A THEOREM OF ABEL AND ITS APPLICATION TO THE DEVELOPMENT OF A FUNCTION IN TERMS OF BESSEL'S FUNCTIONS*

BY

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§ 1. *Introduction.*

THE object of this paper is twofold: first to deduce directly from the properties of double integrals, the formula first given by ABEL,

$$(1)\qquad \frac{2u}{\pi}\int_0^1 dt\int_0^1 \frac{f''(\lambda ut)\lambda d\lambda}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}}=f(u)-f(0);$$

and second to apply this formula to obtain sufficient conditions under which a function is developable in a series of BESSEL'S functions of multiple values of the argument.

PART I. A THEOREM OF ABEL.

§ 2. *Historical remarks.*

In a note on a mechanical problem, published in 1826,† ABEL gave the inversion formula

$$\phi(x)=\int_0^x\frac{\psi'(y)\,dy}{(x-y)^n},\qquad \text{where}\qquad \psi(y)=\frac{\sin n\pi}{\pi}\int_0^y\frac{\phi(x)\,dx}{(y-x)^{1-n}}\quad (n<1),$$

which under sufficiently strong conditions is equivalent to

$$\phi(x)-\phi(0)=\frac{2x\sin n\pi}{\pi}\int_0^1 dy\int_0^1\frac{y^{2n}\phi'(xyz)\,dz}{(1-y^2)^n(1-z^2)^{1-n}}.$$

Relation (1) results from setting $n=\frac{1}{2}$ in this formula. The case $n=\frac{1}{2}$ is the one to which the problem of mechanics considered by ABEL leads, and many authors refer this case alone to him. ABEL'S proof which was founded on properties of the Gamma function is incomplete.

* Presented to the Society December 29, 1905 and February 24, 1906. Received for publication June 8, 1906.

† Crelle's Journal, vol. 1 (1826), p. 153, and *Oeuvres Complètes*, vol. 1, p. 97.

In 1857 SCHLÖMILCH * made use of relation (1) in determining the form of a development in terms of BESSEL's functions of multiple values of the argument. Remarking that ABEL's demonstration is not rigorous, he proceeds to give a proof founded on double integrals, but as he says nothing of any conditions, and as the reader is left to suppose that the theorem is universally true, his proof must be placed in the same category with that of ABEL. SONINE † has given a proof resting on properties of BESSEL's functions, but the work is not rigorous, and he says nothing of the conditions under which it will hold. BELTRAMI ‡ discusses various forms of relation (1), and their application to SCHLÖMILCH's development, but without stating any conditions for its validity. LEVI-CIVITA § investigates a very general inversion formula which contains ABEL's as a special case. He finds that the latter is valid for a function which is representable by FOURIER's double integral, and has an integrable derivative. VOLTERRA ‖ reviews the literature of the subject, and studies more general formulas of inversion. The most recent proof of ABEL's relation seems to be that of NIELSEN.¶ His demonstration, based on properties of the Gamma function, is valid for a function $f(x)$ whose derivative $f'(x)$ can be developed in a series of functions of the form

$$f'(x) = a_1 p_1(x) + a_2 p_2(x) + a_3 p_3(x) + \cdots,$$

where the series is uniformly convergent and each $p_n(x)$ is analytic in the neighborhood of the origin.

The present proof is based on properties of multiple improper integrals as developed by Professor PIERPONT **; and aims to make rigorous the demonstration of SCHLÖMILCH already referred to. It is applicable to a more general class of functions than the proof of NIELSEN, and is much simpler than that of LEVI-CIVITA; though these latter proofs apply to ABEL's general theorem, and not merely to the case $n = \frac{1}{2}$, given in relation (1).

§ 3. *Preliminary theorem.*

THEOREM 1. *Let $f(x)$ be continuous in (a, b) except at the points of an aggregate Δ of the first species.*†† *In any subinterval of (a, b) in which $f(x)$ is not continuous, let it have limited variation. Let $f'(x)$ be integrable in (a, b), and continuous except at the points of an aggregate of the first species,*

* Zeitschrift für Mathematik und Physik, vol. 2 (1857), pp. 156, 157.

† Mathematische Annalen, vol. 16 (1880), p. 48.

‡ Rendiconti del reale Istituto Lombardo, ser. 2, vol. 13 (1880), pp. 327, 402.

§ Atti della reale Accademia delle Scienze di Torino, vol. 3 (1895), p. 25.

‖ Annali di Matematica, ser. 2, vol. 25 (1897), p. 139.

¶ *Handbuch der Theorie der Cylinderfunktionen* (1904), pp. 379–381.

** Transactions of the American Mathematical Society, vol. 7 (1906), p. 155. This paper will be referred to as *Improper Integrals.*

†† PIERPONT, *The Theory of Functions of Real Variables* (1905), § 501, 1.

where it may have finite or infinite discontinuities. Then if $f(x)$ is continuous at the end points of the interval,

$$\int_a^b f'(x)dx = f(b) - f(a) - D.$$

If $f(x)$ is discontinuous at the end points, then

$$\int_a^b f'(x)dx = f(b-0) - f(a+0) - D$$

where D is the sum of the discontinuities of $f(x)$ within the interval.

