialized by the application of a substitution commutative with S. It is shown that normalization must take place in each sum separately, that the normalization of the quadratic function $\sum b_a Q_a$ of certain variables x_j^a must be made by a substitution T on the x_0^a, the same substitution on the x_1^a, the same on x_2^a, etc., with analogous remarks on the sums $\sum a_i H_i$, $\cdots$. But the effect on the b_i is the same as if we had applied to $\sum b_a (x_0^a)^2$ the substitution T on the variables x_0^a alone. Moreover, this partial substitution on the x_0^a may be chosen arbitrarily. Hence the problem of normalizing $\sum_{a=1}^{l} b_a Q_a$ by a substitution commutative with S and cogredient in the various sets of variables is replaced by the problem (treated in Part I) of normalizing an l-ary quadratic form in F by means of unrestricted l-ary substitutions in F. Similarly, the problem on $\sum a_i H_i$ reduces to that on hermitian forms (Part II).

I. *Reduction of quadratic forms* in a general field F.*

1. Within any field F, not having modulus 2, a quadratic form of non-vanishing determinant is linearly reducible to †

$$q \equiv \sum_{i=1}^{n} a_i x_i^2 \qquad \text{(each } a_i \text{ an element} \neq 0 \text{ of } F). \tag{1}$$

Hence for the field of real numbers the canonical types are

$$f_s \equiv \sum_{i=1}^{s} x_i^2 - \sum_{i=s+1}^{n} x_i^2 .$$

For $s \neq \sigma$, f_s cannot be transformed into f_σ by a real n-ary linear substitution; this invariance of s is the JACOBI-SYLVESTER law of inertia of real quadratic forms. ‡

2. Under the transformation

$$x_i = \sum_{j=1}^{n} b_{ij} y_j \qquad (i = 1, \cdots, n),$$

q becomes

$$\sum_{j=1}^{n} A_j y_j^2 + 2 \sum_{j,k,j<k}^{1,\ldots,n} B_{jk} y_j y_k, \qquad A_j \equiv \sum_{i=1}^{n} a_i b_{ij}^2, \qquad B_{jk} \equiv \sum_{i=1}^{n} a_i b_{ij} b_{ik}.$$

We discuss the question: Given $b_{11}, b_{21}, \cdots, b_{n1}$, in the general field F, such that $A_1 \neq 0$, can we determine elements $b_{ij} (j > 1)$ of F, such that every

* The same treatment applies to the reduction of symmetric bilinear forms by cogredient transformations of the two sets of variables.

† The usual proof for the field of all real numbers is valid for F. Or we may proceed as in § 6, identifying every element with its conjugate.

‡ References in BALTZER, *Determinanten*, 5th ed., p. 176.

$B_{jk}=0$ and $\Delta \equiv |b_{ij}| \neq 0$? Since the b_{i1} enter the question symmetrically, we may assume that $b_{11} \neq 0$. The conditions $B_{1k}=0$ are satisfied if we take

$$b_{1k} = -a_1^{-1} b_{11}^{-1} \sum_{i=2}^{n} a_i b_{i1} b_{ik} \qquad (k=2, \cdots, n).$$

If, with these values inserted in Δ, we remove the factor $a_1^{-1} b_{11}^{-1}$ from the first row of Δ, then multiply the ith row by $a_i b_{i1}$ and add the sum to the first row, for $i=2, \cdots, n$, we find that

$$\Delta = a_1^{-1} b_{11}^{-1} A_1 \Delta_{11}, \qquad \Delta_{11} \equiv |b_{is}| \qquad (i, s=2, \cdots, n).$$

If $n=2$, we take $b_{22} \neq 0$ and obtain an affirmative answer to our question. Let next $n>2$. In $B_{jk}=0\,(2 \leqq j<k)$, we replace b_{1j} and b_{1k} by their values and get

$$B'_{jk} \equiv \sum_{i=2}^{n} R_i b_{ij} b_{ik} + \sum_{i,t,\,i\neq t}^{2,\ldots,n} P_{it} b_{ij} b_{tk} = 0, \quad R_i \equiv a_i^2 b_{i1}^2 + a_1 a_i b_{11}^2, \quad P_{it} \equiv a_i a_t b_{i1} b_{t1}.$$

(i) Suppose first that every $R_i=0$ $(i=2, \cdots, n)$. Then every $b_{i1} \neq 0$, $P_{it} \neq 0$. With the exception of b_{32}, we take every $b_{is}=0$ $(i, s=2, \cdots, n;\ i>s)$. We assign any value not zero to b_{ii} $(i>1)$; and any values to b_{23}, b_{32} such that $b_{23} b_{32} = -b_{22} b_{33}$. Then $\Delta_{11} = 2 b_{22} b_{33} \cdots b_{nn} \neq 0$. Finally, for $l=4, \cdots, n$, we set

$$C_{il} \equiv \sum_{t \neq i}^{t=2,\ldots,l} P_{it} b_{tl} = 0 \qquad (i=2, \cdots, l-1).$$

Then conditions $B'_{jk} \doteq 0$ $(j, k=2, \cdots, n;\ j<k)$ are all satisfied, since the coefficient of b_{ij} equals C_{ik} if $k \geqq 4$, while for $k=3$, $j=2$, the condition reduces to $P_{23}(b_{22} b_{33} + b_{32} b_{23}) = 0$. But for a fixed l, $l>3$, equations $C_{il}=0$ serve to express the b_{tl} $(t=2, \cdots, l-1)$ in terms of b_{ll}, since the determinant of the coefficients of the former is*

$$\begin{vmatrix} 0 & P_{23} & P_{24} & \cdots & P_{2\,l-1} \\ P_{32} & 0 & P_{34} & \cdots & P_{3\,l-1} \\ P_{42} & P_{43} & 0 & \cdots & P_{4\,l-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ P_{l-1\,2} & P_{l-1\,3} & P_{l-1\,4} & \cdots & 0 \end{vmatrix} = r \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdots & 0 \end{vmatrix} = (-1)^{l-3}(l-3)r,$$

* To reach the second determinant, replace each P_{it} by $a_i a_t b_{i1} b_{t1}$, remove the factor $a_i b_{i1}$ from the $(i-1)$th row, and the factor $a_t b_{t1}$ from the $(t-1)$th column. To evaluate the second determinant, subtract the first row from each of the remaining rows, then add the 2^d, $\cdots$, $(l-2)$th columns to the first. The resulting determinant has the first row $l-3$, 1, 1, $\cdots$, 1, and zeros elsewhere outside the main diagonal.

where $r \equiv a_2^2 a_3^2 \cdots a_{l-1}^2 b_{21}^2 b_{31}^2 \cdots b_{l-11}^2 \neq 0$. Hence the equations $C_{il} = 0$ can all be satisfied if * F does not have a modulus $\leqq n - 3$.

