ON DERIVATIVES OVER ASSEMBLAGES*

BY

E. R. HEDRICK

§ 1. *Introduction.*

The notion of upper and lower derivatives was introduced by DU BOIS-REYMOND and DINI;† they have been extensively studied by these writers, by SCHEEFFER,‡ BAIRE,§ LEBESGUE‖ and others.¶ It is the purpose of this paper to study a concept which will be called the *derivative with respect to an assemblage*, and to extend the known results concerning derivatives, etc.

A proposition which results from this study is that the existence of a *continuous* assemblage derivative insures the existence of the derivative in the usual sense (theorem 4, § 5); this has been made the center of discussion on account of its beauty and applicability. Some further steps are taken, but the evident extensions, some of which are immediate, have been reserved for possible future presentation.

§ 2. *Upper, lower, and assemblage derivatives.*

Given a function $f(x)$ defined** at the points of an interval $a \leqq x \leqq b$,†† we may extend the notions of continuity, oscillation, approach to limits, and so on, by considering only those values of x which belong to any given assemblage E. Consider in particular the ordinary difference quotient

$$Q(x, h) = \frac{f(x+h) - f(x)}{h}.$$

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† See for example DU BOIS-REYMOND, Mathematische Annalen, vol. 16 (1880), p. 115; DINI, German translation: *Grundlagen für eine Theorie der Functionen einer reellen Grösse*, etc.

‡ Acta Mathematica, vol. 5 (1884), pp. 52, 183, 279.

§ BAIRE, *Thèse, Sur les fonctions de variables réelles*, Annali di Mathematica, ser. 3, vol. 3 (1899); and *Leçons sur les fonctions discontinues*, 1905.

‖ LEBESGUE, *Sur l'intégration et la recherche des fonctions primitives*, 1904.

¶ For example, PASCH, Mathematische Annalen, vol. 30 (1887). I shall refer also to BOREL, *Leçons sur les fonctions de variables réelles*, 1905.

** In this paper it will be understood that the functions used are *single valued* at points at which they are said to be defined.

†† The function may be defined, not for all points of an interval, but for points of any assemblage which includes E.

If E is dense at x_0 both to right and left,* four numbers will be obtained by taking the greatest and least limits of $Q(x_0, h)$ as h approaches zero from the right hand or from the left hand respectively; these will be called the upper [or lower] right hand [or left hand] derived numbers,† with respect to E and will be denoted by

$$D^r_{(E)}f(x_0), \qquad D_{r(E)}f(x_0), \qquad D^l_{(E)}f(x_0), \qquad D_{l(E)}f(x_0).$$

For example, if $M[x_0, d, E, Q(x_0, h)]$ denotes the upper limit of $Q(x_0, h)$ for values of h in the interval $0 < h \leqq d$ for which $x_0 + h$ lies in E, then

$$D^r_{(E)}f(x_0) = \text{lower limit}_{d=0} \{ M[x_0, d, E, Q(x_0, h)] \}$$

If these four derived numbers are all equal, we shall call their common value *the derivative with respect to E*, and we shall denote it by $D_{(E)}f(x_0)$. If E contains no points to the left [right] of x_0, $D^l_{(E)}$ and $D_{l(E)}$ [$D^r_{(E)}$ and $D_{r(E)}$] are meaningless, in which case they are to be neglected in the preceding definition.

We may now state a fundamental theorem:

THEOREM 1. *If $f(x)$ is defined and continuous for all points of an interval $(x_0 - \epsilon, x_0 + \epsilon)$ about $x = x_0$, there exists an assemblage E_r dense at x_0 on the right, for which the derivative of $f(x)$ at x_0 with respect to E_r exists and has any preassigned value λ between $D^r f(x_0)$ and $D_r f(x_0)$, the upper and lower right-hand derived numbers with respect to the continuum, i. e.:*

$$D_{(E_r)}f(x_0) = \lambda, \qquad D^r f(x_0) \geqq \lambda \geqq D_r f(x_0);$$

[or also

$$D_{(E_l)}f(x_0) = \mu, \qquad D^l f(x_0) \geqq \mu \geqq D_l f(x_0).]$$

The truth of this statement is quite obvious from the *continuity* of $Q(x_0, h)$ since $Q(x_0, h)$ actually takes on for some $h < d$ any value between its upper and its lower limits in the interval $0 < h < d$. The cases $\lambda = D_r$ or $\lambda = D^r$ are included even when $D_r = \pm \infty$, etc.