Case 1°. Suppose Δ is of order zero, i. e., $f(x)$ has only a finite number of points of discontinuity $c_1, c_2, \cdots, c_s$.

$$\int_a^b f'(x)dx = \int_a^{c_1-\epsilon} + \int_{c_1-\epsilon}^{c_1+\epsilon} + \int_{c_1+\epsilon}^{c_2-\epsilon} + \int_{c_2-\epsilon}^{c_2+\epsilon} + \cdots + \int_{c_s+\epsilon}^{b}$$

$$= \int_a^{c_1-\epsilon} + \int_{c_s+\epsilon}^{b} + \sum_{k=1}^{s} \int_{c_k-\epsilon}^{c_k+\epsilon} + \sum_{k=1}^{s-1} \int_{c_k+\epsilon}^{c_{k+1}-\epsilon}$$

But *

$$\sum_{k=1}^{s-1} \int_{c_k+\epsilon}^{c_{k+1}-\epsilon} = \sum_{k=1}^{s-1} \{f(c_{k+1}-\epsilon) - f(c_k+\epsilon)\},$$

and

$$\int_a^{c_1-\epsilon} + \int_{c_s+\epsilon}^{b} = f(b) - f(a) + f(c_1-\epsilon) - f(c_s+\epsilon).$$

Hence

$$\int_a^b f'(x)dx = f(b) - f(a) + \sum_{k=1}^{s}\{f(c_k-\epsilon) - f(c_k+\epsilon)\} + \sum_{k=1}^{s} \int_{c_k-\epsilon}^{c_k+\epsilon} f'(x)dx.$$

Since $f'(x)$ is integrable in (a, b),

$$\lim_{\epsilon=0} \sum_{k=1}^{s} \int_{c_k-\epsilon}^{c_k+\epsilon} f'(x)dx = 0.$$

Hence by passing to the limit,

$$\int_a^b f'(x)dx = f(b) - f(a) - \lim_{\epsilon=0} \sum_{k=1}^{s} \{f(c_k+\epsilon) - f(c_k-\epsilon)\}.$$

But

$$\lim_{\epsilon=0} \{f(c_k+\epsilon) - f(c_k-\epsilon)\}$$

* DE LA VALLÉE-POUSSIN, Journal de Mathématiques, ser. 4, vol. 8 (1892), p. 430. See PIERPONT, *Theory of Functions*, §§ 538, 605.

is the discontinuity of $f(x)$ at the point c_k, and

$$\lim_{\epsilon=0}\sum_{k=1}^{s}\{f(c_k+\epsilon)-f(c_k-\epsilon)\}$$

is the sum of the discontinuities of $f(x)$ in (a, b). Hence

$$\int_a^b f'(x)\,dx = f(b)-f(a)-D.$$

Case 2°. Suppose Δ is of order n. The case where Δ is of order zero having been already considered, we suppose the theorem is true for Δ of order $n-1$, and show that it must hold for Δ of order n. In this case $\Delta^{(n)}$ consists of a finite number of points $c_1, c_2, \cdots, c_s$. Then in the intervals $(c_i+\epsilon, c_{i+1}-\epsilon)$ the aggregate of points of discontinuity is of order $n-1$, and we have

$$\int_a^b f'(x)\,dx = \int_a^{c_1-\epsilon} + \int_{c_1-\epsilon}^{c_1+\epsilon} + \int_{c_1+\epsilon}^{c_2-\epsilon} + \cdots + \int_{c_s-\epsilon}^{c_s+\epsilon} + \int_{c_s+\epsilon}^{b}$$

$$= f(b)-f(a)+\sum_{k=1}^{s}\{f(c_k-\epsilon)-f(c_k+\epsilon)\}-\sum_{k=1}^{s+1}d_k+\sum_{k=1}^{s}\int_{c_k-\epsilon}^{c_k+\epsilon}f'(x)\,dx,$$

where d_k is the sum of the discontinuities of $f(x)$ in the interval $(c_{k-1}+\epsilon, c_k-\epsilon)$.* Since $f'(x)$ is integrable in (a, b),

$$\lim_{\epsilon=0}\sum_{k=1}^{s}\int_{c_k-\epsilon}^{c_k+\epsilon}f'(x)\,dx = 0.$$

