THE EXPANSION OF A FUNCTION IN TERMS OF

NORMAL FUNCTIONS.*

BY

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In an article published in these TRANSACTIONS † the author has proved the existence of normal functions for the differential equation

$$(1) \qquad y'' + p(x)y' + [\lambda A(x) - B(x)]y = 0$$

and the boundary conditions

$$(2) \qquad \begin{cases} a_1 y(a) + a_2 y(b) + a_3 y'(a) + a_4 y'(b) = 0, \\ b_1 y(a) + b_2 y(b) + b_3 y'(a) + b_4 y'(b) = 0, \end{cases}$$

by a method which sets in evidence the properties of the normal functions as solutions of minimum problems. It was shown that this point of view is useful in deriving oscillation theorems, i. e., in investigating the zeros of the functions.

It is the object of this paper to show that the minimal properties of the normal functions may be used to prove in a simple manner the possibility of expanding a function in terms of normal functions. ‡

The coefficients p, A, B are supposed to be continuous in the interval (a, b). In addition the following assumptions are made, the determinants $a_i b_k - a_k b_i$ of the coefficients of the equations (2) being denoted by d_{ik}:

(α) $$B(x) \geqq 0 \qquad\qquad (a \leqq x \leqq b),$$

(β) $$d_{24} = d_{13} e^{\int_a^b p\,dx},$$

(γ)   *No two of the quantities d_{12}, d_{14}, d_{23}, d_{43} have opposite signs.*

Equation (β) insures that any two functions u, v which satisfy the boundary conditions (2) also satisfy the equation *

(3) $$[P(uv' - vu')]_a^b = 0 \qquad \left(P = e^{\int_a^x p\,dx}\right)$$

On account of (γ) the quantity

$$- [Pyy']_a^b$$

is positive or zero for any function y which satisfies (2).   Under these assumptions the following theorem holds : †

   There exists an infinite series of normal parameter values λ_n and corresponding solutions (normal functions) y_n of the differential equation (1) and the boundary conditions (2). If A changes sign in (a, b), the values λ_n include an infinite series of positive terms, $\lambda_1 \leqq \lambda_2 \leqq \lambda_3 \leqq \lambda_4 \leqq \cdots$, increasing without limit, and an infinite series of negative terms, $\lambda_{-1} \geqq \lambda_{-2} \geqq \lambda_{-3} \geqq \lambda_{-4} \geqq \cdots$, decreasing without limit.‡   The function y_n satisfies the conditions §

$$\int_a^b PAy^2\,dx = \pm 1,$$

$$\int_a^b PAy_i y\,dx = 0 \qquad [i = \pm 1, \pm 2, \cdots, \pm (n-1)],$$

and gives to the expression

$$J(y) = \int_a^b (y'^2 + By^2)P\,dx - [yy'P]_a^b$$

its least possible value consistent with these conditions and with equations (2). This minimum value of J is $\pm \lambda_n$.
   On the basis of these properties the following theorem will be proved :

   A function $f(x)$ which satisfies the boundary conditions (2), is continuous within (a, b), and has a derivative which is continuous except at a finite number of points, may be expanded in a uniformly and absolutely converging series of the form

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   * See page 349 of the article cited.
   † Ibid., page 356.
   ‡ If A does not change sign, there are no normal parameter values of sign opposite to that of A.   In case $B \equiv 0$, λ_1 or λ_{-1} may be zero.
   § The upper or lower signs are to be taken according as λ_n is positive or negative.

$$c_1 y_1 + c_2 y_2 + c_3 y_3 + \cdots$$
$$+ c_{-1} y_{-1} + c_{-2} y_{-2} + c_{-3} y_{-3} + \cdots,$$

where

$$c_{\pm n} = \pm \int_a^b fAy_{\pm n}\, dx.$$

§ 1. *A property of the constants* c_i.