In any case, if $f(x)$ is merely *defined*, an assemblage E_r exists for which $D_{(E_r)}f(x) = D^r f(x)$.‡ And evidently the theorem holds when $D^r f(x)$, etc. are taken with respect to any assemblage E whatever, instead of with respect to the continuum.

If the derivative of $f(x)$ exists at a point $x = x_0$ with respect to each of a *finite* number of assemblages $E_1, E_2, \cdots, E_n$, and has the same value for each

* See DINI, loc. cit., p. 244; BOREL, loc. cit., p. 28; LEBESGUE, loc. cit., p. 67, etc. These definitions are usually given only for the case in which E is the continuum.

† We shall allow the ideal values $\pm \infty$ for D^r, etc., except when specifically excluded. In most instances the word *limit*, without special mention of such possibilities as $D^r = -\infty$, will be used to avoid long circumlocutions.

‡ But not necessarily any value *between* $D^r f(x_0)$ and $D_r f(x_0)$.

of them, the derivative of $f(x)$ exists for the total assemblage ΣE formed by combining the given assemblages and has the same value; but this is *not* always true if the number of given assemblages is infinite. In fact, given a continuous function $f(x)$, all the values of x in an interval about $x=x_0$ can always be arranged into a sequence of assemblages $E_1, E_2, \cdots, E_n, \cdots$ for each of which the derivative of $f(x)$ exists and has the same value, which may be chosen at random as any number between $D^r f(x_0)$ and $D_r f(x_0)$ [or D^l and D_l]. *

On the other hand, a necessary and sufficient condition that the derivative of a function $f(x)$ defined in any interval exist with respect to the total assemblage ΣE at a point $x=x_0$, is that the derivative of $f(x)$ exist at $x=x_0$ and have the same value with respect to each of the component assemblages E (finite or infinite in number) and *that the corresponding oscillation* $\omega_{(E)}(x_0, d)$ *approach its limit zero uniformly*, i. e., $|\omega_{(E)}(x_0, d)| < \epsilon$ whenever $|d| < \delta$ where δ is independent of the assemblage E under consideration. Or again a necessary and sufficient condition that $f(x)$ have a derivative at $x=x_0$ with respect to any assemblage H is that the derivative of $f(x)$ should exist and have the same value *for every possible sequence in H.*

§ 3. *The law of the mean.*

Various generalizations of the law of the mean have been stated in terms of the four fundamental derived numbers.† All of these are special cases or corollaries of the following somewhat more general statement:

THEOREM 2. *Let $f(x)$ be defined on an assemblage H. If $f(x)$ is at a maximum with respect to H at a point $x=x_0$ of H, i. e., if $f(x_0) \geqq f(x)$ for all x in H, then*

$$D^l_{(E)} f(x_0) \geqq D_{l(E)} f(x_0) \geqq 0 \geqq D^r_{(E)} f(x_0) \geqq D_{r(E)} f(x_0)$$

where E is any subassemblage of H for which x_0 is a limiting point.

The proof is immediate since $M[x_0, d, E, Q(x_0, h)]$, for example, is not positive for sufficiently small positive values of d; hence the lower limit of M, that is $D^r_{(E)} f(x_0)$, is not positive.

This theorem, combined with theorem 1 and with the well-known theorem that any *continuous* function actually assumes its maximum on any *closed* assemblage, leads to many corollaries. Of these corollaries I shall mention only the following:

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* These and the following statements immediately result from theorems on limits for assemblages in general; see for example my paper: *On a Function, etc.*, Annals of Mathematics, 2d series, vol. 7 (1906), p. 177.

† See for example LEBESGUE, loc. cit., p. 70, etc., where a rather complete summary of the usual theorems is given; and BOREL, loc. cit., p. 28.

COROLLARY A. *If $f(x)$ is defined in an interval* $a \leqq x \leqq b$ and if there exists at each point $a \leqq x < b$ an assemblage R_x having points near x to the right and such that $D_{(R_x)}f(x)$ exists and is positive, then $f(x)$ is at a maximum, if at all, at b.*

It is understood that R_x may be chosen differently at each point x. If $f(x)$ is at a maximum at $x = k$, $a \leqq k < b$, then $D_{(R_x)}f(x) \leqq 0$ by theorem 2.