Since $f(x)$ has limited variation in any interval when it is not continuous,

$$\lim_{\epsilon=0}\sum_{k=1}^{s+1}d_k$$

is finite, and we set

$$\lim_{\epsilon=0}\sum_{k=1}^{s+1}d_k = \sum_{k=1}^{s+1}D_k,$$

* That d_k has a meaning when the number of discontinuities in the interval $(c_{k-1}+\varepsilon, c_k-\varepsilon)$ infinite, is seen as follows: If the aggregate Δ of points of discontinuity is finite, or in other words if Δ is of order zero, d_k is obviously a well defined quantity. Hence we suppose that d_k is defined when Δ is of order $n-1$, and show that it is defined when Δ is of order n. If Δ is of order n, $\Delta^{(n)}$ consists of a finite number of points $p_1, p_2, \cdots, p_t$. Enclose these in little intervals $(p_i-\eta, p_i+\eta)$. In the remaining intervals Δ is of order $n-1$, hence d_k is defined. Let δ_{i1} be the absolute value of the first discontinuity falling in the interval $(p_i-\eta, p_i+\eta)$; let δ_{i2} be the sum of the absolute values of the first and second, etc. Then $\delta_{i1}, \delta_{i2}, \delta_{i3}, \cdots$ is an infinite increasing sequence, which, as f has limited variation, must always remain less than some fixed quantity. Then from *Theory of Functions*, §§ 109, 101, 102 the sequence $\delta_{i1}, \delta_{i2}, \delta_{i3} \cdots$ has a limit, which we define as the value of d_k in the interval $(p_i-\eta, p_i+\eta)$. Hence d_k is defined throughout the interval $(c_{k-1}+\varepsilon, c_k-\varepsilon)$. The same reasoning holds for $D_k = \lim_{\epsilon=0} d_k$.

where D_k is the sum of the discontinuities of $f(x)$ in the interval (c_{k-1}, c_k) excluding the end points.

Also since $f(x)$ has limited variation,

$$\lim_{\epsilon=0} \sum_{k=1}^{s} \{f(c_k - \epsilon) - f(c_k + \epsilon)\} = \sum_{k=1}^{s} \{f(c_k - 0) - f(c_k + 0)\}$$

exists and is finite.* Set

$$\sum_{k=1}^{s+1} D_k + \sum_{k=1}^{s} \{f(c_k - 0) - f(c_k + 0)\} = D.$$

Then D is the sum of the discontinuities of $f(x)$ in (a, b), and

$$\int_a^b f'(x)\, dx = f(b) - f(a) - D.$$

Suppose now $f(x)$ is discontinuous at $x = a$. Then obviously

$$\int_a^b f'(x) dx = f(b) - f(a + 0) - D.$$

Similar reasoning holds if $f(x)$ is discontinuous at $x = b$.

§ 4. *Proof of Abel's relation.*

THEOREM 2.† 1°. *Let $f(x)$ be continuous in the interval $(0, \pi)$, except at the points of an aggregate Δ of the first species. In any subinterval of $(0, \pi)$ in which $f(x)$ is not continuous let it have limited variation.*

2°. *Let $f'(x)$ be integrable in $(0, \pi)$, and continuous except at the points of an aggregate of the first species where it may have finite or infinite discontinuities.*

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* JORDAN, *Cours d'Analyse*, vol. 1, p. 55.

† Added by the author, December, 1906. In a paper presented to the American Mathematical Society in New York, December 28, 1906, Professor PIERPONT gave the following theorem: Let the infinities of $|f(x)|$ be discrete in the measurable aggregate $\mathfrak{R}$ and let

$$\int_{\mathfrak{B}} \bar{\int_{\mathfrak{C}}} |f|$$

be convergent. Then

$$\int_{\mathfrak{A}} f = \int_{\mathfrak{B}} \int_{\mathfrak{C}} f.$$

By virtue of this theorem, conditions 3°, 4° of theorem 2 may be replaced by the following: 3°. Let

$$\int_0^1 dt \int_0^1 \frac{|f'(\lambda u t)|}{\sqrt{1-t^2} \cdot \sqrt{1-\lambda^2}}\, d\lambda$$

be convergent. Either set of these conditions is sufficient for the validity of theorems 3 to 10.

3°. *Let the function*

$$\frac{f'(\lambda u t)}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}} \qquad (0 \leqq u \leqq \pi),$$

be integrable in the square S, bounded by the lines $\lambda = 0$, $\lambda = 1$, $t = 0$, $t = 1$.

4°. *Let the integral*

$$\int_0^1 \frac{f'(\lambda u t)}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}}\, d\lambda$$

be regular and integrable in $0 \leqq t \leqq 1$. Then*

$$(1)\quad \frac{2u}{\pi}\int_0^1 d\lambda \int_0^1 \frac{f'(\lambda u t)\lambda\, dt}{\sqrt{1-t^2}\ \sqrt{1-\lambda^2}} = \frac{2u}{\pi}\int_0^1 dt \int_0^1 \frac{f'(\lambda u t)\lambda\, d\lambda}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}}$$

$$= f(u-0) - f(0+0) - D(u),$$

where $D(u)$ is the sum of the discontinuities of $f(u)$ in the interval $(0, u)$.