(ii) If not every R_i vanishes, we may set $R_2 \neq 0$ in view of the symmetry. We determine the b_{2k} to make the coefficient of b_{2j} in B'_{jk} vanish:

$$b_{2k} = - R_2^{-1} \sum_{t=3}^{n} P_{2t} b_{tk} \qquad (k = 3, \cdots, n).$$

We give to b_{22} any value $\neq 0$ and set $b_{i2} = 0$ $(i = 3, \cdots, n)$. Then $B'_{2k} \equiv 0$ and

$$\Delta_{11} = b_{22} \Delta_{22}, \qquad \Delta_{22} \equiv |b_{is}| \qquad (i, s = 3, \cdots, n).$$

If $n = 3$, we take $b_{33} \neq 0$ and obtain an affirmative answer to our question. Let next $n > 3$. In $B'_{jk} = 0$ $(3 \leqq j < k)$, we replace b_{tk} for $t = 2$ by its value and obtain

$$B''_{jk} \equiv \sum_{i=3}^{n} b_{ij} \left\{ R'_i b_{ik} + \sum_{t \neq i}^{t=3, \ldots, n} P'_{it} b_{tk} \right\} = 0, \quad R'_i \equiv R_2 R_i - P_{2i}^2, \quad P'_{it} \equiv a_1 a_2 b_{11}^2 P_{it},$$

the value of P'_{it} being initially $R_2 P_{it} - P_{2i} P_{2t}$. The present problem — to determine the b_{is} $(i \geqq 3, s \geqq 3)$ so that their determinant $\Delta_{22} \neq 0$ and each $B''_{jk} = 0$ $(3 \leqq j < k)$ — is the exact analogue of the former problem — to determine the b_{is} $(i \geqq 2, s \geqq 2)$ so that $\Delta_{11} \neq 0$ and each $B'_{jk} = 0$ $(2 \leqq j < k)$. After suitable repetitions of the argument, the final problem is to determine the b_{is} $(i, s = n - 1, n)$ so that $|b_{is}| \neq 0$ and

$$B^*_{n-1\,n} \equiv \sum_{i=n-1}^{n} b_{i\,n-1} \left\{ R^*_i b_{in} + \sum_{t \neq i}^{t=n-1,\, n} P^*_{it} b_{tn} \right\} = 0, \qquad P^*_{n-1\,n} = P^*_{n\,n-1} \equiv P.$$

If R^*_{n-1}, R^*_n and P all vanish, we may choose the b's to be any elements of determinant not zero. In the contrary case, we take

$$b_{n-1\,n-1} = R^*_n b_{nn} + P_{n-1\,n}, \qquad b_{n\,n-1} = - R^*_{n-1} b_{n-1\,n} - P b_{nn}.$$

Then $B^*_{n-1\,n} \equiv 0$, $|b| = R^*_n b_{nn}^2 + 2 P b_{nn} b_{n-1\,n} + R^*_{n-1} b_{n-1\,n}^2 \not\equiv 0$.

THEOREM. *If F is any field not* † *having a modulus $\leqq n - 3$, there exists an n-ary linear transformation (b_{ij}) in F, with preassigned values of $b_{11}, b_{21}, \cdots, b_{n1}$ making $\sum_{i=1}^n a_i b_{i1}^2 \neq 0$, which replaces a given quadratic form $\sum_{i=1}^n a_i x_i^2$ by one of the type $\sum_{i=1}^n A_i x_i^2$ with $A_1 = \sum_{i=1}^n a_i b_{i1}^2$.*

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* This condition is necessary for the solvability of the $C_{il} = 0$. If F has a modulus which divides $l - 3$, the above determinant vanishes. Call $M_2, M_3, \cdots, M_{l-1}$ the minors of the elements $0, P_{32}, \cdots, P_{l-1\,2}$. Then must $b_{ll}(P_{2l} M_2 - P_{3l} M_3 + \cdots) = 0$. The determinant whose expansion is the second factor is seen as above to have the value 1.

† It is unnecessary for our applications to inquire whether or not this restriction on F is necessary for the validity of the theorem.

3. If F is a finite field the equation $a_1 b_{11}^2 + a_2 b_{21}^2 = 1$ has solutions b_{11}, b_{21} in F. Hence, applying the theorem of § 2 for $n = 2$, we see that any form $\sum_{i=1}^{m} a_i x_i^2$ in the $GF[p^k]$, $p > 2$, can be reduced by a succession of binary transformations to $\sum_{i=1}^{m-1} x_i^2 + a x_m^2$. Since one-half of the marks, not zero, of the $GF[p^k]$, $p > 2$, are squares, while the ratio of any two not-squares is a square, we can make $a = 1$ or a particular not-square ν.

THEOREM.* *In the $GF[p^k]$, $p > 2$, any m-ary quadratic form of non-vanishing determinant can be reduced to $\sum_{i=1}^{m} x_i^2$ or else to $\sum_{i=1}^{m-1} x_i^2 + \nu x_m^2$, ν being a particular not-square.*

4. Let F be the field R of all rational numbers. If $a_1 = \alpha_1/d$, $a_1 x_1^2 = \alpha_1 d y_1^2$, where $y_1 = x_1/d$. Hence we may assume that each a_1 in (1) is an integer. If $a_1, \cdots, a_4$ are not all negative, the equation

$$\sum_{i=1}^{4} a_i b_i^2 = 1 \tag{2}$$

can be satisfied by rational values of $b_1, \cdots, b_4$. Indeed, there exist† integers $\beta_1, \cdots, \beta_4, \sigma$, not all zero, for which $\sum_{i=1}^{4} a_i \beta_i^2 - \sigma^2 = 0$. But if $\sigma = 0$ and say $\beta_1 \neq 0$, $\beta_2 \neq 0$, (2) is satisfied by

$$b_1 = \frac{\beta_1}{2\beta_2}\left(1 - \frac{1}{a_2}\right), \qquad b_2 = \tfrac{1}{2}\left(1 + \frac{1}{a_2}\right), \qquad b_3 = \frac{b_1}{\beta_1}\beta_3, \qquad b_4 = \frac{b_1}{\beta_1}\beta_4.$$

It now follows that, if $a_1, \cdots, a_4$ are all negative, there exist rational solutions b_i of $\sum a_i b_i^2 = -1$. Hence, by the theorem of § 2, $\sum_{i=1}^{4} a_i x_i^2$ can be transformed by a quaternary substitution with rational coefficients into

$$\pm x_1^2 + \sum_{i=2}^{4} a_i' x_i^2,$$

the sign being $+$ unless $a_1, \cdots, a_4$ are all negative. We thus obtain the

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* The reduction of quadratic forms in the $GF[p^k]$ was effected by the writer in a memoir on the linear groups defined by a quadratic invariant, American Journal of Mathematics, vol. 21 (1899), pp. 194, 222. (Cf. *Linear Groups*, pp. 158, 197.) For $p = 2$, there exist m-ary quadratic forms of non-vanishing discriminant D only for $m = 2n$, and then the two canonical types are

$$F_\lambda = \lambda x_1^2 + \lambda y_1^2 + \sum_{i=1}^{n} x_i y_i,$$

where $\lambda = 0$ or a particular one of the marks for which $\lambda x_1^2 + \lambda y_1^2 + x_1 y_1$ is irreducible in the $GF[2^k]$. For $k = 1$, JORDAN states in his memoir cited above that a quadratic form is reducible to F_0 or F_1 according as $\left(\frac{2}{D}\right) = +1$ or -1. This is clearly an oversight since D is the mark 1 and thus can be taken to be an arbitrary odd integer. The same oversight occurs in DE SÉGUIER's *Groupes Abstraits*, p. 51, footnote.

† A. MEYER, Vierteljahrschrift der naturforschenden Gesellschaft in Zurich, vol. 29 (1884), pp. 209–222. He shows that $a_1 x_1^2 + \cdots + a_5 x_5^2 = 0$ has integral solutions x_i not all zero if $a_1, \cdots, a_5$ are integers $\neq 0$, not all of like sign.

THEOREM. *Within the field of all rational numbers any n-ary quadratic form of non-vanishing determinant can be transformed into one of the forms*

$$-\sum_{i=1}^{s} x_i^2 + \sum_{i=s+1}^{n-3} x_i^2 + ax_{n-2}^2 + bx_{n-1}^2 + cx_n^2,$$

where $s = 0$ unless a, b, c are all negative.

The difficult problem of ultimate canonical forms is not undertaken here. I pass to hermitian forms and effect a complete reduction to canonical forms.