COROLLARY B. (*Rolle's theorem.*) *If $f(x)$ is defined in the interval† $a \leqq x \leqq b$ and if $f(x)$ is at a maximum or at a minimum at $x = x_0$, $(a < x_0 < b)$, then*

$$L \geqq 0 \geqq l$$

where L and l are respectively the upper and lower limits of $D_{(R_x)}f(x)$ [or of $D_{(L_x)}f(x)$] in the interval $a \leqq x \leqq b$, where $R_x\,[L_x]$ is a random assemblage at the right [left] of x, for every x for which $a \leqq x < b$ $[a < x \leqq b]$.

For if, for example, $D_{(R_x)}f(x) > 0$ (i. e., if $l > 0$), $f(x)$ is at a maximum at $x = b$, by corollary A.

A consideration of the usual function

$$\phi(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)$$

leads to an important corresponding generalization of the Law of the Mean, of which a special case is well known:‡

COROLLARY C. (*Law of the Mean.*) *If $f(x)$ is defined and continuous in the interval $a \leqq x \leqq b$, then*

$$L \geqq \frac{f(b) - f(a)}{b - a} \geqq l$$

where L and l are the upper and lower limits, respectively, of the values of $D_{(R_x)}f(x)$ in the interval, where R_x is an assemblage chosen at random as above, and differently for each value of x.

It is now clear that the upper and lower limits of any fixed set of right-hand assemblage derivatives $D_{(R_x)}f(x)$ in any interval are the same as those for any other such set, or for any left-hand set; in particular they are the same

* Instead of interval we may read *perfect assemblage*, in which case the maximum lies at a right-hand boundary point.

† Or in a perfect assemblage, if x_0 is an interior point.

‡ See LEBESGUE, loc. cit., p. 70. The theorem as stated usually applies only to the four fundamental members. The theorem here stated seems not much more general until it is noticed that R_x may be chosen *and then held fixed*, independently at each x. It should be noticed that these theorems really apply to cases in which $D_{(R_x)}f(x)$ does not exist; if it does not we can choose a new R_x, say R'_x, so that $D_{(R'_x)}f(x)$ exists and has any desired value between $D^r_{(R_x)}$ and $D_{r(R_x)}$.

as the upper and lower limits of any one of the four fundamental numbers in the same interval.* For a subinterval (a_1, b_1) surely exists for which $[f(b_1)-f(a_1)]/[b_1-a_1]$ is as near L (or l) for any preassigned set of $D_{(R_x)}f(x)$ as we please. It follows that the upper and lower limits of any fixed set of $D_{(R_x)}f(x)$ as the interval (a, b) enclosing x_0 is diminished toward zero in any manner, that is the maximum and minimum of any fixed set of $D_{(R_x)}f(x)$ at any point $x = x_0$, are always the same; *hence the oscillation of any fixed set of $D_{(R_x)}f(x)$ is the same as that of any other fixed set.*†

§ 4. *Determination of a function by given assemblage derivatives.*

A well known method ‡ may now be used to prove the following proposition:

THEOREM 3. *If $D_{(R_x)}f(x) = 0$ for some right-hand assemblage R_x at each point x of an interval $a \leqq x \leqq b$ in which $f(x)$ is defined and continuous, $f(x)$ is a constant.*

For let

$$\phi(x) = f(x) + \lambda(x-a), \qquad (\lambda > 0),$$

$$\psi(x) = f(x) - \lambda(x-a);$$

then $D_{(R_x)}\phi(x) = \lambda > 0$ and $D_{(R_x)}\psi(x) = -\lambda < 0$ for $a \leqq x < b$; hence $\phi(x)$ is at a maximum at $x = b$ and $\psi(x)$ is at a minimum at $x = b$; that is,

$$\phi(b) \geqq \phi(x) \geqq f(x) \geqq \psi(x) \geqq \psi(b) \qquad (a \leqq x < b),$$

or

$$f(b) + \lambda(b-a) \geqq f(x) \geqq f(b) - \lambda(b-a) \qquad (a \leqq x < b).$$

It follows that $f(x)$ differs from $f(b)$ by at most the arbitrarily small quantity $\lambda(b-a)$.