Using conditions 3°, 4° we have †

$$\int_S \frac{f'(\lambda u t)\lambda\, dt\, d\lambda}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}} = \int_0^1 dt \int_0^1 \frac{f'(\lambda u t)\lambda\, d\lambda}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}}.$$

Since the integrand of

$$\int_0^1 \frac{f'(\lambda u t)\, dt}{\sqrt{1-t^2}\ \sqrt{1-\lambda^2}}$$

is symmetrical in λ and t, it follows from condition 4° that the integral is regular and integrable in $0 \leqq \lambda \leqq 1$. Hence

$$(2)\quad \int_S \frac{f'(\lambda u t)\lambda\, dt\, d\lambda}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}} = \int_0^1 dt \int_0^1 \frac{f'(\lambda u t)\lambda\, d\lambda}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}} = \int_0^1 d\lambda \int_0^1 \frac{f'(\lambda u t)\lambda\, dt}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}}.$$

Let us apply to the integral on the left of (2) the transformation

$$T: \qquad t = \frac{x}{\sqrt{x^2+y^2}}, \qquad \lambda = \frac{\sqrt{x^2+y^2}}{u}$$

whose determinant

$$J = \begin{vmatrix} \dfrac{y^2}{(x^2+y^2)^{3/2}} & \dfrac{-xy}{(x^2+y^2)^{3/2}} \\ \dfrac{x}{u\sqrt{x^2+y^2}} & \dfrac{y}{u\sqrt{x^2+y^2}} \end{vmatrix} = \frac{y}{u(x^2+y^2)}.$$

* PIERPONT, *Improper Integrals*, p. 167.

† PIERPONT, *Improper Integrals*, p. 168, theorem 27.

The image of the square S is the quadrant Q, bounded by the lines $x=0$, $y=0$, $x^2+y^2=u^2$. The transformation is regular* in S except at the points of a discrete aggregate D, which consists of the boundaries of S and a set of equilateral hyperbolas. The image of D is also discrete, consisting of the boundaries of Q and a set of lines parallel to the y axis. Hence† we have

$$\int_S \frac{f'(\lambda ut)\lambda dt\, d\lambda}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}} = \frac{1}{u}\int_Q \frac{f'(x)\,dx\,dy}{\sqrt{u^2-x^2-y^2}}. \tag{3}$$

By condition 2°, $f'(x)$ is integrable in the interval $(0, u)$, $0 \leqq u \leqq \pi$, and is continuous except at the points of an aggregate of the first species. Hence since the integral

$$\int_0^{\sqrt{u^2-x^2}} \frac{f'(x)\,dy}{\sqrt{u^2-x^2-y^2}} = \frac{\pi}{2} f'(x),$$

it is regular and integrable in $0 \leqq x \leqq u$. Then‡ we have

$$\int_Q \frac{f'(x)\,dx\,dy}{\sqrt{u^2-x^2-y^2}} = \int_0^u dx \int_0^{\sqrt{u^2-x^2}} \frac{f'(x)\,dy}{\sqrt{u^2-x^2-y^2}} = \frac{\pi}{2}\int_0^u f'(x)\,dx. \tag{4}$$

From theorem 1,

$$\int_0^u f'(x)\,dx = f(u-0) - f(0+0) - D(u). \tag{5}$$

Then (2), (3), (4), (5) give (1).

Corollary 1. *If $f(x)$ is continuous in $(0, \pi)$, and conditions 2°, 3°, 4° of the preceding theorem are satisfied, then*

$$\frac{2u}{\pi}\int_0^1 dt \int_0^1 \frac{f'(\lambda ut)\lambda d\lambda}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}} = \frac{2u}{\pi}\int_0^1 d\lambda \int_0^1 \frac{f'(\lambda ut)\lambda dt}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}} = f(u) - f(0).$$

Corollary 2. *If $f(x)$ is continuous in $(0, \pi)$, and $f'(x)$ is limited and becomes discontinuous only at the points of an aggregate of the first species, then*

$$\frac{2u}{\pi}\int_0^1 dt \int_0^1 \frac{f'(\lambda ut)\lambda d\lambda}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}} = \frac{2u}{\pi}\int_0^1 d\lambda \int_0^1 \frac{f'(\lambda ut)\lambda dt}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}} = f(u) - f(0).$$

For, $f'(x)$ is integrable § in $(0, \pi)$, and $f'(\lambda ut)$ is integrable § in S. Then

* Pierpont, *Theory of Functions*, § 742.

† Pierpont, *Improper Integrals*, p. 174, theorem 32.

‡ Pierpont, *Improper Integrals*, p. 168, theorem 27.

§ Pierpont, *Theory of Functions*, § 719, 2.

since $1/\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}$ is integrable in S, $f'(\lambda ut)/\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}$ is also integrable * in S. Also the integral

$$\int_0^1 \frac{f'(\lambda ut)\,d\lambda}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}}$$

is uniformly convergent † in $0 \leqq t \leqq 1$. Hence it is limited ‡ in the same interval, and continuous § except at the points of a discrete aggregate. It is therefore integrable in $0 \leqq t \leqq 1$, and the conditions of theorem 2 are satisfied.

