A PROBLEM IN THE REDUCTION OF HYPERELLIPTIC INTEGRALS*

BY

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The reduction of hyperelliptic integrals to elliptic has been treated in numerous memoirs. In the more general case of the reduction of abelian (non-hyperelliptic) integrals, the properties of the table of normal periods have been investigated by WEIERSTRASS † and M. POINCARÉ,‡ while the case $p = 3$ has been treated in detail by Mme. KOWALEWSKY † by WEIERSTRASS' methods.

Professor KÖNIGSBERGER § has drawn attention to the resemblance between JACOBI's transformation theory of elliptic integrals and the reduction theory of hyperelliptic integrals, which is such that the latter appears as a generalization of the former.

A fundamental memoir by M. PICARD || contains new results and suggests new problems. It is shown that when, to the relation $y^2 = R_6(x)$, $R_6(x)$, denoting a polynomial of the sixth degree, there belongs a reducible integral, there exists also a second one, that these two form a system of normal integrals and that the table of periods corresponding to them is

$$\begin{matrix} 1 & 0 & G & \frac{1}{D} \\ 0 & 1 & \frac{1}{D} & G' \end{matrix} \qquad (D \text{ an integer})$$

M. PICARD shows also that these integrals are both reducible by transformations of degree D. The condition for the existence of reducible integrals belonging to $y^2 = R_6(x)$ is an algebraic relation among the roots of $R_6(x) = 0$.

The following problem presents itself: Given one reducible integral and its reducing transformation, to determine the second one. This problem has given

* Presented to the Society (San Francisco) February 24, 1906. Received for publication July 11, 1906.

† Mme. KOWALEVSKY, Acta Mathematica, vol. 4 (1884), p. 393.

‡ POINCARÉ, Bulletin de la Société Mathématique de France, vol. 12 (1884), p. 124.

§ KÖNIGSBERGER, Crelle, vol. 67 (1867), p. 58, p. 97.

|| PICARD, Bulletin de la Société Mathématique de France, vol. 11 (1883), p. 25.

rise to numerous researches. With the help of the theory of transformation of ϑ-functions, it has been solved when $D=4$ by Professor BOLZA.* By algebraic means, it has been solved when $D=3$ by M. GOURSAT,† and when $D=2$ by Professor HUTCHINSON.‡ The algebraic side has been studied by the writer § when $D=4$. In the case $D=3$, an elegant geometrical investigation has been given by Professor BOLZA; || while a portion of his results had been previously given by M. HUMBERT.¶

I propose in this paper to give the solution of the problem for the transformation of any order. The method which I shall use differs from any used by the writers mentioned and depends on an application of the theory of residues. Further the result will be so stated as to show its independence of linear transformation.

The reducing substitutions of reducible integrals may be studied by the methods that have been applied to the transforming substitutions of elliptic integrals. Further, there remains the problem of deriving the second reducible integral when one is given, while the hyperelliptic relation is not assumed to be in the normal form $y^2 = R_6(x)$. To these problems I hope to return.

I. Algebraic formulation of the problem.

Let $R_6(x)$ be a polynomial of the 6th degree and let $y^2 = R_6(x)$. Then

$$\int \frac{P+Qx}{y}\,dx \tag{1}$$

is a hyperelliptic integral of genus 2 and of the first kind.

Such an integral is called reducible if there exists a rational function $f(x)/F(x)$ such that when the transformation of variable $z = f(x)/F(x)$ is made, the integral (1) is transformed into

$$\int \frac{dz}{\sqrt{R_4(z)}},$$

$R_4(z)$ being a polynomial of degree 4.

In order that the function $f(x)/F(x)$ shall exist, the polynomial $R_6(x)$ must satisfy a condition. The degree of this fraction is called the degree of the transformation.

* BOLZA, Mathematische Annalen, vol. 28 (1887), p. 447.

† GOURSAT, Bulletin de la Société Mathématique de France, vol. 13 (1885), p. 143.