Since

$$D_{(E)}[f_1(x) + f_2(x)] = D_{(E)}f_1(x) + D_{(E)}f_2(x)$$

if $D_{(E)}f_1(x)$ and $D_{(E)}f_2(x)$ both exist, we may conclude that if $f_1(x)$ and $f_2(x)$ are each continuous in an interval $a \leqq x \leqq b$ and if $D_{(R_x)}f_1(x) = D_{(R_x)}f_2(x)$ where (R_x) is the same for $f_1(x)$ and $f_2(x)$, but is possibly different for any two values of x, then $f_1(x) = f_2(x) + \text{const.}$ In case the ordinary derivative of one of the functions, say $f_2(x)$, exists, we shall have $D_{(R_x)}f_2(x) = df_2(x)/dx$ for any R_x whatever; hence in this case it is sufficient to know that $D_{(R_x)}f_1(x) = df_2(x)/dx$, where R_x is wholly unrestricted.

* Compare DINI, loc. cit., p. 264, etc.; LEBESGUE, loc. cit., p. 71. The usual statements hold only for the four fundamental numbers.

† The obvious consequences of this remark will not be dwelt upon here. It may be remarked that theorem 4 results at once from this; but the total argument necessary is more complex than that which follows. Again the upper and lower integrals of any fixed set of $D_{(R_x)}f(x)$ are evidently equal to those for any other set, in particular for any of the four fundamental numbers and these various functions are integrable whenever any one set of $D_{(R_x)}f(x)$ is so.

‡ See DINI, loc. cit., p. 113.

These statements may be somewhat generalized by using a slightly different proof, following that of SCHEEFFER or that of LEBESGUE, who have stated similar theorems *with respect to the four fundamental numbers.** These methods lead to the result: if $f(x)$ is defined and continuous in an interval $a \leqq x \leqq b$ and *if* $D_{(R_x)} f(x) = 0$ *for some assemblage* R_x *which has points near* x *to the right, for all* x *in the interval except at the points of a countable*† *assemblage* E, *then* $f(x)$ *is a constant;* or again, E *need only be of measure zero if we also know that the two right-hand fundamental numbers are not both* $+\infty$ *or both* $-\infty$ *at points of* E.‡ The proofs, being similar to those given by SCHEEFFER and LEBESGUE (l. c.) are here omitted. It should be noticed that *a continuous function is determined by a knowledge of* $D_{(R_x)} f(x)$ *for all except the points of* E. A large part of LEBESGUE's work may be slightly generalized — in some cases the generalization being an essential one — by means of these remarks. One such case will be treated in the next section (p. 352).

§ 5. *Continuous assemblage derivatives.*

From what precedes we may easily deduce the result:

THEOREM 4. *If* $f(x)$ *is defined and continuous in an interval* $a \leqq x < b$, *and if some right-hand assemblage* R_x *exists at every point* x *such that* $D_{(R_x)} f(x) = \phi(x)$ *exists and is continuous, then the ordinary derivative* $df(x)/dx$ *of* $f(x)$ *exists and is equal to* $\phi(x)$; *that is, the existence of a continuous right-hand assemblage derivative insures the existence and continuity of the ordinary derivative.*§

For, if $D_{(R_x)} f(x) = \phi(x)$ exists and is continuous, then the integral

$$\psi(x) = \int_a^x \phi(x)\, dx,$$

exists, is continuous, and has a derivative in the ordinary sense which is

$$\frac{d\psi(x)}{dx} = \phi(x) \qquad (a < x < b),$$

$$\left.\frac{d\psi(x)}{dx}\right|_{(r)} = \phi(a) \qquad (x = a),$$

where (r) indicates the right-hand derivative in the ordinary sense. Hence $f(x)$ and $\psi(x)$ are two continuous functions for which

* SCHEEFFER, loc. cit., p. 282; DINI, loc. cit., p. 274. Both methods are given by LEBESGUE, loc. cit., pp. 76–78. No essential change in method is necessary.

† What is really proved is that the assemblage E may be any assemblage whose power is less than that of the continuum.

‡ This is slightly more general than to say that not both are infinite.