II. *On the reduction of hermitian forms.*

5. Let F be any field* for which there is an equation

$$x^2 + ux + v = 0 \qquad (\text{roots } \omega,\ \bar{\omega},\ \bar{\omega} \neq \omega)\dagger,$$

belonging to and irreducible in F. Denote the field $F(\omega)$ by Q. Any element of Q may be given the form $e = a + b\omega$, a and b in F. Set $\bar{e} = a + b\bar{\omega}$. Then

$$H_\alpha \equiv \sum_{i,j}^{1,\ldots,n} \alpha_{ij} \xi_i \bar{\xi}_j \qquad (\text{each } \alpha_{ij} \text{ in } Q,\ \bar{\alpha}_{ij} = \alpha_{ji})$$

will be called a *hermitian form in the field* Q, and $|\alpha_{ij}|$ will be called the *determinant* of H_α. Under a linear transformation

$$(3) \qquad \xi_i = \sum_{k=1}^{n} \beta_{ik} \eta_k, \qquad \bar{\xi}_i = \sum_{k=1}^{n} \bar{\beta}_{ik} \bar{\eta}_k \qquad (i = 1, \cdots, n),$$

in which the β_{ik} are elements of Q and $B \equiv |\beta_{ik}| \neq 0$, H_α becomes a hermitian form H_γ whose determinant $|\gamma_{ij}|$ equals $B\bar{B}|\alpha_{ij}|$. ‡

6. THEOREM. *By a transformation* (3) *in Q of determinant unity, any hermitian form H_α in Q of non-vanishing determinant Δ can be reduced to*

$$(4) \qquad \sum_{i=1}^{n} \gamma_i \eta_i \bar{\eta}_i \qquad (\textit{each } \gamma_i \textit{ in } F).$$

We first reduce H_α to a form $H_{\alpha'}$ having $\alpha'_{11} \neq 0$. If $\alpha_{11} = 0$ and $\alpha_{jj} \neq 0$, we apply the transformation $\xi_1 = \eta_j$, $\xi_j = -\eta_1$. If every $\alpha_{ii} = 0$, we may take $\alpha_{12} \neq 0$; under the transformation $\xi_1 = \eta_1$, $\xi_2 = \eta_2 + \mu\eta_1$, H_α becomes $H_{\alpha'}$ with

* In particular, F shall not be the field of all complex numbers $x + yi$, nor a field $F_{m,p}$ defined as the aggregate of the Galois fields of orders p^m, p^{2m}, p^{4m}, p^{8m}, $\cdots$. If in a field F of modulus 2 every equation $x^2 + ux + v = 0$ $(u \neq 0)$ is reducible, F contains $F_{1,2}$.

† In the field of all rational functions of a variable z with integral coefficients taken modulo 2, the equation $x^2 = z$ is irreducible but has equal roots. We exclude such a case $\bar{\omega} = \omega$, since the problem is then that of bilinear forms in F subject to cogredient transformations.

‡ The simplest proof follows by use of generators of the types

$$\xi_1 = \eta_1 + \lambda\eta_2,\ \xi_i = \eta_i \qquad \text{and} \qquad \xi_1 = B\eta_1,\ \xi_i = \eta_i \qquad (i = 2, \ldots, n).$$

$\alpha'_{11} = \alpha_{12}\bar{\mu} + \bar{\alpha}_{12}\mu$. Take $\mu = (a + b\omega)/\bar{\alpha}_{12}$; then $\alpha'_{11} \equiv 2a - bu$ can be made different from zero, since we have excluded the case $u = 0$ when F has modulus 2 by assuming that $\bar{\omega} \neq \omega$.

In $H_{a'}$ with $\alpha'_{11} \neq 0$, set $\xi_1 = \eta_1 - \bar{\alpha}'_{12}\alpha'^{-1}_{11}\eta_2$, $\xi_2 = \eta_2$; there results $H_{a''}$ with $\alpha''_{11} = \alpha'_{11}$, $\alpha''_{12} = \alpha''_{21} = 0$. Similarly, we make every α_{1i} and α_{i1} zero ($i > 1$) and reach $H_{a''} = \alpha'_{11}\eta_1\bar{\eta}_1 + f$, where f is a hermitian form on η_i, $\bar{\eta}_i (i = 2, \cdots, n)$ of determinant $\neq 0$.

7. Let F be the $GF[p^m]$, so that Q is the $GF[p^{2m}]$. For any mark g of F the equation $c\bar{c} = g$, viz., $c^{p^m+1} = g$, is solvable in Q. *Hence there is an unique canonical form* $\Sigma\xi_i\bar{\xi}_i$.

8. Let F be the field of all real numbers and set $\omega = \sqrt{-1}$. Then $\gamma_i = \pm c_i^2$, so that (4) can be reduced to one of the forms

$$h_r \equiv \sum_{i=1}^{r} \xi_i\bar{\xi}_i - \sum_{l=r+1}^{n} \xi_i\bar{\xi}_i \qquad (r = 0, 1, \cdots, n). \tag{5}$$

Now r is an invariant, i. e., h_r can not be reduced to $h_\rho (\rho \neq r)$ by a transformation (3). Indeed, for $\xi_i = x_i + y_i\sqrt{-1}$, (3) becomes a special $2n$-ary real linear transformation, and (5) becomes

$$\sum_{i=1}^{r} (x_i^2 + y_i^2) - \sum_{i=r+1}^{n} (x_i^2 + y_i^2), \tag{6}$$

so that $2r$ is an invariant by § 1. *There are* $n + 1$ *canonical forms* (5).

9. Consider for a general field F the possible normalizations of a binary hermitian form $h = a(x\bar{x} + ry\bar{y})$, a and r in F, and each $\neq 0$. Set

$$x = \lambda X + \mu Y, \qquad y = \rho X + \sigma Y, \qquad D \equiv \lambda\sigma - \mu\rho \neq 0.$$

Then h becomes

$$h' = a(\lambda\bar{\lambda} + r\rho\bar{\rho})X\bar{X} + a(\mu\bar{\mu} + r\sigma\bar{\sigma})Y\bar{Y} + AX\bar{Y} + \bar{A}\bar{X}Y,$$

where $A = a(\lambda\bar{\mu} + r\rho\bar{\sigma})$. We desire that $A = 0$. For $\lambda \neq 0$, we take $\mu = -r\bar{\rho}\sigma/\bar{\lambda}$. Then $D = (\lambda\bar{\lambda} + r\rho\bar{\rho})\sigma/\bar{\lambda}$, and

$$h' = a(\lambda\bar{\lambda} + r\rho\bar{\rho})(X\bar{X} + r\tau\bar{\tau}Y\bar{Y}), \qquad \tau \equiv \frac{\sigma}{\lambda}.$$

The same form with $\tau = \mu/r\rho$ results if $\lambda = 0$, whence $\sigma = 0$. Hence r can be changed only by a factor $\tau\bar{\tau}$. By a preliminary unary transformation on y, we can restrict r to the series of multipliers $1, m_1, m_2, \cdots$ in a rectangular array of the elements of F with the various distinct elements $1, \kappa_1\bar{\kappa}_1, \kappa_2\bar{\kappa}_2, \cdots$ in the first row, where the κ's are arbitrary in Q. To retain this normalization, we set $\tau = 1$.

THEOREM. *Let* $1, m_1, m_2, \cdots$ *be the multipliers in a rectangular array of elements of* F, *the elements of the first row being the distinct elements* $\kappa\bar{\kappa}$, κ *ranging over* Q. *Any binary hermitian form can be reduced linearly to to* $a(x\bar{x} + ry\bar{y})$, *where* a *and* r *belong to the set* $1, m_1, m_2, \cdots$. *Two such reduced forms can be transformed into each other if and only if they have the same* r, *and the ratio of their* a*'s is a mark, not zero, expressible in the form* $\lambda\bar{\lambda} + r\rho\bar{\rho}$.