It should be noted that in the general case of the theorem conditions 3°, 4° do not follow from 2°. For if $f'(x)$ has a single point of infinite discontinuity, $f'(\lambda ut)$ becomes infinite at all the points of an hyperbola, and hence has points of infinite discontinuity in common with $1/\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}$. Then $f'(\lambda ut)/\sqrt{1-\lambda^2}\cdot\sqrt{1-t^2}$ may not be integrable.

The conditions of theorem 2 obviously include those of Nielsen ||, since in his theorem $f'(x)$, being representable by a uniformly convergent series of continuous functions, must be a continuous function in the vicinity of the origin. His theorem, however, covers a more general set of inversion formulæ.

Part II. Development of an arbitrary function in terms of Bessel's functions.

§ 5. *Historical remarks.*

In 1857 Schlömilch ¶ in the paper already cited outlined a method for obtaining a development of the form

$$f(x) = a_0 + a_1 J_0(x) + a_2 J_0(2x) + a_3 J_0(3x) + \cdots, \tag{1}$$

where

$$\begin{aligned} a_0 &= f(0) + \frac{1}{\pi}\int_0^\pi u\,du \int_0^1 \frac{f'(uv)\,dv}{\sqrt{1-v^2}}, \\ a_n &= \frac{2}{\pi}\int_0^\pi u\cos nu\,du \int_0^1 \frac{f'(uv)\,dv}{\sqrt{1-v^2}}, \qquad (u \neq 0), \end{aligned} \tag{2}$$

and where J_0 is a Bessel's function of order zero. By termwise differentiation of this series, and by use of the relation

$$\frac{dJ_0(x)}{dx} = -J_1(x),$$

* Pierpont, *Improper Integrals*, p. 159.

† Pierpont, *Theory of Functions*, § 615.

‡ Pierpont, *Theory of Functions*, § 611, 4.

§ Jordan, loc. cit., vol. 2, § 156.

|| Loc. cit., pp. 379–381.

¶ Loc. cit., pp. 155–158.

he obtained also a development in terms of $J_1(x)$,

$$f(x) = b_1 J_1(x) + b_2 J_1(2x) + b_3 J_1(3x) + \cdots, \tag{3}$$

where

$$b_n = -\frac{2n}{\pi}\int_0^\pi u \cos nu\, du \int_0^1 \frac{f(uv)\,dv}{\sqrt{1-v^2}}. \tag{4}$$

He however specified no conditions sufficient for the proof of the validity of either development. BELTRAMI,* GEGENBAUER,† and VOLTERRA ‡ study developments of this and more general forms, considering the multipliers of the variable x in the respective terms as the roots of a transcendental equation. This throws the consideration of these developments back upon the very general problem studied by DINI,§ which he solved by use of the complex variable. NIELSEN || gives as sufficient conditions for the development of $f(x)$ in a SCHLÖMILCH series, that $f'(x)$ shall exist and be such that $f(\alpha x) + \alpha x f'(\alpha x)$ may be developed in a FOURIER series, which shall be uniformly convergent for $-\pi < x < \pi$, $-1 \leqq \alpha \leqq 1$. It follows from this that $f(\alpha x) + \alpha x f'(\alpha x)$ is a continuous function in the interval, a condition which is not imposed in the following theorems. NIELSEN's theorem applies, however, to a more general class of developments.

§ 6. *Developments in terms of $J_0(x)$.*

THEOREM 3. *Let $f(x)$, $f'(x)$ be subject to the conditions of theorem 2. Set*

$$f(0+0) + \lambda u \int_0^1 \frac{f'(\lambda u t)\,dt}{\sqrt{1-t^2}} = F(\lambda u) \quad (0 \leqq \lambda \leqq 1,\ 0 \leqq u \leqq \pi), \tag{1}$$

Then

$$f(u-0) = \frac{2}{\pi}\int_0^1 \frac{F(\lambda u)\,d\lambda}{\sqrt{1-\lambda^2}} + D(u), \tag{2}$$

where $D(u)$ is the sum of the discontinuities of $f(u)$ in $(0, u)$.

Multiplication of (1) by $2/\pi\sqrt{1-\lambda^2}$ gives

$$\frac{2f(0+0)}{\pi\sqrt{1-\lambda^2}} + \frac{2\lambda u}{\pi}\int_0^1 \frac{f'(\lambda u t)\,dt}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}} = \frac{2F(\lambda u)}{\pi\sqrt{1-\lambda^2}}.$$

By condition 4° in theorem 2 this is integrable in $0 \leqq \lambda \leqq 1$. Hence, integrating, we get

* Loc. cit., pp. 327, 402.

† Wiener Sitzungsberichte, vol. 88, II (1883), p. 975.