‡ HUTCHINSON, Chicago dissertation, 1897.

§ McDONALD, Transactions American Mathematical Society, vol. 2 (1901), p. 437.

|| BOLZA, Mathematische Annalen, vol. 50 (1898), p. 313; vol. 51 (1899), p. 478.

¶ HUMBERT, American Journal of Mathematics, vol. 16 (1894), p. 238.

Supposing (1) to be reducible, we know from the theorem of M. PICARD that there exists a second integral

$$\int \frac{P' + Q'x}{y}\,dx \qquad (PQ' - P'Q \neq 0),$$

which is reducible by a transformation $z = \phi(x)/\Phi(x)$, and that the degrees of $f(x)/F(x)$ and $\phi(x)/\Phi(x)$ are the same.

The reduction problem may be put in an algebraic form from which appears the resemblance to JACOBI's transformation problem of elliptic integrals. Let homogeneous variables be introduced by writing $x = x_1/x_2$. The differential of x is then

$$\frac{x_2\,dx_1 - x_1\,dx_2}{x_2^2} \equiv \frac{(xdx)}{x_2^2}$$

in the current notation of the theory of invariants. The integral of the first kind appears as

$$\int \frac{(x\delta)(xdx)}{\sqrt{R_6(x_1x_2)}}, \qquad R_6(x_1x_2) \equiv (x\alpha)(x\beta)(x\gamma)(x\alpha')(x\beta')(x\gamma').$$

For reducibility we must be able to find two polynomials,

$$z_1 = f_1(x_1x_2), \qquad z_2 = f_2(x_1x_2)$$

of degree D such that the transformation of x into z gives

$$\int \frac{(x\delta)(xdx)}{\sqrt{R_6(x_1x_2)}} = \int \frac{(zdz)}{\sqrt{R_4(z_1z_2)}},$$

where

$$R_4(z_1z_2) = (z\lambda)(z\mu)(z\nu)(z\sigma).$$

In this form the problem of constructing reducible integrals or of determining whether or not a given integral is reducible is purely algebraic and is a special case of the construction of a binary involution of given discriminant, an important but unsolved problem.

On carrying out the transformation we obtain results analogous to those in the elliptic integral transformation theory.* We have to make a distinction between the cases D odd and D even. Supposing that $D = 2k + 1$ and letting H_{k-1}, ϕ_k, ψ_k, χ_k be forms of degree shown by their subscripts, we must have

$$\lambda_2 f_1 - \lambda_1 f_2 = (x\alpha)(x\beta)(x\gamma)H_{k-1}^2, \qquad \mu_2 f_1 - \mu_1 f_2 = (x\alpha')\phi_k^2,$$

$$\nu_2 f_1 - \nu_1 f_2 = (x\beta')\psi_k^2, \qquad \sigma_2 f_1 - \sigma_1 f_2 = (x\gamma')\chi_k^2,$$

$$(zdz) = \vartheta(xdx), \quad / \vartheta = H\phi\psi\chi(x\delta),$$

* JACOBI, *Fundamenta Nova*, *Werke*, vol. 1, p. 57.

where ϑ denotes the Jacobian of the forms f_1, f_2. We see then that $(x\delta)$, the numerator of the reducible integral, must be a factor of the Jacobian of f_1 and f_2, that is, a double element in the involution $\lambda f_1 + \mu f_2$.

Supposing that $D = 2k$, we must have (if the transformation is indecomposible) *

$$\lambda_2 f_1 - \lambda_1 f_2 = (x\alpha)(x\alpha')\phi^2_{k-1}, \qquad \mu_2 f_1 - \mu_1 f_2 = (x\beta)(x\beta')\psi^2_{k-1},$$

$$\nu_2 f_1 - \nu_1 f_2 = (x\gamma)(x\gamma')\chi^2_{k-1}, \qquad \sigma_2 f_1 - \sigma_1 f_2 = H^2_k,$$

$$\vartheta = \phi\psi\chi H(x\delta), \qquad (zdz) = \vartheta(xdx).$$

Again $(x\delta)$ is a factor of the Jacobian of f_1, f_2.