10. Let F be the field R of all rational numbers. Then $Q = R(\epsilon)$, where $\epsilon^2 = \nu$, ν being a fixed integer $\neq 1$ having no square factor. Thus $\bar{\epsilon} = -\epsilon$. By § 9, we can transform $ax\bar{x} + by\bar{y}$ into $Lx\bar{x} + My\bar{y}$, where $L = a\lambda\bar{\lambda} + b\rho\bar{\rho}$, λ and ρ being any elements of Q for which $L \neq 0$. If $a = \alpha/d$, α and d being integers, then $ax\bar{x} = \alpha dz\bar{z}$, $z = x/d$. Hence we may assume that a and b are integers. Set $\lambda = \alpha + \beta\epsilon$, $\rho = \gamma + \delta\epsilon$, $\alpha, \beta, \gamma, \delta$ in R. Then

$$L \equiv a\lambda\bar{\lambda} + b\rho\bar{\rho} = a(\alpha^2 - \nu\beta^2) + b(\gamma^2 - \nu\delta^2). \tag{7}$$

As* in § 4, we can choose $\alpha, \beta, \gamma, \delta$ in R to make $L = +1$ if ν, a, b are not all negative, and $L = -1$ in the contrary case. Hence if ν is positive, any n-ary hermitian form is reducible to

$$\sum_{i=1}^{n-1} \xi_i\bar{\xi}_i + r\xi_n\bar{\xi}_n; \tag{8}$$

while, if ν is negative, it can be reduced to one of the types

$$f_s \equiv -\sum_{i=1}^{s} \xi_i\bar{\xi}_i + \sum_{i=s+1}^{n-1} \xi_i\bar{\xi}_i + r\xi_n\bar{\xi}_n \qquad (s = 0, 1, \cdots, n-1). \tag{9}$$

If $r > 0$ and $s > 0$, we can transform $-\xi_s\bar{\xi}_s + r\xi_n\bar{\xi}_n$ into $+\xi_s\bar{\xi}_s + r'\xi_n\bar{\xi}_n$, and hence transform f_s into f_{s-1}. The reduced forms for ν negative are thus

$$f_0 \text{ with } r > 0;\ f_s(s = 0, 1, \cdots, n-1) \text{ with } r < 0, \tag{10}$$

falling into $n + 1$ types, each characterized by the number of its negative terms. Hence by § 8, a form of one type cannot be reduced to one of a different type by a transformation (3). But by § 5, under a transformation (3) of determinant B, the determinant $(-1)^s r$ of f_s is multiplied by $B\bar{B}$. Hence $f_s(r)$ is reducible to $f_s(\rho)$ if and only if ρ/r is expressible in the form $B\bar{B}$.

THEOREM. *Any n-ary hermitian form in* $R(\sqrt{\nu})$ *with non-vanishing determinant can be reduced by a linear transformation in* $R(\sqrt{\nu})$ *to one and but one of the canonical forms:*

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* The case $\sigma = 0$ may now be treated more naturally. There are then solutions $\lambda_1 \neq 0$, $\rho_1 \neq 0$ in $R(\epsilon)$ of $a\lambda_1\bar{\lambda}_1 + b\rho_1\bar{\rho}_1 = 0$. Let $\tau = \rho\lambda_1/\rho_1$. Then $L = 1$ becomes $\lambda\bar{\lambda} - \tau\bar{\tau} = a^{-1}$, and is satisfied by $\lambda = \frac{1}{2}(a^{-1} + 1)$, $\tau = \frac{1}{2}(a^{-1} - 1)$.

(I) *for* $\nu > 0$, $\sum_{i=1}^{n-1}\xi_i\bar{\xi}_i + r\xi_n\bar{\xi}_n$, *where* r *ranges over the multipliers (integers)* $1, m_1, m_2, \cdots$, *in a rectangular array of all rational numbers with those representable by* $x^2 - \nu y^2$ *in the first row;*

(II) *for* $\nu < 0$, $\sum_{i=1}^{n-1}\xi_i\bar{\xi}_i + \rho\xi_n\bar{\xi}_n$, $-\sum_{i=1}^{s}\xi_i\bar{\xi}_i + \sum_{i=s+1}^{n-1}\xi_i\bar{\xi}_i - \rho\xi_n\bar{\xi}_n$, *where* $s = 0, 1, \cdots, n-1$, *and* ρ *ranges over the multipliers (positive integers) in a rectangular array of all positive rational numbers with those representable by* $x^2 - \nu y^2$ *in the first row.*

In the examples,* (k) denotes all primes of the form k; $q_1, q_2, q_3, \cdots$, denote distinct primes; $p_1, p_2, p_3, \cdots$, denote distinct primes.

For $\nu = 2$, $r = 1, q_1, q_1q_2, q_1q_2q_3, \cdots$, q_i ranging over $(8m+3), (8m+5)$.

For $\nu = 3$, $r = \pm 1, \pm q_1, \pm q_1q_2, \cdots$, q_i ranging over $(12m+5), (12m+7)$.

For $\nu = 5$, $r = 1$, q_1, q_1q_2, $\cdots$, q_i ranging over $(20m+3)$, $(20m+7)$, $(20m+13)$, $(20m+17)$.

For $\nu < 0$, $\rho = 1, q_1, q_1q_2, q_1q_2q_3, \cdots$, where for $\nu = -1$, $q_i = (4m+3)$; for $\nu = -2$, $q_i = (8m+5)$, $(8m+7)$; for $\nu = -3$, $q_i = (3m+2)$; for $\nu = -5$, the only limitations on the primes q_i are that no one is 5, $20m+1$, or $20m+9$, while at most one is chosen from the set 2, $(20m+3)$, $(20m+7)$; for $\nu = -6$, no q_i is $24m+1$ or $24m+7$, while at most one is chosen from the set $2, 3, (24m+5), (24m+11)$; for $\nu = -7$, $q_i = (28m+t)$, $t = 3, 5, 13, 17, 19, 27$.

III. *The bilinear forms invariant under a given substitution S.*

11. Let F be an arbitrarily given field. We seek all bilinear functions $\Phi = \sum_{ij}\gamma_{ij}\xi_i\eta_j$ with discriminant $D \equiv |\gamma_{ij}| \neq 0$ and coefficients in F, such that Φ is invariant under a given substitution, cogredient in the two sets of variables,

$$S: \qquad \xi_i' = \sum_{j=1}^{n}\alpha_{ij}\xi_j, \qquad \eta_i' = \sum_{j=1}^{n}\alpha_{ij}\eta_j \qquad (i = 1, \cdots, n),$$

with coefficients also in F. In the canonical form of S (with the initial variables ξ_i), the new variables fall into as many classes as there are distinct roots of the characteristic equation $\Delta(\rho) = 0$ of S. Each class is composed of one or more series, the variables of any series being transformed by S into linear functions of themselves, as follows:

$$| x_0, x_1, \cdots, x_t \qquad \rho x_0, \rho(x_1 + x_0), \cdots, \rho(x_t + x_{t-1}) |.$$

* The integers representable by $x^2 - 2y^2$ are, aside from square factors, ± 1, ± 2, $(8n+1)$, $(8n+7)$, and their products two, three, four, $\cdots$, at a time. For $x^2 - 3y^2$, they are -2, -3, $(12m+1)$, $-(12m+11)$. For $x^2 + 5y^2$, they are 5, $(20m+1)$, $(20m+9)$, $2p_1$, p_1p_2, and the products of these expressions two, three, $\cdots$, at a time, where $p_i = (20m+3)$, $(20m+7)$. For $x^2 + 6y^2$, they are $(24m+1)$, $(24m+7)$, p_1p_2, and their products, where $p_i = 2, 3, 24m+5$ or $24m+11$.