§ For the similar result for the four fundamental numbers, see, e. g., DINI, loc. cit., p. 267.

$$D_{(R_x)}f(x) = D_{(R_x)}\psi(x) = \frac{d\psi(x)}{dx} \qquad (a \leqq x < \xi < b).$$

It follows that

$$f(x) = \psi(x) + \text{const.} \qquad (a \leqq x < b),$$

and hence $f(x)$ has an ordinary derivative.

Several corollaries suggest themselves. One restatement deserving mention is the following: If a continuous curve $y = \phi(x)$ can be drawn which passes between $y_1 = D^r f(x)$ and $y_2 = D_r f(x)$, i. e., $y_1 \leqq y \leqq y_2$, at every point of an interval $a \leqq x \leqq b$ in which $f(x)$ is continuous, then $f(x)$ has an ordinary derivative equal to $\phi(x)$ in that interval. For theorem 1 shows that this statement is equivalent to theorem 4.

As mentioned in the footnote, p. 349, this theorem also results from the fact that the oscillation of any set of $D_{(R_x)}f(x)$ is the same at each point as the oscillation of any of the four fundamental numbers. Indeed the continuity of any set of $D_{(R_x)}f(x) = \phi(x)$ at a single point $x = x_0$ insures the existence of the ordinary derivative at that point, for the oscillation of $D_{(R_x)}f(x)$ at $x = x_0$ and therefore the oscillation of the function $Q(x_0, h)$ is zero at the point $h = 0$.*

Resuming the argument of § 3 we may write

$$L \geqq \frac{f(\beta) - f(\alpha)}{\beta - \alpha} \geqq l \qquad (a \leqq \alpha < \beta \leqq b),$$

where L and l are the upper and lower limits of $D_{(R_x)}f(x)$ for a continuous function $f(x)$ in the interval $a \leqq x \leqq b$. Therefore

$$\Sigma\,[(\beta - \alpha)L] \geqq \Sigma\,[f(\beta) - f(\alpha)] \geqq \Sigma\,[(\beta - \alpha)l]$$

where the sums indicated are extended over any finite set of subintervals in the interval $a \leqq x \leqq b$. Passing to the limit we have

$$\overline{\int_a^x} D_{(R_x)}f(x)\,dx \geqq f(x) - f(a) \geqq \underline{\int_a^x} D_{(R_x)}f(x)\,dx \quad (a \leqq x \leqq b),$$

whenever the upper and lower integrals $\overline{\int}$ and $\underline{\int}$ exist, which is surely true, for example, if $D_{(R_x)}f(x)$ is limited, i. e., if $f(x)$ is continuous and not both $D^r f(x)$ and $D_r f(x)$ are $+\infty$ or $-\infty$ at any point. If now

$$\overline{\int_a^x} D_{(R_x)}f(x)\,dx = \underline{\int_a^x} D_{(R_x)}f(x)\,dx,$$

i. e., if $D_{(R_x)}f(x)$ is integrable in the RIEMANN-CAUCHY sense, in which case any assemblage derivative is also integrable and has the same integral, the pre-

* However the derivative may still be discontinuous at $x = x_0$ or even not defined near $x = x_0$ in this case.

ceding inequalities become equations; hence we may say that *if $f(x)$ is continuous in an interval $a \leqq x \leqq b$ and if a right-hand assemblage R_x exists $(a \leqq x < b)$ for which $D_{(R_x)} f(x)$ exists and is integrable, then*

$$\int_a^x D_{(R_x)} f(x)\,dx = f(x) - f(a) \qquad (a \leqq x \leqq b);$$

*that is, the indefinite integrals of $D_{(R_x)} f(x)$ coincide with its primitive functions.**

In order that $\int D_{(R_x)} f(x)\,dx$ exist, $D_{(R_x)} f(x)$ must be continuous except at an assemblage of points of measure zero. Since the ordinary derivative of $f(x)$ exists at every point where $D_{(R_x)} f(x)$ is continuous, it follows that *if $D_{(R_x)} f(x)$ is integrable, $f(x)$ has a derivative except at points of an assemblage E of measure zero, and*

$$\int_a^x D_{(R_x)} f(x)\,dx = \int_a^x \frac{df(x)}{dx}\,dx = f(x) + \text{const.} \qquad (a \leqq x \leqq b),$$

it being understood that the second integral is taken neglecting the points of E. It is easily seen that this is the actual state of affairs whenever $D_{(R_x)} f(x)$ exists and is continuous except at points of an assemblage E whose power is less than that of the continuum; or also whenever $D_{(R_x)} f(x)$ is limited everywhere and is continuous except at points of an assemblage E of measure zero.