The particular problem to be treated in this paper is the following: Given that

$$\int \frac{(x\delta)(xdx)}{\sqrt{R_6(x_1 x_2)}}$$

is reducible by the transformation

$$z_1 = f_1(x_1\, x_2), \qquad z_2 = f_2(x_1 x_2),$$

to determine a linear form $(x\delta')$, so that

$$\int \frac{(x\delta')(xdx)}{\sqrt{R_6(x_1 x_2)}}$$

shall also be reducible.

II. The normal integrals and the reducing substitution.

Let us now return to non-homogeneous variables and suppose such a linear transformation made as shall convert the two numerators into 1 and x, so that the reducible integrals are

$$u = \int_{x_0 y_0}^{xy} \frac{dx}{y} \qquad \text{and} \qquad v = \int_{x_0 y_0}^{xy} \frac{xdx}{y}.$$

Let $a_1, a_2, a_3, a_4, a_5, a_6$ be the six zeros of $R_6(x) = 0$ and suppose them so numbered and the point x_0 so chosen that the a's may be joined to x_0 by lines which do not interfere and which succeed each other in the order one to six.

Let A_i denote the integral u taken from x_0, with the initial determination y_0 of y, along the line from x_0 to a_i, and let B_i denote the integral v along the same path. Then the following represent a system of normal periods.†

* It may be shown that those decompositions of $R_6(x_1 x_2)$ into four factors, two or more of which are of zero degree, arise from a reduction of degree k combined with a quadratic transformation of the elliptic integral.

† Briot, *Fonctions Abéliennes*, p. 64.

$$2A_1-2A_2,\ 2A_2-2A_3,\ 2A_1-2A_2+2A_3-2A_4,\ 2A_4-2A_5,$$
$$2B_1-2B_2,\ 2B_2-2B_3,\ 2B_1-2B_2+2B_3-2B_4,\ 2B_4-2B_5.$$

If we suppose that u and v are a pair of reducible integrals, then by the theorem of M. PICARD we may suppose this table identical with

$$\begin{matrix} 1 & G & 0 & \frac{1}{D} \\ 0 & \frac{1}{D} & 1 & G'. \end{matrix}$$

Let $sn(u)$ be an elliptic function constructed with the periods

$$2K=\frac{1}{D},\qquad 2iK'=G;$$

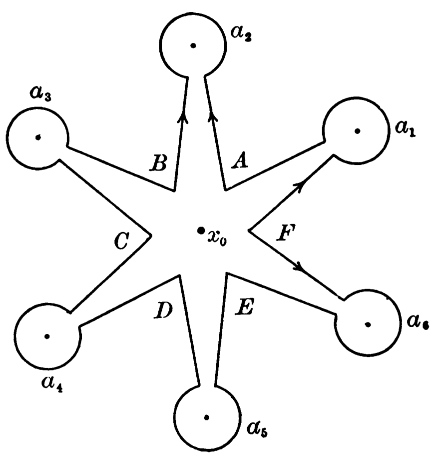
then according to M. PICARD, $sn(u)sn(2A_1-u)$ is a rational function of x of degree D:

$$sn(u)sn(2A_1-u)=\frac{f(x)}{F(x)}.$$

Expressing $sn(u)$ in terms of ϑ-functions, we have

$$sn(u)=\frac{1}{\sqrt{k}}\frac{H(u)}{\Theta(u)},$$

$H(u)$ and $\Theta(u)$ being JACOBI's ϑ-functions and k a modulus corresponding to



the periods $1/D$ and G, the ϑ-functions being supposed constructed with the same periods. The following relations hold:

$$\Theta\left(u+\frac{1}{D}\right)=\Theta(u), \qquad \Theta(u+G)=-e^{-2i\pi D\left(u+\frac{G}{2}\right)}\Theta(u),$$

$$H\left(u+\frac{1}{D}\right)=-H(u), \qquad H(u+G)=-e^{-2i\pi D\left(u+\frac{G}{2}\right)}H(u).$$

We shall now find the sums of the zeros and infinities of the function

$$sn(u)sn(2A_1-u)=\frac{f(x)}{F(x)}.$$

To this end we suppose a star cut out of the x plane in the manner before described and illustrated in the figure.