By consideration of the expression J it will be proved that the series

$$\sum_{i=1}^{\infty} c_i^2 \lambda_i, \qquad \sum_{i=-1}^{-\infty} c_i^2 \lambda_i$$

are convergent.

If u and v are two functions satisfying the boundary conditions, then

$$J(u + v) = J(u) + J(v) + 2J_1(u, v),$$

where

$$J_1(u, v) = \int_a^b (u'v' + Buv) P\, dx - [uv'P]_a^b.$$

Then, since $J(y_i) = \lambda_i$ for positive subscripts,

$$(4) \quad J\left(f - \sum_{i=1}^{n} c_i y_i \right) = J(f) + \sum_{i=1}^{n} c_i^2 \lambda_i - 2 \sum_{i=1}^{n} c_i J_1(f, y_i) + 2 \sum_{i,\, k=1}^{n} c_i c_k J_1(y_i, y_k).$$

Now the function y_i satisfies the differential equation (1) for $\lambda = \lambda_i$, or on multiplying (1) by $P = e^{\int_a^x p\, dx}$,

$$(Py_i')' + (\lambda_i A - B) Py_i = 0.$$

Therefore,

$$J_1(f, y_i) = \int_a^b (f'y_i' + Bfy_i) P\, dx - [fy_i' P]_a^b$$

$$= \int_a^b f[BPy_i - (Py_i')']\, dx = \lambda_i \int_a^b PAfy_i\, dx = c_i \lambda_i.$$

Similarly,

$$J_1(y_i, y_k) = \lambda_i \int_a^b PAy_i y_k\, dx = 0.$$

Since the left member of equation (4) is positive or zero, it follows that whatever be n,

$$J(f) \geqq \sum_{i=1}^{n} c_i^2 \lambda_i.$$

But the terms of the series are positive. Therefore the series

$$\sum_{i=1}^{\infty} c_i^2 \lambda_i$$

is convergent.   The convergence of the series with negative subscripts is proved in the same way by considering $J(f - \sum_{i=-1}^{-n} c_i y_i)$.

§ 2. *Convergence of the series.*

Consider the terms with positive subscripts, and write

$$\phi_{n, m} = \sum_{i=n}^{n+m} |c_i y_i|.$$

If n be taken sufficiently large, each function y_i will have at least one zero α_i in the interval (a, b),* and

$$\phi_{n, m} = \sum_{i=n}^{n+m} \left| c_i \int_{\alpha_i}^{x} y_i' \, dx \right|.$$

Now $r^2 + s^2 \geqq 2rs$ for any two numbers r, s, and therefore

$$\phi_{n, m}^2 \leqq 2 \sum_{i=n}^{n+m} c_i^2 \left(\int_{\alpha_i}^{x} y_i' \, dx \right)^2.$$

Furthermore, for all values of the parameter μ, the expression

$$\int_{\alpha_i}^{x} (\mu + y_i')^2 \, dx = \mu^2 (x - \alpha_i) + 2\mu \int_{\alpha_i}^{x} y_i' \, dx + \int_{\alpha_i}^{x} y_i'^2 \, dx$$

has the sign of $x - \alpha_i$. The discriminant of this form is then not negative, that is,

$$\left(\int_{\alpha_i}^{x} y_i' \, dx \right)^2 \leqq (x - \alpha_i) \int_{\alpha_i}^{x} y_i'^2 \, dx \leqq (b - a) \int_{a}^{b} y_i'^2 \, dx.$$

But

$$\int_{a}^{b} y_i'^2 \, dx \leqq J(y_i) = \lambda_i,$$

and therefore

$$\phi_{n, m}^2 \leqq 2 \sum_{i=n}^{n+m} c_i^2 \lambda_i.$$

Since the series

$$\sum_{i=1}^{\infty} c_i^2 \lambda_i$$

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* This follows at once from one of the simplest of STURM's well-known results. It may be proved for the case here considered as follows : Let u be a function having α and β for consecutive zeros in an interval where $A > 0$, and satisfying (1) for $\lambda = \bar{\lambda}$ inside (α, β). The existence of such a function is assured by the theorem stated in the introduction. From the differential equations for u and y_i it follows that

$$[y_i u' P]_\alpha^\beta = (\lambda_i - \bar{\lambda}) \int_\alpha^\beta P A u y_i \, dx.$$