Finally let us consider a *fixed* assemblage R at each point x of an interval $a \leqq x \leqq b$ in which $f(x)$ is continuous. For example, let R be a *fixed* sequence, i. e., let R be the sequence

$$x + h_1,\ x + h_2,\ x + h_3,\ \cdots,\ x + h_n,\ \cdots \qquad (h_i > 0),$$

where the sequence $(h_1, h_2, h_3, \cdots, h_n, \cdots)$ is the *same* for every value of x. Then the sequence of continuous functions

$$Q(x, h_1),\ Q(x, h_2),\ \cdots,\ Q(x, h_n),\ \cdots$$

has as its limit $D_{(R)} f(x)$ whenever $D_{(R)} f(x)$ exists; and the upper and lower limits of $Q(x, h_i)$ are in any case equal to the upper and lower limits of $D^r f(x)$ or of any other assemblage derivative, in any interval. Suppose that $D_{(R)} f(x)$ exists for the *fixed* assemblage R; then by a theorem due to BAIRE † $D_{(R)} f(x)$, since it is the limit of a sequence of continuous functions, is pointwise discontinuous ‡ on every perfect assemblage. But if $D_{(R)} f(x)$ is continuous, the derivative of $f(x)$ exists in the ordinary sense.

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* This generalizes a result given by LEBESGUE, loc. cit., p. 81. See also DUBOIS-REYMOND, loc. cit., p. 115; PASCH, loc. cit., p. 153.

† See BAIRE, *Thèse*, loc. cit., p. 62 or also *Leçons sur les fonctions discontinues*, p. 98. A simpler proof is that given by LEBESGUE; see BOREL, *Leçons*, note II, p. 149.

‡ That is, the function is continuous at least once in every interval.

THEOREM 5. *If there exists any fixed sequence R (or other assemblage) for which $D_{(R)}f(x)$ exists, then that derivative is pointwise discontinuous and moreover the derivative of $f(x)$ exists in the ordinary sense at all points of an assemblage of the second category.* *

For BAIRE has pointed out that a function which is pointwise discontinuous is continuous at points of an assemblage of the second category.

It is not difficult to see that if the above sequence $Q(x, h_i)$ converges to a continuous limit, the convergence is uniform, and the converse is obviously true.† In case it does, the ordinary derivative of $f(x)$ exists, is continuous, and has the same value as $D_{(R)}f(x)$. This convergence is closely allied to what DINI called simple uniform convergence ‡ in the case of any sequence of continuous functions; but the result in this case is much more far-reaching on account of the special nature of sequences which arise from the process of attempted differentiation. For whereas the simple uniform convergence of DINI is sufficient but not necessary for the continuity of the sum of any given sequence, and whereas moreover it is necessary to assume that the given sequence converges, we may announce the result:

THEOREM 6. *A necessary and sufficient condition that the quotient $Q(x, h)$ approach a limit and that that limit be continuous in an interval $a \leqq x \leqq b$ is that $Q(x, h)$ should be simply uniformly convergent § in that interval;* i. e., *that there should exist a fixed sequence $(h_1, h_2, \cdots, h_n, \cdots)$ for which $Q(x, h_i)$ converges uniformly.*

In the preceding statements it is evident that the numbers h_i need not be constant; if they merely are continuous functions of x all the statements made still hold.

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* An assemblage is of the first category if each point of it belongs to at least one of a *countable* set of non-dense assemblages. Otherwise it is of the second category. BAIRE, *Thèse*, p. 65; *Leçons*, p. 87.

† See my paper in the Annals of Mathematics, 2d series, vol. 7 (1906).

‡ See DINI, loc. cit., p. 147. The statement is equally applicable of course to the convergence of any function $F(x, h)$ as h approaches zero. For if $F(x, h)$ approaches a limit $\phi(x)$ as h approaches zero and if a sequence of values of h can be found for which $F(x, h_i)$ converges uniformly, then $\phi(x)$ is evidently continuous. I shall say that $F(x, h)$ has simple uniform convergence if h_i exist for which $F(x, h_i)$ converges uniformly.

§ This does not here *postulate* that $Q(x, h)$ *itself* approaches any limit whatever.