The points $ABCDEF$ are supposed to be indefinitely near x_0.

If we suppose the integration carried out from a definite one of the inner points of the star, then u is a uniform function of x in the plane as cut. We shall choose the point F as initial point in integration.

Then if period multiples be ignored, to any value of x in the uncut plane correspond two values of the integral, viz., u and $2A_1-u$, and the sum of the zeros of $f(x)/F(x)$ is given according to the formula of CAUCHY by

$$\frac{1}{2\pi i}\int xd\log H(u)H(2A_1-u),$$

and the sum of the poles by

$$\frac{1}{2\pi i}\int xd\log\Theta(u)\Theta(2A_1-u),$$

the integrations being taken around the star in such a way as to leave the point ∞ on the left, or in the clockwise sense if the roots are arranged as in the figure.

The most convenient way to carry out these integrations* is to write

$$\int xd\log H(u)H(2A_1-u)=\int x\left[d\log H(2A_1-u)+d\log H(u)\right].$$

Reversing the order of the second integration, we get

$$\int xd\log H(2A_1-u)-\int xd\log H(u),$$

the first integration being performed from F towards E and the second from F towards A.

If we let u_1 represent the value of the integral at a point on the opposite side

* PICARD, loc. cit., p. 50.

of the loop from where the value u is attained, as exhibited on loop 2, we may combine the two integrals into one:

$$\int xd \log \frac{H(2A_1 - u_1)}{H(u)} \equiv \int xd \log \frac{H(u')}{H(u)} \quad (u' = 2A_1 - u_1).$$

This integral may be considered as the sum of six others taken along the six loops. We shall call these $I_i \ (i = 1, \cdots, 6)$.

To find these we observe that the difference $u' - u$ is constant along any loop, since u' and u are integrals with the same integrand function, as may be seen from counting the number of reversals of sign that have taken place in reaching the loop.

It suffices to calculate this difference at the beginning of the loop, that is, at corresponding points such as A and F.

We may conveniently denote by $u_A, u_B, \cdots$ the values of the integral at the points $A, B, \cdots$. Accordingly,

$$u'_A = 2A_1 - u_A = 2A_1 - 2A_1 = 0, \qquad u_F = 0,$$

and the difference along the first loop is $u'_A - u_F = 0$.

On loop 2 the difference is $u'_B - u_A = 2A_2 - 2A_1$, since

$$u'_B = 2A_1 - u_B = 2A_1 - 2A_1 + 2A_2, \qquad u_A = 2A_1.$$

In this way the successive differences may be found. They are here tabulated and expressed in terms of the periods $1/D$ and G.

Loop.	$u' - u$	$u' - u$
1	0	0
2	$2A_2 - 2A_1$	-1
3	$2A_2 - 2A_3 - (2A_1 - 2A_2)$	$G - 1$
4	$2A_2 - 2A_3$	G
5	$2A_2 - 2A_3 + 2A_4 - 2A_5$	$G + \frac{1}{D}$
6	$2A_2 - 2A_3 + 2A_4 - 2A_5$	$G + \frac{1}{D}$