‡ Loc. cit., p. 139.

§ *Serie di Fourier e altre rappresentazioni analitiche delle funzioni di una variabile reale*, 1880.

|| Loc. cit., p. 348.

$$f(0+0)+\frac{2u}{\pi}\int_0^1 d\lambda\int_0^1 \frac{f'(\lambda ut)\lambda dt}{\sqrt{1-t^2}\cdot\sqrt{1-\lambda^2}}=\frac{2}{\pi}\int_0^1\frac{F(\lambda u)d\lambda}{\sqrt{1-\lambda^2}};$$

whence by theorem 2,

$$f(u-0)=\frac{2}{\pi}\int_0^1\frac{F(\lambda u)d\lambda}{\sqrt{1-\lambda^2}}+D(u).$$

THEOREM 4. 1°. *Let $f(x)$, $f'(x)$ be subject to the conditions of theorem* 2. 2°. *Let the integral*

$$\int_0^1\frac{f'(tx)}{\sqrt{1-t^2}}dt$$

represent a function having limited variation in $0\leqq x\leqq\pi$. Then SCHLÖMILCH's *development in terms of $J_0(x)$ is valid in the same interval.*

By condition 2° the function

$$F(x)=f(0+0)+x\int_0^1\frac{f'(tx)}{\sqrt{1-t^2}}dt$$

has limited variation in $(0, \pi)$, and hence can be developed in FOURIER's series,*

$$F(x)=\frac{a_0}{2}+a_1\cos x+a_2\cos 2x+a_3\cos 3x+\cdots\qquad(0\leqq x\leqq\pi),$$

where

$$a_n=\frac{2}{\pi}\int_0^\pi F(u)\cos nu\,du. \tag{3}$$

Replacing x by $x\sin\lambda$, where $0\leqq x\leqq\pi$, $0\leqq\lambda\leqq\pi/2$, we have

$$\frac{2}{\pi}F(x\sin\lambda)=\frac{a_0}{\pi}+\frac{2a_1}{\pi}\cos(x\sin\lambda)+\frac{2a_2}{\pi}\cos(2x\sin\lambda)+\cdots.$$

By ARZELÀ's condition † this series is integrable termwise in the interval $0\leqq\lambda\leqq\pi/2$ if it is "in general uniformly convergent by segments" and is determinate for every value of λ in the interval, if each term is integrable, and the sum of s terms is less than a finite number L for all s and all λ. Now F is continuous ‡ except, perhaps, for a set of points of the first species. These can be included in a finite number of intervals of total length ϵ, where ε is arbitrarily small. Since F is continuous in the remaining subintervals, the series converges uniformly § in each of these. Hence in the interval $(0, \pi)$ it is according to ARZELÀ ‖ "in general uniformly convergent by segments." Fur-

* JORDAN, loc. cit. vol., 2, § 231.

† *Sulle serie di funzioni*, Memorie dell' Accademia delle Scienze dell' Istituto di Bologna, (5), 8 (1900), p. 724–725.

‡ JORDAN, loc. cit., vol. 2, § 156.

§ JORDAN, loc. cit., vol. 2, § 231.

‖ Loc. cit., p. 712.

thermore each of the functions

$$\frac{2a_n}{\pi}\cos(nx\sin\lambda)$$

is integrable in $0 \leqq \lambda \leqq \pi/2$; the sum

$$\sum_{n=1}^{\infty}\frac{2a_n}{\pi}\cos(nx\sin\lambda)$$

is determinate for every value of λ in the interval $(0, \pi/2)$; and by known properties* of FOURIER's series

$$\left|\sum_{n=1}^{s}\frac{2a_n}{\pi}\cos(nx\sin\lambda)\right| < L,$$

where L is a finite number independent of s and of λ. Hence

$$\int_0^{\pi}\sum_{n=1}^{\infty}\frac{2a_n}{\pi}\cos(nx\sin\lambda)\,d\lambda = \sum_{n=1}^{\infty}\int_0^{\pi}\frac{2a_n}{\pi}\cos(nx\sin\lambda)\,d\lambda.$$

Then

$$\frac{2}{\pi}\int_0^{\pi/2}F(x\sin\lambda)\,d\lambda = \frac{a_0}{2} + \frac{2a_1}{\pi}\int_0^{\pi/2}\cos(x\sin\lambda)\,d\lambda + \frac{2a_2}{\pi}\int_0^{\pi/2}\cos(2x\sin\lambda)\,d\lambda + \cdots.$$

Substitution from the formula

$$J_0(nx) = \frac{2}{\pi}\int_0^{\pi/2}\cos(nx\sin\lambda)\,d\lambda$$

gives

$$\frac{2}{\pi}\int_0^{\pi/2}F(x\sin\lambda)\,d\lambda = a_0 + a_1J_0(x) + a_2J_0(2x) + a_3J_0(3x) + \cdots.$$