Similarly, S with the initial variables η_i affects the general series of canonical variables as follows:

$$|y_0, y_1, \cdots, y_\tau \quad \rho_1 y_0, \rho_1(y_1 + y_0), \cdots, \rho_1(y_\tau + y_{\tau-1})|.$$

In the new variables, Φ becomes a sum of functions each separately invariant, the general one, f, being bilinear in x_i, y_j $(i = 0, \cdots, t;\ j = 0, \cdots, \tau)$. Let ax_iy_j be any term of the latter, $i + j$ its *rank*, i being the rank of x_i and j the rank of y_j. Since the increment obtained from any term is of lower rank, the set of terms of maximum rank in the transform of f by S is derived from the set of terms of maximum rank in f by multiplication by $\rho\rho_1$. Unless f is identically zero, $\rho\rho_1 = 1$. But if none of the variables x appeared in the new form of Φ, its discriminant would vanish. Hence there is at least one series of variables y which S_η multiplies by ρ^{-1}. With the class C_x of all the variables which S_ξ multiplies by ρ is associated a class C_y of the variables which S_η multiplies by ρ^{-1}. Then $\Phi = [C_xC_y] + \Psi$, where $[C_xC_y]$ is bilinear in the variables of C_x, C_y, while Ψ does not contain those variables. The discriminant of $[C_xC_y]$ is a factor of D and hence is not zero; this requires that the classes C_x and C_y shall contain the same number of variables. Thus ρ^{-1} must be a root of $\Delta(\rho) = 0$ of the same multiplicity as the root ρ. A first necessary condition for an invariant Φ under S is:

(11) *The characteristic equation of S must be a reciprocal equation.*

A second condition for the existence of $[C_xC_y]$ is (JORDAN, § 13):

(12) *Classes C_x, C_y must be of like type as to number and length of series.*

12. When these conditions on S are satisfied, invariants $[C_xC_y]$ of non-vanishing discriminant exist (JORDAN, § 12, case I alone occurs for bilinear functions), the general one being $\sum_{\alpha,\beta,r} a_r^{\alpha\beta} f_{\alpha\beta r}$, where f is a definite bilinear function and the a's are any polynomials in $\rho, \rho_1, \cdots, \rho_{\nu-1}$, for which the discriminant of $[C_xC_y]$ is not zero and the following "reality" conditions hold: Since Φ is to be equal to a function of the initial variables with coefficients in F, we must have $\Phi = \sum + \Phi_1$, where

$$\sum = [C_xC_y] + [C'_xC'_y] + \cdots + [C_x^{\nu-1}C_y^{\nu-1}],$$

$[C'_xC'_y]$, $\cdots$, being derived from $[C_xC_y]$ by interchanging ρ and $\rho_1, \cdots$, ρ and $\rho_{\nu-1}$, respectively, $\rho_1, \cdots, \rho_{\nu-1}$, being the remaining roots of the same irreducible factor of $\Delta(\rho)$. Thus $a_r^{\alpha\beta}$ must equal a polynomial in ρ. Let these conditions be satisfied. Then (JORDAN,* § 14–16) by a linear transformation leaving the canonical form of S unaltered (not necessarily the same transformation on the y's as on the x's), we can reduce Φ to a unique canonical form.

*An obvious correction (not altering the argument) is to be made on p. 237, l. 11. The expression in brackets should be

$$y'_0(x'_m + C^1_{m-1}x'_{m-1} + \cdots + x'_1) + y''_0(x''_m + C^1_{m-1}x''_{m-1} + \cdots + x''_1) + \cdots.$$

13. Conditions (11) and (12) depend upon quantities irrational in general with respect to the initial field F. It seems desirable to proceed further and exhibit purely rational conditions for the solvability of the problem. We obtain * the

THEOREM. *The necessary and sufficient conditions that a given substitution* (α_{ij}) *with coefficients in* F *shall leave invariant one or more bilinear forms with coefficients in* F *are:* (i) *the characteristic determinant* $\Delta(\rho)$ *of* S *has a decomposition into factors irreducible in* F *of the type*

$$\Delta(\rho) = S_k^a S_l'^b \cdots R_t^e R_t^{*e} \cdots R_s'^f R_s'^{*f} \cdots,$$

where S_k, S'_l, $\cdots$ *are self-reciprocal, viz.,* $S_k(\rho) \equiv \rho^k S_k(\rho^{-1})$, *while* R_t *and* R_t^* *are reciprocal;* (ii) *the invariant-factors of the matrix* $\Delta(\rho)$ *are of the form*

$$S_k^{a_1} S_l'^{b_1} \cdots R_t^{e_1} R_t^{*e_1} \cdots R_s'^{f_1} R_s'^{*f_1}, \quad S_k^{a_2} S_l'^{b_2} \cdots R_t^{e_2} R_t^{*e_2} \cdots R_s'^{f_2} R_s'^{*f_2}, \quad \cdots,$$

a pair of reciprocal factors occurring always to the same power.

When these conditions are satisfied there exist in F *bilinear forms* $\sum b_{ij} \xi_i \eta_j$ *invariant under* S_ξ, S_η. *All such invariant forms are reducible to a single one by a transformation on the* ξ's *commutation with* S_ξ, *and a (possibly different) transformation on the* η's *commutative with* S_η.

IV. *The quadratic forms invariant under a given substitution* S.

14. JORDAN's treatment of the case of a field of order p, a prime, can be extended immediately to any finite field, and with certain essential modifications to any infinite field. Let F be any field.

Let ρ be a root of $f(x) = 0$, where $f(x)$ is a factor of degree k of $\Delta(x)$ and is irreducible in F. If ρ^{-1} is not a root of $f(x) = 0$, the question is essentially the same as that for bilinear forms, discussed above. Let next ρ^{-1} be a root of $f(x) = 0$, $\rho^{-1} \neq \rho$. Then $f(x) = 0$ and $x^k f(x^{-1}) = 0$ are equations belonging to and irreducible in F with a root ρ in common; hence all their roots are common and $f(x) = 0$ is a reciprocal equation. Since no root equals its reciprocal, $k = \text{even} = 2\nu$, and the roots may be designated $\rho \equiv \rho_0, \rho_1, \cdots, \rho_{2\nu-1}$ with $\rho_\nu = \rho_0^{-1}, \rho_{\nu+1} = \rho_1^{-1}, \cdots, \rho_{2\nu-1} = \rho_{\nu-1}^{-1}$. In the canonical form of S all the variables corresponding to a root ρ_i are said to form the class C_i. Thus an invariant quadratic form Φ must equal $[C_0 C_\nu] + \Psi$, where $[C_0 C_\nu]$ is bilinear in the variables of classes C_0 and C_ν, while Ψ does not contain them. Now $[C_0 C_\nu]$ which is itself invariant must have the form

$$(13) \qquad [C_0 C_\nu] = \sum_{\alpha,\beta,r} a_r^{\alpha\beta} F_{\alpha\beta r},$$

where $a_r^{\alpha\beta}$ are any constants satisfying the conditions later specified, while $F_{\alpha\beta r}$

* Cf. Transactions, vol. 3 (1902), pp. 290–292.

are perfectly definite bilinear forms derived from those given by JORDAN, pp. 240, 241, by making the following changes. For b_i, $b_i^{p^\nu}$, c_i, $c_i^{p^\nu}$, $\cdots$ write $b_i(\rho)$, $b_i(\rho^{-1})$, $c_i(\rho)$, $c_i(\rho^{-1})$, $\cdots$, respectively; for e of § 21 write $\rho - \rho^{-1}$. In § 20, take $b_0 = \frac{1}{2}$ if F does not have modulus 2, while for modulus 2 take

$$b_0 = -\frac{1}{\alpha_\nu}(\rho^\nu + \alpha_1\rho^{\nu-1} + \alpha_2\rho^{\nu-2} + \cdots + \alpha_{\nu-1}\rho),$$

where the α's refer to the irreducible reciprocal equation with root ρ:

$$(14)\quad E \equiv y^\nu + y^{-\nu} + \alpha_1(y^{\nu-1} + y^{-\nu+1}) + \cdots + \alpha_{\nu-1}(y + y^{-1}) + \alpha_\nu = 0.$$

If a_ν vanished, there would be a factor $y - y^{-1}$ modulo 2. Since the variables entering $F_{\alpha\beta r}$ are linear functions of the initial variables ξ_i whose coefficients are polynomials in ρ with coefficients in F, $F_{\alpha\beta r}$ can be expressed as a function of ρ and the ξ's with coefficients in F. In particular $F_{\alpha\beta r}$ is unaltered by any permutation of the roots $\rho_1, \rho_1^{-1}, \cdots, \rho_{\nu-1}, \rho_{\nu-1}^{-1}$. The same must be true of $[C_0 C_\nu]$, which is composed of all the terms of Φ involving the variables of classes C_0 and C_ν (viz., those corresponding to the roots ρ and ρ^{-1}), since Φ is to be equal to a function of the ξ's with coefficients in F. Hence the a's in (13) must be symmetric functions of those $2\nu - 2$ roots; but the latter are the roots of $E/(y + y^{-1} - \rho - \rho^{-1}) = 0$, E being given by (14). *Hence the coefficients in* (13) *are rational functions of* ρ *with coefficients in* F. By its construction $F_{\alpha\beta r}$ becomes $F_{\beta\alpha r}$ when ρ is replaced by ρ^{-1}. Hence

$$(15)\qquad a_r^{\alpha\beta}(\rho) = a_r^{\beta\alpha}(\rho^{-1}),$$

whence $a_r^{\alpha\alpha}$ is a polynomial in $\rho + \rho^{-1}$.