Referring to the relations for the addition of periods to the argument of H, we find

$$I_1 = \int xd \log 1 = 0, \qquad I_2 = \int xd \log (\pm 1) = 0,$$

$$I_3 = \int xd \log \frac{H(u + G - 1)}{H(u)} = \int xd \log \left(\pm e^{-2\pi i D\left(u + \frac{G}{2}\right)} \right)$$

$$= -2\pi i D\int x du = -2\pi i D\int dv,$$

$$I_4 = \int x d\log\frac{H(u+G)}{H(u)} = \int -2\pi i D x(-du) = 2\pi i D\int dv,$$

$$I_5 = \int x d\log\frac{H\left(u+G+\frac{1}{D}\right)}{Hu} = -2\pi i D\int dv,$$

$$I_6 = \int x d\log\frac{H\left(u+G+\frac{1}{D}\right)}{Hu} = 2\pi i D\int dv.$$

These integrals are supposed to be taken on their respective loops with the initial determination y_0 of y.

Referring to the table of periods of the integrals u and v we can express the integrals I in terms of B_i as follows:

$$I_1 = 0, \qquad I_2 = 0, \qquad I_3 = -2\pi i D\cdot 2B_3,$$

$$I_4 = 2\pi i D\cdot 2B_4, \qquad I_5 = -2\pi i D\cdot 2B_5, \qquad I_6 = 2\pi i D\cdot 2B_6.$$

Hence

$$\Sigma I_i = 2\pi i D(-2B_3 + 2B_4 - 2B_5 + 2B_6) = 2\pi i D(2B_1 - 2B_2) = 0,$$

since

$$2B_1 - 2B_2 + 2B_3 - 2B_4 + 2B_5 - 2B_6 = 0.$$

Since $\Sigma I_i = 0$, we have proved that

$$\int x d\log H(u)H(2A_1 - u) = 0$$

or the sum of the zeros of the rational function $f(z)/F(z)$ vanishes.

In essentially the same manner, we may prove that the sum of its infinities also vanishes.

If we write explicitly

$$\frac{f(x)}{F(x)} = \frac{a_0x^D + a_1x^{D-1} + \cdots + a_D}{b_0x^D + b_1x^{D-1} + \cdots + b_D},$$

then our results are simply that $a_1 = 0$, $b_1 = 0$.

We can now solve the problem which was proposed, viz., given a reducible hyperelliptic integral

$$\int\frac{P+Qx}{y}dx$$

and its reducing substitution $z = f(x)/F(x)$, to find the second integral

$$\int \frac{P' + Q'x}{y}\, dx$$

reducible by a transformation of the same degree.

Solution: Make the linear transformation

$$x = \frac{\alpha x' + \beta}{\gamma x' + \delta}$$

and so determine α, β, γ, δ that the first reducible integral shall have a constant numerator and the reducing fraction $f(x)/F(x)$ shall be transformed into one wanting its terms of degree $D - 1$. Then the second reducible integral has a numerator proportional to x'.

We proceed to define this process in a way independent of linear transformation, which will at the same time show its uniqueness and thus complete the demonstration of its validity.

III. Application of the theory of invariants.

Let $f = a_x^n$ and $\phi = b_x^n$ be two binary forms of degree n. Then if the nth transvectant of f over ϕ or $(f, \phi)_n = (ab)^n$ vanishes, the forms f and ϕ are said to be apolar.

The apolarity condition extended is

$$a_0 b_n - n a_1 b_{n-1} + \frac{n(n-1)}{2} a_2 b_{n-2} - \cdots \pm a_n b_0 = 0.$$

Let $\lambda_1 f_1 + \mu_2 f_2$ be a pencil of binary forms of degree n denoted by (Σ). Then the forms apolar to (Σ) form an $n - 2$-fold linear system $(\bar{\Sigma})$:

$$\mu_1 \phi_1 + \mu_2 \phi_2 + \cdots + \mu_{n-1} \phi_{n-1}.$$

Every form of either system is apolar to every form of the other.

