EXISTENCE PROOF FOR A FIELD OF EXTREMALS TANGENT TO A GIVEN CURVE*

BY

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In a recent paper,† Professor BLISS has given sufficient conditions for a minimum of the integral

$$(1) \qquad J = \int_{t_0}^{t_1} F(x, y, x', y')\, dt$$

with respect to one-sided variations. His proof is based upon the construction of a *field of extremals tangent to a given curve.* He establishes the existence of such a field first for the special case where all curves considered are representable in the form $y = f(x)$, and then reduces the general case of parameter representation to the former by a point-transformation of the plane.

The object of the following note is to give a *direct proof for the existence of these fields* which play an important part also in other investigations of the calculus of variations. ‡

§ 1. *The set of extremals tangent to a given curve.*

The terminology and assumptions concerning the function F being the same as in § 24 of my *Lectures on the Calculus of Variations*, we consider a curve *of class* C''

$$\tilde{\mathfrak{C}}: \qquad x = \tilde{x}(a), \qquad y = \tilde{y}(a), \qquad A_1 \leqq a \leqq A_2,$$

without multiple points, which lies *in the interior of the region of the* x, y-*plane in which the function* F *is supposed to be of class* C''' for every $(x', y') \neq (0, 0)$, and *satisfies the inequality*

* Presented to the Society March 30, 1907. Received for publication February 27, 1907.

† Transactions of the American Mathematical Society, vol. 5 (1904), p. 477.

‡ Compare LINDEBERG, Mathematische Annalen, vol. 59 (1904), p. 321.

$$F_1[\tilde{x}(a),\ \tilde{y}(a),\ \tilde{x}'(a),\ \tilde{y}'(a)] > 0 \text{ in } (A_1 A_2), \tag{2}$$

where $\tilde{x}' = d\tilde{x}/da$, $\tilde{y}' = d\tilde{y}/da$.

For simplicity, we suppose that the parameter a is the arc of the curve $\tilde{\mathfrak{C}}$ measured from some fixed initial point.

Under these conditions it follows from the general existence theorems* for differential equations applied to the differential equation of the extremals† for the integral (1), that through every point $P(a)$ of the curve $\tilde{\mathfrak{C}}$ one and but one extremal $\mathfrak{E}_a$ can be drawn which is tangent to $\tilde{\mathfrak{C}}$ at P in such a manner that the positive tangents of the two curves coincide. For the parameter t on the extremal $\mathfrak{E}_a$ we may choose the arc of the extremal measured from the point P so that for every value of a the point P corresponds on $\mathfrak{E}_a$ to the value $t = 0$.

If we vary a, we thus obtain a set of extremals

$$x = \phi(t, a), \qquad y = \psi(t, a), \tag{3}$$

for which the functions ϕ, ψ have the following properties:

1) The functions

$$\phi,\ \phi_t,\ \phi_{tt};\ \psi,\ \psi_t,\ \psi_{tt}$$

are as functions of t and a of class C' in the domain

$$0 \leqq t \leqq l, \qquad A_1 \leqq a \leqq A_2, \tag{4}$$

where l is a sufficiently small positive quantity *independent of* a.‡

2) The functions ϕ, ψ satisfy the following initial conditions:

$$\begin{aligned} \phi(0, a) &= \tilde{x}(a), & \psi(0, a) &= \tilde{y}(a), \\ \phi_t(0, a) &= \tilde{x}'(a), & \psi_t(0, a) &= \tilde{y}'(a). \end{aligned} \tag{5}$$

From (5) we obtain by differentiation

$$\begin{aligned} \phi_a(0, a) &= \tilde{x}'(a), & \psi_a(0, a) &= \tilde{y}'(a), \\ \phi_{ta}(0, a) &= \tilde{x}''(a), & \psi_{ta}(0, a) &= \tilde{y}''(a). \end{aligned} \tag{6}$$

From these equations we derive for the Jacobian

$$\Delta(t, a) = \frac{\partial(\phi, \psi)}{\partial(t, a)}$$

* Compare BLISS, Annals of Mathematics, ser. 2, vol. 6 (1905), pp. 49–67.

† Compare KNESER, *Lehrbuch der Variationsrechnung*, §§ 27, 29 and BOLZA, *Lectures on the Calculus of Variations*, § 25, *b*).