Finally, since $[C_0 C_\nu] + \Psi$ is to be equal to a function Φ of the ξ's with coefficients in F, Ψ must contain the bilinear forms $[C_1 C_{\nu+1}], \cdots, [C_{\nu-1} C_{2\nu-1}]$ derived from $[C_0 C_\nu]$ by replacing ρ by $\rho_1, \cdots, \rho_{\nu-1}$, respectively. Hence must $\Phi = Q + \Phi_1$, where Φ_1 involves no variable in the classes $C_0, \cdots, C_{2\nu-1}$, while

$$(16)\qquad Q \equiv \sum_{i=0}^{\nu-1} [C_i C_{\nu+i}] = \sum_{\alpha,\beta,r,i} a_r^{\alpha\beta}(\rho_i) F_{\alpha\beta r}^{(i)},$$

$F_{\alpha\beta r}^{(i)}$ being the same function of ρ_i and the variables of C_i, $C_{\nu+i}$ that $F_{\alpha\beta r}^0 \equiv F_{\alpha\beta r}$ is of ρ_0 and the variables of C_0, C_ν. In (16), the a's are arbitrary rational functions of ρ_i satisfying (15) and making the descriminant of $[C_0 C_\nu]$ not zero (requiring that certain determinants of the a's be $\neq 0$). The same argument is to be repeated for Φ_1 with reference to each new irreducible factor of $\Delta(\rho)$.

15. The next problem is the reduction of $[C_0 C_\nu]$ to one or more normal forms by means of a transformatian of variables leaving S unaltered. JORDAN shows (§§ 23–28) that a unique normal form results when F is a field of order

p. The same is true for any finite field, but not for an arbitrary infinite field. For an arbitrary field F we can proceed with JORDAN's normalization by observing the following modifications.* The constants a, a^{p^ν}, λ, λ^{p^ν}, $\cdots$ are to be interpreted as $a(\rho)$, $a(\rho^{-1})$, $\lambda(\rho)$, $\lambda(\rho^{-1})$, $\cdots$, respectively.

To prove that the argument at the bottom of p. 243 remains valid, we have to show that, for $a_m^{21} \neq 0$,

$$\lambda(\rho)a_m^{21}(\rho) + \lambda(\rho^{-1})a_m^{12}(\rho) + \lambda(\rho)\lambda(\rho^{-1})a_m^{22}(\rho)$$

is not identically zero for every rational function λ. But if the sum vanished for $\lambda(\rho) = 1, -1$, and ρ, then would

$$\begin{vmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ \rho & \rho^{-1} & 1 \end{vmatrix} \equiv -2(\rho^{-1} - \rho) = 0,$$

which is impossible if F does not have modulus 2. For modulus 2, we employ $\lambda(\rho) = 1$, ρ, and $\rho + 1$, obtaining as the determinant $(\rho^{-1} + \rho)(\rho^{-1} - \rho)$.

The argument to make $a_m^{11} = 1$ (top of p. 244) must now be abandoned. For the present we allow a_m^{11} to remain arbitrary, but $\neq 0$. To make $a_m^{12} = a_m^{21} = 0$, apply the transformation which replaces x'_i and y'_i by $x'_i - a(\rho)x''_i$ and $y'_i - a(\rho^{-1})y''_i$, respectively, for $i = m, \cdots, 0$, taking $a(\rho) = a_m^{21}(\rho)/a_m^{11}(\rho)$. By analogous transformations we can make every $a_m^{\alpha\beta} = 0\ (\alpha \neq \beta)$ and reach

$$(17) \qquad \phi_a = A \sum_{a=1}^{l} \{ a_m^{aa}[x_0^a y_m^a + (-1)^m x_m^a y_0^a] \} \quad (\text{each } a_m^{aa} \neq 0,\ A \neq 0).$$

Consequently we insert the factors a_m^{aa} in the formulæ of §§ 26–28. For λ on p. 248 we now take

$$\lambda(\rho) = -c(\rho^{-1}) \div \{A(\rho^{-1})a_m^{aa}(\rho)\}.$$

For case 1°, p. 250, the condition to be satisfied is now

$$(-1)^{m'-r'}[\lambda(\rho^{-1}) + \lambda(\rho)]\,a_m^{aa} + a_r^{aa} = 0,$$

where each a is a rational function of $\rho + \rho^{-1}$. To this end we apply the

LEMMA. *If ρ is a root of an irreducible reciprocal equation* (14), *we can determine a polynomial $\lambda(\rho)$ such that $\lambda(\rho^{-1}) + \lambda(\rho) = f(\rho + \rho^{-1})$, f being any given polynomial, where the coefficients of λ and f belong to the arbitrary field F.*

In view of (14), we may set

$$f = g_1(\rho^{\nu-1} + \rho^{-\nu+1}) + \cdots + g_{\nu-1}(\rho + \rho^{-1}) + g_\nu \qquad (g\text{'s in } F).$$

* We do not consider the numerical results of §§ 24, 25, peculiar to finite fields. On p. 245, line 6, the term $c_1^{p^\nu}d_1$ should be deleted. The group considered in § 24 occurs in the literature, Mathematische Annalen, vol. 52 (1899) p. 561, and vol. 55 (1902), p. 521, as the hyperorthogonal group in the $GF[p^{2\nu}]$.

We proced to exhibit a solution λ of the form

$$\lambda(\rho) = c_0\rho^\nu + c_1\rho^{\nu-1} + \cdots + c_\nu \qquad (c\text{'s in } F).$$

If F does not have modulus 2 we may take

$$c_0 = 0,\ c_1 = g_1,\ \cdots,\ c_{\nu-1} = g_{\nu-1},\ c_\nu = \tfrac{1}{2}g_\nu.$$

If F has modulus 2, we apply (14) to eliminate $\rho^\nu + \rho^{-\nu}$ and get

$$\lambda(\rho^{-1}) + \lambda(\rho) = \sum_{i=1}^{\nu-1}(c_i - c_0\alpha_i)(\rho^{\nu-i} + \rho^{-\nu+i}) - c_0\alpha_\nu.$$

Hence this equals f if we set

$$c_0 = \frac{g_\nu}{\alpha_\nu}, \qquad c_i = g_i + \frac{\alpha_i g_\nu}{\alpha_\nu} \qquad (i = 1, \cdots, \nu - 1).$$

For case 2°, page 250, the condition to be satisfied is now*

$$(-1)^{m'-r'-1}e[\lambda(\rho^{-1}) - \lambda(\rho)]\,a_m^{aa} + a_r^{aa} = 0,\ e \equiv \rho - \rho^{-1}.$$

Now $(-1)^{m'-r'}a_r^{aa}/a_m^{aa}$ equals a polynomial $f(\rho+\rho^{-1})$ by (15). Set $\mu(\rho) = (\rho^{-1} - \rho)\lambda(\rho)$. The resulting condition $\mu(\rho^{-1}) + \mu(\rho) = f(\rho+\rho^{-1})$ may be satisfied by the Lemma.