In the pencil $\lambda_1 f_1 + \lambda_2 f_2$ occur $2n - 2$ double elements or linear forms $(x\delta)$ whose squares divide a form $\lambda_1 f_1 + \lambda_2 f_2$. These forms $(x\delta)$ occur $n - 1$-fold in the system $(\bar{\Sigma})$ or $(x\delta)^{n-1}$ divides a form of $(\bar{\Sigma})$, and the complementary factor or quotient is unique. Denote this by $(x\delta')$. Then $(x\delta)^{n-1}(x\delta')$ is a form of $(\bar{\Sigma})$. Further, $(x\delta')$ is distinct from $(x\delta)$ unless $(x\delta)$ is a factor in every form $\lambda_1 f_1 + \lambda_2 f_2$, a possibility which we exclude.

If we make a linear transformation

$$x_1' = (x\delta'), \qquad x_2' = (x\delta).$$

then x_2' is a double element in (Σ) and $x_1' x_2'^{n-1}$ is a form of $(\bar{\Sigma})$.

When f_1 and f_2 are written explicitly in terms x'_1, x'_2,

$$f_1 \equiv a_0 x_1'^n + a_1 x_1'^{n-1} x'_2 + \cdots, \qquad f_2 = b_0 x_1'^n + b_1 x_1'^{n-1} x'_2 + \cdots,$$

the conditions that $x'_1 x_2'^{n-1}$ is apolar to f_1 and f_2 are $a_1 = 0$, $b_1 = 0$. Conversely, if $a_1 = 0$, $b_1 = 0$, $x'_1 x_2'^{n-1}$ will be apolar to f_1 and f_2.

Hence in order that the forms of a pencil may lack the terms next the highest it is necessary that the variables represent a double element in the pencil and its complementary element in that form of the conjugate system in which the double elements occur $n-1$-fold.

Returning to the reduction problem, let us introduce homogeneous variables by putting $x = x_1/x_2$. Then the integrals and the reducing substitution become:

$$\int \frac{x_2(xdx)}{y}, \qquad \int \frac{x_1(xdx)}{y}, \qquad z = \frac{a_0 x_1^D + a_1 x_1^{D-1} x_2 + \cdots + a_D x_2^D}{b_0 x_1^D + b_1 x_1^{D-1} x_2 + \cdots + b_D x_2^D}.$$

Now we have proved that $a_1 = 0$, $b_1 = 0$.

Hence x_1 is the element conjugate to x_2^{n-1} in the system apolar to the forms

$$\lambda_1(a_0 x_1^D + \cdots + a_0 x_2^D) + \lambda_2(b_0 x_1^D \cdots + b_0 x_2^D).$$

Without using a special normal form let

$$\int \frac{(x\delta)(xdx)}{\sqrt{R_6(x_1 x_2)}}, \qquad \int \frac{(x\delta')(xdx)}{\sqrt{R_6(x_1 x_2)}},$$

$$\begin{cases} z_1 = f_1(x_1 x_2), \\ z_2 = f_2(x_1 x_2), \end{cases} \qquad \begin{cases} z_1 = f'_1(x_1 x_2), \\ z_2 = f'_2(x_1 x_2), \end{cases}$$

be two reducible integrals and the substitutions reducing them; then $(x\delta)^{D-1}(x\delta')$ is apolar to $\lambda_1 f_1 + \lambda_2 f_2$ and $(x\delta')^{D-1}(x\delta)$ is apolar to $\lambda_1 f'_1 + \lambda_2 f'_2$. We have now proved the

THEOREM. *Given that*

$$\int \frac{(x\delta)(xdx)}{\sqrt{R_6(x_1 x_2)}}$$

is reducible by a transformation $z_1 = f_1(x_1 x_2)$, $z_2 = f_2(x_1 x_2)$ of order D. Determine $(x\delta')$ so that $(x\delta)^{D-1}(x\delta')$ belongs to the system apolar to $\lambda_1 f_1 + \lambda_2 f_2$. Then

$$\int \frac{(x\delta')(xdx)}{\sqrt{R_6(x_1 x_2)}}$$

is reducible by a transformation of order D.

The problem proposed is completely solved and the result expressed in a manner independent of linear transformation.

TORONTO, *July* 6, 1906.