‡ Compare the corollary given by BLISS in the article on differential equations just referred to, p. 53, at the end of section 1.

the result:

$$\Delta(0, a) = 0, \qquad \Delta_t(0, a) = \frac{1}{r} - \frac{1}{\tilde{r}}, \tag{7}$$

if we denote by $1/r$ and $1/\tilde{r}$ the curvature at the point P of the curve $\mathfrak{C}_a$ and of the curve $\tilde{\mathfrak{C}}$ respectively.

We make the *further assumption* that

$$\frac{1}{r} - \frac{1}{\tilde{r}} \neq 0$$

along $\tilde{\mathfrak{C}}$, and in order to fix the ideas we suppose * that

$$\frac{1}{r} - \frac{1}{\tilde{r}} > 0. \tag{8}$$

From this additional assumption it follows that two positive quantities $l_0 \leqq l$ and m can be determined so that

$$\Delta(t, a) \geqq tm \tag{9}$$

in the domain

$$0 \leqq t \leqq l_0, \qquad A_1 \leqq a \leqq A_2. \tag{10}$$

For if we define the function $\chi(t, a)$ for the domain (4) by the equations

$$\chi(t, a) = \begin{cases} \dfrac{\Delta(t, a)}{t}, & \text{when } t \neq 0, \\ \Delta_t(0, a), & \text{when } t = 0, \end{cases}$$

it is easily seen that $\chi(t, a)$ is continuous in the domain (4), and since moreover $\chi(0, a) > 0$ in $(A_1 A_2)$, it follows that a positive quantity $l_0 \leqq l$ can be assigned such that $\chi(t, a) > 0$ in the domain (10). If we denote by m the minimum of $\chi(t, a)$ in the domain (10), we obtain (9).

§ 2. *Proof that the set of extremals* (3) *furnishes a field.*

We now choose two quantities a_1, a_2 so that

$$A_1 < a_1 < a_2 < A_2$$

and propose to prove that under the assumptions enumerated in § 1, *a positive quantity* $k \leqq l_0$ *can be assigned such that the equations* (3) *define a one-to-one correspondence between the rectangle*

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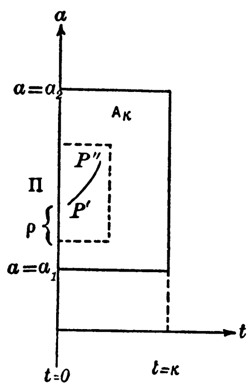
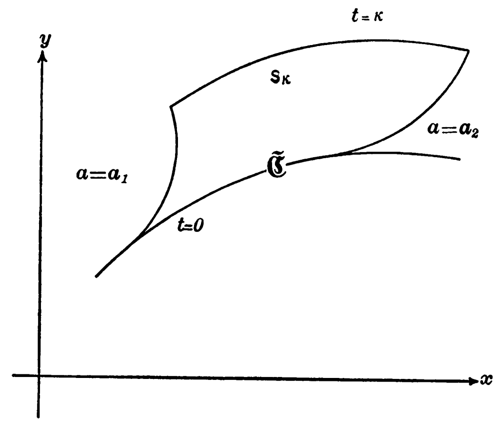
* In order that the curve $\tilde{\mathfrak{C}}$ may furnish a minimum for the integral (1) with respect to one-sided variations on the left of $\tilde{\mathfrak{C}}$, it is necessary that $1/r - 1/\tilde{r} \geqq 0$; compare BOLZA, *Lectures*, p. 194.

$$\mathbf{A}_\kappa: \qquad 0 \leqq t \leqq \kappa, \qquad a_1 \leqq a \leqq a_2$$

in the t, a-plane and its image $\mathbf{S}_\kappa$ in the x, y-plane.

We suppose that it were not so; that is we suppose that, however small κ may be taken, there always exists in $\mathbf{A}_\kappa$ at least one pair of distinct points whose images in the x, y-plane coincide. Reasoning then exactly as in the proof for the exist-





ence of a field which I have given in § 34 of my *Lectures* for the case where $\Delta(t, a) \neq 0$, we reach the result that under this hypothesis there would exist a point $\Pi\,(t = 0,\, a = \alpha)$ in the rectangle $\mathbf{A}_\kappa$ such that every vicinity of Π contains at least one pair of distinct points of $\mathbf{A}_\kappa$ whose images in the x, y-plane coincide.

We are going to prove that this leads to a contradiction with the inequality (9).