Cases 3_1°, 3_2° are analogous to 2°, 1°, respectively.

Hence $[C_0\,C_\nu]$ *can be reduced to the semi-normal form*

$$(18) \qquad \sum_{\alpha=1}^{l} a_m^{\alpha\alpha} F_{\alpha\alpha m} + \sum_{\beta=l+1}^{l+l'} a_{m'}^{\beta\beta} F_{\beta\beta m'} + \sum_{\gamma=l+l'+1}^{l+l'+l''} a_{m''}^{\gamma\gamma} F_{\gamma\gamma m''} + \cdots,$$

where each $a = a(\rho) = a(\rho^{-1}) \neq 0$, *while* $F_{\alpha\alpha m}$ *is a bilinear function with fixed coefficients of* x_i^α *and* y_i^α $(i = 0, 1, \cdots, m;\ \alpha = 1, \cdots, l)$, $F_{\beta\beta m'}$ *a bilinear function of* x_i^β *and* y_i^β $(i = 0, 1, \cdots, m';\ \beta = l+1, \cdots, l+l')$, *etc. Also* $y(\rho) = x(\rho^{-1})$, $m > m' > m'' \cdots$.

16. First let F be a finite field, the $GF[p^k]$. Then $F(\rho)$ is the $GF[p^{2\nu k}]$ and $\rho^{p^{\nu k}} = \rho^{-1}$ The a's belong to the $GF[p^{\nu k}]$. Apply the transformation (commutative with S) which multiplies each x_i^α by λ and each y_i^α by $\lambda^{p^{\nu k}}$. The new coefficients of $F_{\alpha\alpha m}$ is $a_m^{\alpha\alpha}\lambda^{p^{\nu k}+1}$ and hence can be made unity by choice of λ in the $GF[p^{2\nu k}]$. *For a finite field every* a *in* (18) *can be made unity, so that there is a unique normal form.*

17. For a general field F the question is not so simple; we shall have to

* By a misprint the wrong sign is given by JORDAN. On p. 249, 5th line from the bottom, $\lambda^{p^\nu}y_{m-r}^a$ should read $\lambda^{p^\nu}y_0^a$. On p. 250, 3d line from the bottom, $\lambda^{p^\nu} + \lambda$ should read $\lambda^{p^\nu} - \lambda$.

consider normalizations not used or needed in JORDAN's case. Now S affects * the variables entering (18) as follows:

$$
(19)\left\{\begin{array}{ll}
x_0^\alpha, x_1^\alpha, \cdots, x_m^\alpha, \quad \rho x_0^\alpha, \rho(x_1^\alpha+x_0^\alpha), \cdots, \rho(x_m^\alpha+x_{m-1}^\alpha) & \\
y_0^\alpha, y_1^\alpha, \cdots, y_m^\alpha, \quad \rho^{-1} y_0^\alpha, \rho^{-1}(y_1^\alpha+y_0^\alpha), \cdots, \rho^{-1}(y_m^\alpha+y_{m-1}^\alpha) & (\alpha=1, \cdots, l) \\
x_0^\beta, x_1^\beta, \cdots, x_{m'}^\beta, \quad \rho x_0^\beta, \rho(x_1^\beta+x_0^\beta), \cdots, \rho(x_{m'}^\beta+x_{m'-1}^\beta) & \\
y_0^\beta, y_1^\beta, \cdots, y_{m'}^\beta, \quad \rho^{-1} y_0^\beta, \rho^{-1}(y_1^\beta+y_0^\beta), \cdots, \rho^{-1}(y_{m'}^\beta+y_{m'-1}^\beta) & (\beta=l+1, \cdots, l+l') \\
\text{analogously for } x_0^\gamma, y_0^\gamma, \cdots, x_{m''}^\gamma, y_{m''}^\gamma & (\gamma=l+l'+1, \cdots, l+l'+l'').
\end{array}\right.
$$

Any substitution P commutative with (19) is the product of a substitution T on the x's by the conjugate substitution $\bar{T}$ on the y's. Further,

$$
T = T_m T_{m'} T_{m''} \cdots W,
$$

where each factor is commutative with S, T_m denoting a substitution of the form

$$
(20) \qquad \left| x_i^\alpha \quad \sum_{\delta=1}^{l} b_{\alpha\delta} x_i^\delta \right| \qquad (\alpha=1, \cdots, l;\ i=0, 1, \cdots, m),
$$

$T_{m'}$ being cogredient on $x_i^{l+1}, \cdots, x_i^{l+l'}$, for $i = 0, 1, \cdots, m'$, etc., while W is derived from substitutions, the general one of which replaces x_i^α by $x_i^\alpha + \lambda x_{i-m+r}^\beta$ $(i = m, \cdots, m - r)$ and leaves fixed $x_j^\alpha (j = m - r - 1, \cdots, 0)$, r being $< m$. Hence † T replaces each x_0^α by a linear function of $x_0', \cdots, x_0^l$ only, while

* Here and henceforth it is to be understood that the remaining variables $x(\rho_1), y(\rho_1), \cdots, x(\rho_{\nu-1}), y(\rho_{\nu-1})$, conjugate to $x(\rho) \equiv x$, undergo the transformation conjugate to that on x. Thus the complete substitution can be expressed as a substitution on the initial variables with coefficients in F.

† The explicit form of T is not essential to the argument; it is moreover fairly complex *Linear Groups*, § 218). For $m=2$, $m'=1$, $l=2$, $l'=1$, S is

$$
\left| \begin{array}{ll} x_0^\alpha, x_1^\alpha, x_2^\alpha & \rho x_0^\alpha, \rho(x_1^\alpha + x_0^\alpha), \rho(x_2^\alpha + x_1^\alpha) \\ x_0''', x_1''' & \rho x_0''', \rho(x_1''' + x_0''') \end{array} \right| \qquad (\alpha = 1, 2),
$$

the explicit form of T is the following (zero coefficients not being entered):

	x_0'	x_0''	x_1'	x_1''	x_2'	x_2''	x_0'''	x_1'''
x_0'	b_{11}	b_{12}						
x_0''	b_{21}	b_{22}						
x_1'	c_1	c_2	b_{11}	b_{12}			e_1	
x_1''	c_3	c_4	b_{21}	b_{22}			e_2	
x_2'	d_1	d_2	c_1	c_2	b_{11}	b_{12}	f_1	e_1
x_2''	d_3	d_4	c_3	c_4	b_{21}	b_{22}	f_2	e_2
x_0'''	g_1	g_2					b_{33}	
x_1'''	h_1	h_2	g_1	g_2			k	b_{33}

$x'_m, \cdots, x'_m$ appear in the functions by which T replaces x_i^a only when $i = m$. Analogous remarks on the y's hold for $\overline{T}$. Hence in the function by which $P \equiv T\overline{T}$ replaces (18), the terms involving the variables x_m^a and y_m^a of maximum rank m come only from the terms of (18) involving these variables. But the latter terms are given by ϕ_a of (17). Hence in order that P shall transform f_a given by (18) into a similar function $f_{a'}$, it is necessary that P shall transform ϕ_a into $\phi_{a'} + \psi$, where the variables of ψ are of rank $< m$. Hence the factor T_m, given by (20), of T must transform ϕ_a into $\phi_{a'}$. The necessary and sufficient conditions for this are

$$\sum_{a=1}^{l} a_m^{aa} b_{a\delta} \overline{b}_{a\epsilon} = \begin{cases} 0 & \text{if } \delta \neq \epsilon, \\ a_m'^{\delta\delta} & \text{if } \delta = \epsilon. \end{cases} \tag{21}$$

We assume that these conditions on T_m are satisfied. Then T_m replaces $\sum_{a=1}^{l} a_m^{aa} x_i^a y_j^a$ by

$$\sum_{\delta,\epsilon=1}^{l} \left(\sum_{a=1}^{l} a_m^{aa} b_{a\delta} \overline{b}_{a\epsilon} \right) x_i^\delta y_j^\epsilon = \sum_{\delta=1}^{l} a_m'^{\delta\delta} x_i^\delta y_j^\delta .$$

Hence T_m replaces $\sum_{a=1}^{l} a_m^{aa} F_{aam}$ by $\sum_{a=1}^{l} a_m'^{aa} F_{aam}$. The same reasoning applies to $T_{m'}, T_{m''}, \cdots$. Hence* if $T\overline{T}$ replaces f_a by $f_{a'}$, the product $T_m \overline{T}_m T_{m'} \overline{T}_{m'} \cdots$ must replace f_a by $f_{a'}$.