Determine the sign of u to be positive in (α, β). Then $u'(\beta) < 0$, $u'(\alpha) > 0$, since u cannot vanish together with its derivative. Now $A > 0$ in (α, β), and therefore it follows from the above equation that y_i must change sign in (α, β) if $\lambda_i > \bar{\lambda}$.

is convergent, it follows that the series

$$c_1 y_1 + c_2 y_2 + c_3 y_3 + \cdots$$

converges uniformly and absolutely. The convergence of the series of terms with negative subscripts is proved in the same way.

§ 3. *The series represents the function.*

Since the series converges uniformly, the difference

$$f - \sum_{i=-\infty}^{\infty} c_i y_i = g$$

is a continuous function. On multiplying this equation by PAy_i and integrating from a to b, the equation

$$(5) \qquad \int_a^b PAy_i g\,dx = 0$$

results, whatever be i. This equation is sufficient to insure that g is identically zero. It will be shown that on assumption of the contrary a function y may be formed, which has a continuous derivative and satisfies the boundary conditions (2) and the equations

$$(6) \qquad \int_a^b PAy_i y\,dx = 0 \qquad (i = \pm 1, \pm 2, \pm 3, \cdots),$$

$$(7) \qquad \int_a^b PAy^2\,dx = \pm 1.$$

Suppose such a function exists. The value of the expression

$$J = \int_a^b P(y'^2 + By^2)\,dx - [yy'P]_a^b$$

is less than λ_n and $-\lambda_{-n}$ if n be taken sufficiently large, since these values increase without limit with n. Now y satisfies all the conditions of the minimum problem of which either y_n or else y_{-n} is the solution, according as (7) holds with upper or lower sign. But it gives to J a value less than the smallest value λ_n or $-\lambda_{-n}$ consistent with the conditions. Such a function y can therefore not exist.

It remains to show that if the difference g is not identically zero, a function y satisfying the boundary conditions (2) and the equations (6) and (7) can be formed. Let $\bar\lambda$ be a value which is not a normal parameter value. Then there exists a solution of the equation

$$(Pu')' + (\bar\lambda A - B)Pu = PAg$$

under the boundary conditions (2). From this equation and

$$(Py_i')' + (\lambda_i A - B)Py_i = 0$$

it follows, on account of (5), that

$$\int_a^b \{y_i(Pu')' - u(Py_i')'\}\, dx = (\lambda_i - \bar{\lambda}) \int_a^b PAy_i u\, dx.$$

The left member vanishes after an integration by parts, on account of equation (3). Therefore, since $\bar{\lambda} \neq \lambda_i$, the equations (6) are satisfied by the function u. On multiplying u by a constant the equation (7) may be satisfied also, unless

$$\int_a^b PAu^2\, dx = 0.$$

Suppose, however, that this equation holds. Solve the equation

$$(Pv')' + (\bar{\lambda} A - B)Pv = PAu$$

under the boundary conditions (2). This function satisfies (6), the proof being the same as before, except that g is replaced by u. Hence the desired function y may be formed by multiplying v by a constant so that (7) will hold, unless

$$\int_a^b PAv^2\, dx = 0.$$

Suppose again that this is the case. Consider then the function

$$w = u - v,$$

which satisfies (2) and (6). Now

$$\int_a^b PAw^2\, dx = -2\int_a^b PAuv\, dx = -2\int_a^b \{v(Pv')' - BPv^2\}\, dx = 2J(v) > 0.$$

The required function y may, therefore, be formed in any case, unless g is identically zero. It follows that the series represents the function.

At a later time I hope to extend this method to more general cases.

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