Setting λ in place of $\sin\lambda$, and substituting from (2), we have

$$(4)\qquad f(x-0) = D(x) + a_0 + a_1J_0(x) + a_2J_0(2x) + a_3J(3x) + \cdots.$$

From (1) and (3) we have

$$a_0 = f(0+0) + \frac{1}{\pi}\int_0^{\pi}u\,du\int_0^1\frac{f'(ut)\,dt}{\sqrt{1-t^2}},\qquad a_n = \frac{2}{\pi}\int_0^{\pi}u\cos nu\,du\int_0^1\frac{f'(ut)\,dt}{\sqrt{1-t^2}}$$
$$(n \neq 0).$$

In essentially the same manner as theorems 3 and 4 we may prove the following theorems:

* JORDAN, loc. cit., vol. 2, § 231.

THEOREM 5. *Let $\phi(x) \equiv xf(x)$ and $\phi'(x)$ be subject to the conditions of theorem* 2. *Set*

$$\int_0^1 \frac{\lambda f(\lambda x) + \lambda^2 x f'(\lambda x)}{\sqrt{1-\lambda^2}} d\lambda = F(x).$$

Then

$$f(u) = \frac{2}{\pi}\int_0^1 \frac{F(\lambda u)}{\sqrt{1-\lambda^2}} d\lambda + \frac{D(u)}{u}.$$

THEOREM 6. *Let $\phi(x) \equiv xf(x)$ and $\phi'(x)$ be subject to the conditions of theorem* 2. *Let the integral*

$$\int_0^1 \frac{\lambda^2 f'(\lambda x)\, d\lambda}{\sqrt{1-\lambda^2}}$$

represent a function having limited variation in $0 \leqq x \leqq \pi$. Then $f(x)$ may be developed in the form

$$(5)\quad f(x-0) = \frac{D(x)}{x} + \frac{\alpha_0}{2} + \alpha_1 J_0(x) + \alpha_2 J_0(2x) + \alpha_3 J_0(3x) + \cdots \quad (0 \leqq x \leqq \pi),$$

where

$$\alpha_n = \frac{2}{\pi}\int_0^\pi \cos nu\, du \int_0^1 \frac{\lambda f(\lambda u) + \lambda^2 u f'(\lambda u)}{\sqrt{1-\lambda^2}} d\lambda.$$

It will be noted that this development may be valid when $f(x)$ has an isolated infinity of order less than one at the origin, a fact which we are unable to affirm concerning the corresponding development of SCHLÖMILCH.

The question of the identity of these two forms of development (4), (5) is an interesting one. It seems probable that for any function which is regular at the origin they are the same, as is the case in the following examples.

Example 1. $\qquad f(x) = \sin x, \qquad f'(x) = \cos x.$

$$\alpha_n - a_n = \frac{2}{\pi}\int_0^\pi \cos nu du \int_0^1 \frac{\lambda \sin \lambda u + (\lambda^2 - 1) u \cos \lambda u}{\sqrt{1-\lambda^2}} d\lambda$$

$$= \frac{2}{\pi}\int_0^\pi \cos nu \left[\sum_{k=1}^\infty \frac{(-1)^k u^{2k-1}}{(2k-1)!}\int_0^1 \frac{(2k-1)\lambda^{2k-2} - 2k\lambda^{2k}}{\sqrt{1-\lambda^2}} d\lambda\right] du.$$

But

$$2k\int_0^1 \frac{\lambda^{2k} d\lambda}{\sqrt{1-\lambda^2}} = (2k-1)\int_0^1 \frac{\lambda^{2k-2} d\lambda}{\sqrt{1-\lambda^2}}.$$

Hence $\alpha_n - a_n = 0$, and the two developments are identical.

Example 2. $\qquad f(x) = x^m, \qquad f'(x) = mx^{m-1},$

$$\alpha_n - a_n = \frac{2}{\pi}\int_0^\pi u^m \cos nu\, du \int_0^1 \frac{m\lambda^{m-1} - (m+1)\lambda^{m+1}}{\sqrt{1-\lambda^2}} d\lambda.$$

But

$$(m+1)\int_0^1 \frac{\lambda^{m+1}}{\sqrt{1-\lambda^2}}\,d\lambda = m\int_0^1 \frac{\lambda^{m-1}}{\sqrt{1-\lambda^2}}\,d\lambda.$$

Hence $\alpha_n = a_n$, and the two developments are identical for any integral rational function of x.

§ 7. *Development in terms of $J_1(x)$.*

In the same manner as theorem 3, we may prove the following theorems.