For this purpose we notice that our assumptions concerning the curve $\tilde{\mathfrak{C}}$ imply * that $\tilde{x}'(\alpha)$, $\tilde{y}'(\alpha)$ are not both zero; let $\tilde{x}'(\alpha) \neq 0$, or as we may write on account of (6),

$$\phi_a(0, \alpha) \neq 0.$$

We may then apply Dini's theorem on implicit functions to the function $\phi(t, a)$ and the point $t = 0$, $a = \alpha$. From this theorem it follows † that below any

* Compare the definition of "curve of class C''" on p. 116 of my *Lectures*.

† Choose $d > 0$ so that $\phi(t, a)$ is of class C' and $\phi_a(t, a) \neq 0$ for $|t| \leqq d$, $|a - \alpha| \leqq d$. Let A be the maximum of $|\phi_t(t, a)|$, B the minimum of $|\phi_a(t, a)|$ in this domain. Choose $0 < d_0 < d$ and

$$\sigma < d_1 - d_0, \quad \rho < d_0,\ \frac{\sigma}{2},\ \frac{B\sigma}{2A}.$$

Compare Peano-Genocchi, *Differentialrechnung und Grundzüge der Integralrechnung*, pp. 138–141.

preassigned positive quantity δ two positive quantities ρ and σ can be determined having the following properties: If $P'(t', a')$ and $P''(t'', a'')$ be any two distinct points of the vicinity (ρ) of the point $\Pi(0, \alpha)$ for which

$$\phi(t', a') = \phi(t'', a''), \tag{11}$$

then in the first place $t'' \neq t'$ (say $t' < t''$) and in the second place the two points P', P'' can be joined by a curve representable in the form

$$a = \mathfrak{a}(t), \qquad t' \leqq t \leqq t'',$$

such that

$$\phi[t, \mathfrak{a}(t)] = \phi(t', a') \text{ for } t' \leqq t \leqq t''. \tag{12}$$

The function $\mathfrak{a}(t)$ is of C', and satisfies the inequality

$$|\mathfrak{a}(t) - a'| < \sigma \text{ for } t' \leqq t \leqq t''$$

and the initial conditions

$$\mathfrak{a}(t') = a', \qquad \mathfrak{a}(t'') = a''. \tag{13}$$

Differentiating (12) we obtain

$$\phi_t[t, \mathfrak{a}(t)] + \phi_a[t, \mathfrak{a}(t)]\, \mathfrak{a}'(t) = 0. \tag{14}$$

On the other hand, it follows from the characteristic property of the point Π that there exists at least one pair of distinct points P', P'' in the domain

$$0 \leqq t < \rho, \qquad |a - \alpha| < \rho$$

for which not only (11) holds but at the same time

$$\psi(t', a') = \psi(t'', a''). \tag{15}$$

For such a pair of points the function $\psi[t, \mathfrak{a}(t)]$ is of class C' in $(t't'')$ and takes, according to (13) and (15), the same value for $t = t'$ and $t = t''$. Hence its derivative must vanish at least for one value $t = \tau$ between t' and t'':

$$\psi_t(\tau, \mathfrak{a}(\tau)] + \psi_a[\tau, \mathfrak{a}(\tau)]\, \mathfrak{a}'(\tau) = 0.$$

Combining this equation with the equation derived from (14) by putting $t = \tau$, we obtain the result:

$$\Delta[\tau, \mathfrak{a}(\tau)] = 0.$$

But if we take ρ and σ sufficiently small, the point $t = \tau$, $a = \mathfrak{a}(\tau)$ lies in the domain (10); moreover, τ is positive since $0 \leqq t' < \tau < t''$. Hence we have indeed reached a contradiction with the inequality (9), and therefore the statement enunciated at the beginning of this section is proved.

The image $\mathfrak{L}$ of the boundary of the rectangle $\mathbf{A}_\kappa$ is a continuous closed curve without multiple points. According to a theorem due to SCHÖNFLIES,* the point-set $\mathbf{S}_\kappa$ is therefore identical with the interior of $\mathfrak{L}$ together with the curve $\mathfrak{L}$ itself.

THE UNIVERSITY OF CHICAGO,
February 27, 1907.

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*Göttinger Nachrichten, 1899, p. 282; compare also OSGOOD, ibid., 1900, p. 94; and BERNSTEIN, ibid., 1900, p. 98.