THEOREM. *Any possible transformation of* (18) *into a similar function by means of a substitution commutative with S can be effected by the simple substitutions* (20) *subject to conditions* (21). *The normalization of* (18) *must take place in the individual sums independently.*

18. In the actual normalization of $\sum_a a_m^{aa} F_{aam}$, the plan of § 9 is to apply, instead of a single l-ary substitution (20), a succession of binary substitutions [special cases of (20)]:

$$|x'_i \quad x''_i \quad b_{11}x'_i + b_{12}x''_i,\ b_{21}x'_i + b_{22}x''_i| \quad (i = 0, 1, \cdots, m), \tag{22}$$

where by (21), $a_m^{11} b_{11}\overline{b}_{12} + a_m^{22} b_{21}\overline{b}_{22} = 0$. It follows as in § 9 that $a(F_{11m} + rF_{22m})$ can be multiplied by $\lambda\overline{\lambda} + r\mu\overline{\mu}$ by means of a binary substitution commutative with S, where λ and μ are any rational functions of ρ such that $\lambda\overline{\lambda} + r\mu\overline{\mu} \neq 0$.

19. Finally, for a root $\rho = \pm 1$, the operations take place in the initial field F. We may therefore follow JORDAN's developments † (§§ 33–35), to obtain the most general invariant $[C]$ involving the variables with multiplier ± 1. But the reduction of $[C]$ to normal forms by means of substitutions commuta-

* For our normalization, we may therefore dispense with the substitutions W. It may be be noted in passing that W replaces $f_{a'}$ by $f_{a''}$ only when $f_{a'} \equiv f_{a''}$. This may be proved directly by noting that the first term in JORDAN's expression for F_{aam} appears with the same coefficient after transformation by W.

† On p. 258, l. 7, read $2n - 2$ in the second term; at the end of $G_{a''}$ read $x_{n-k-k'}$.

tive with S requires essential modifications for the generalization to an arbitrary field F. The first step (§ 37) is now impossible in general. Instead,* if F is any field not having modulus 2, we employ the preliminary normalization reducing ϕ to $\sum_{a=1}^{l} a_n^{aa} x_n^a x_n^a$, each $a \neq 0$. Although a_n^{11} is not necessarily unity, the argument in §§ 38–40 holds after an evident modification, so that we reach

$$[C] = \sum_{a=1}^{l} a_n^{aa} G_{an} + \Psi \qquad (\text{if } m = \text{even} = 2n).$$

No changes are necessary in §§ 41–43, so that there results

$$[C] = \sum_{k=1}^{l/2} f_{2k-1,\,2k,\,2n-1} + \Psi \qquad (\text{if } m = \text{odd} = 2n - 1).$$

Indeed, all alternate bilinear forms of determinant $\neq 0$,

$$\sum_{i,\,k}^{1,\,\ldots,\,2m} c_{ik} x_i X_k \qquad (c_{ik} = -c_{ki},\ c\text{'s in field } F),$$

are reducible by a linear substitution in F, congredient on the x's and X's, to a unique normal form $\sum_{k=1}^{m} (x_{2k-1} X_{2k} - x_{2k} X_{2k-1})$.

Continuing similarly the reduction of Ψ, we obtain as a semi-canonical form of $[C]$ a sum of terms G and f, affecting different variables. The factors a of the G are as yet any marks not zero of F, but otherwise the coefficients in $[C]$ are all fixed constants. As shown above for hermitian forms, so here further normalization must take place in the separate sums

$$(23) \qquad \sigma_a \equiv \sum_{a=1}^{l} a_n^{aa} G_{an}, \qquad \sum_{\beta=l+1}^{l+l'} a_{n'}^{\beta\beta} G_{\beta n'}, \cdots \qquad (n > n' > n'', \cdots),$$

the only substitution effective in normalizing σ_a being of the form

$$(24) \qquad \left| x_i^a \qquad \sum_{\delta=1}^{l} b_{a\delta} x_i^\delta \right| \qquad (a = 1, \cdots, l;\ i = 0, 1, \cdots, n).$$

Now $G_{an} = x_n^a x_n^a + \sum c_{ij} x_i^a x_j^a (i \neq j)$. Further, (24) replaces $x_i x_j (i \neq j)$ by a sum of such terms. Hence if (24) replaces σ_a by $\sigma_{a'}$ it must replace $\sum a_n^{aa} x_n^a x_n^a$, the only terms in σ_a with like subscripts to the two x's, by $\sum a_n'^{aa} x_n^a x_n^a$, the only terms in $\sigma_{a'}$ with like subscripts to the two x's. Hence, since F does not have modulus 2, we derive the necessary conditions

$$(25) \qquad \sum_{a=1}^{l} a_n^{aa} b_{a\delta}^2 = a_n'^{\delta\delta}, \qquad \sum_{a=1}^{l} a_n^{aa} b_{a\delta} b_{a\epsilon} = 0 \quad (\delta, \epsilon = 1, \cdots, l;\ \delta \neq \epsilon).$$

*Some corrections make the reading of §§ 37–39 easier. In §§ 37, 38, read $[C]$ for $[aa]$. In § 38 the use of a_{2n}^{11} instead of a_n^{11} does not conform to the earlier notations. For T' in § 39 read

$$T' = \left| x'_{2n}, \cdots, x'_{2n-2k} \qquad x'_{2n} + \lambda x'_{2k}, \cdots, x'_{2n-2k} + \lambda x'_0 \right|.$$

But these are also sufficient conditions that (24) shall replace σ_a by $\sigma_{a'}$. Indeed, if (25) hold, (24) replaces $\sum_a a_n^{aa} x_i^a x_j^a \, (i \neq j)$ by

$$\sum_{\delta,\epsilon}^{1,\ldots,l} \left(\sum_{a=1}^{l} a_n^{aa} b_{a\delta} b_{a\epsilon} \right) x_i^\delta x_j^\epsilon = \sum_{\delta=1}^{l} a_n'^{\delta\delta} x_i^\delta x_j^\delta .$$

Hence for the purposes of normalization, G_{an} of σ_a may be replaced by its leading term $x_n^a x_n^a$ and the subscript i in (24) restricted to the value n. *The problem is therefore essentially the normalization of an l-ary quadratic form within the group of all l-ary substitutions in the field F.*

20. With regard to fields having modulus 2, we restrict our attention to a finite field, say the $GF[2^k]$. Then every mark is a square. The developments of JORDAN (§§ 46–53) hold for the $GF[2^k]$ after slight changes. In § 46 we must multiply $x_n'^2 + x_n''^2$ by λ, where λ is any fixed mark such that $\lambda x^2 + \lambda y^2 + xy$ is irreducible in the $GF[2^n]$ (cf. *Linear Groups*, § 199). Hence in § 47, $G_{1n} + G_{2n}$ must be multiplied by λ; similarly in the latter sections. In § 48 the case $b^a \neq 0$ can be reduced to the case $b^a = 1$ by applying the substitution which multiplies x_i^a by $(b^a)^{-\frac{1}{2}}$ for $i = 0, 1, \cdots, m$.

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