THEOREM 7. *Let $f(x)$, $f'(x)$ be subject to the conditions of theorem 2. Set*

$$\int_0^1 \frac{f'(tx)}{\sqrt{1-t^2}}\,dt = F(x).$$

Then

$$f(u-0) = f(0+0) + \frac{2u}{\pi}\int_0^1 \frac{\lambda F(\lambda u)}{\sqrt{1-\lambda^2}}\,d\lambda + D(u).$$

THEOREM 8. *Let $\phi(x) \equiv xf(x)$ and $\phi'(x)$ be subject to the conditions of theorem 2. Set*

$$\int_0^1 \frac{f(tx)+txf'(tx)}{\sqrt{1-t^2}}\,dt = F(x).$$

Then

$$f(u) = \frac{2}{\pi}\int_0^1 \frac{\lambda F(\lambda u)}{\sqrt{1-\lambda^2}}\,d\lambda + \frac{D(u)}{u}.$$

THEOREM 9. *Let $f(x)$, $f'(x)$ be subject to the conditions of theorem 2. Let the integral*

$$\int_0^1 \frac{f'(tx)}{\sqrt{1-t^2}}\,dt$$

represent a function having limited variation in $0 \leqq x \leqq \pi$. Then $f(x)$ may be developed in the form

$$(6)\quad f(x+0) = f(0-0) + D(x) + x[b_1' J_1(x) + b_2' J_1(2x) + b_3' J_1(3x) + \cdots], \qquad (0 < x \leqq \pi),$$

where

$$b_n' = \frac{2}{\pi}\int_0^\pi \sin nu\,du\int_0^1 \frac{f'(ut)}{\sqrt{1-t^2}}\,dt.$$

Since

$$F(x) = \int_0^1 \frac{f'(tx)}{\sqrt{1-t^2}}\,dt$$

has limited variation in $0 \leqq x \leqq \pi$, it may be developed in Fourier's series,

$$F(x) = b_1' \sin x + b_2' \sin 2x + b_3' \sin 3x \cdots \qquad (0 < x \leqq \pi),$$

where

$$b_n' = \frac{2}{\pi} \int_0^\pi F(u) \sin nu \, du.$$

Proceeding as in theorem 4, and substituting from the formula

$$J_1(nx) = \frac{2}{\pi} \int_0^{\pi/2} \sin(nx \sin \lambda) \sin \lambda d\lambda$$

we obtain the development (6).

In a similar manner we prove the following theorem.

THEOREM 10. *Let $\phi(x) \equiv xf(x)$ and $\phi'(x)$ be subject to the conditions of theorem 2. Let the integral*

$$\int_0^1 \frac{\lambda f'(\lambda x)}{\sqrt{1-\lambda^2}} d\lambda$$

represent a function having limited variation in $0 \leqq x \leqq \pi$. Then $f(x)$ may be developed in the form

$$(7) \quad f(x-0) = \frac{D(x)}{x} + \beta_1 J_1(x) + \beta_2 J_1(2x) + \beta_3 J_1(3x) + \cdots \quad (0 < x \leqq \pi),$$

where

$$\beta_n = \frac{2}{\pi} \int_0^\pi \sin nu \, du \int_0^1 \frac{f(\lambda u) + \lambda u f'(\lambda u)}{\sqrt{1-\lambda^2}} d\lambda.$$

As in the case of $J_0(x)$, if SCHLÖMILCH's development given in § 5, (3) is valid, it is identical with (7) in simple cases. We note the following examples:

Example 1. $\qquad f(x) = \sin x, \qquad f'(x) = \cos x.$

$$\beta_n - b_n$$
$$= \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \cdot 2 \cdot 4 \cdot 6 \cdots (2k-2)}{1 \cdot 3 \cdot 5 \cdots (2k-1) \cdot (2k-1)!} \left[2k \int_0^\pi u^{2k-1} \sin nu \, du + u \int_0^\pi u^{2k} \cos nu \, du \right].$$

But

$$\int_0^\pi u^{2k} \cos nu \, du = -\frac{2k}{u} \int_0^\pi u^{2k-1} \sin nu \, du.$$

Hence $\beta_n - b_n = 0$, and the two developments are identical.

Example 2. $\qquad f(x) = x^m, \qquad f'(x) = mx^{m-1}.$

$$\beta_n - b_n = \frac{2}{\pi} \cdot \frac{2 \cdot 4 \cdot 6 \cdots (m-1)}{1 \cdot 3 \cdot 5 \cdots m} \left[(m+1) \int_0^\pi u^m \sin nu\, du + n \int_0^\pi u^{m+1} \cos nu\, du \right]$$

(*m* odd),

or

$$\beta_n - b_n = \frac{1 \cdot 3 \cdot 5 \cdots (m-1)}{2 \cdot 4 . 6 \cdots m} \left[(m+1) \int_0^\pi u^m \sin nu\, du + n \int_0^\pi u^{m+1} \cos nu\, du \right]$$

(*m* even).

But

$$\int_0^\pi u^{m+1} \cos nu\, du = -\frac{m+1}{n} \int_0^\pi u^m \sin nu\, du .$$

Hence $\beta_n - b_n = 0$, and the two developments are the same for any integral rational function of x.

NORTHFORD, CONN.,
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