GROUPS DEFINED BY THE ORDERS OF TWO GENERATORS AND THE ORDER OF THEIR COMMUTATOR*

BY

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§ 1. *Introduction.*

The commutator of two operators (s_1, s_2) may be represented by $s_1^{-1} s_2^{-1} s_1 s_2$. If the four elements of this commutator are permuted in every possible manner there result 24 operators. Sixteen of these are equal to the identity irrespective of the choice of s_1, s_2, while the other eight are commutators and may all be distinct.† These eight have the following forms:

$$s_1^{-1} s_2^{-1} s_1 s_2, \quad s_2^{-1} s_1 s_2 s_1^{-1}, \quad s_2 s_1^{-1} s_2^{-1} s_1, \quad s_1 s_2 s_1^{-1} s_2^{-1};$$

$$s_2^{-1} s_1^{-1} s_2 s_1, \quad s_1 s_2^{-1} s_1^{-1} s_2, \quad s_1^{-1} s_2 s_1 s_2^{-1}, \quad s_2 s_1 s_2^{-1} s_1^{-1}.$$

The last four are the inverses of the first four, and these are the transforms of the first with respect to 1, s_1^{-1}, s_2^{-1}, $s_2^{-1} s_1^{-1}$, respectively. In particular, *the eight commutators which may be obtained from a single commutator by permuting its elements are of the same order, and if this order is two not more than four of them are distinct.* While the order of a commutator is independent of the particular form by which it is represented, we shall always use the first one of these forms to represent the commutator of s_1 and s_2. Its inverse represents the commutator of s_2 and s_1.

It is sometimes convenient to employ the order of the commutator in the abstract definition of a group. The simplest instance with respect to non-abelian groups is exhibited by the following theorem: *If any two operators of order two have a commutator of even order they generate a group whose order is four times the order of their commutator.* This theorem follows directly from the facts that two operators of order 2 generate a dihedral group and that the commutator of two operators of order 2 is the square of their product. From these facts it follows also that two operators of order 2 whose commutator is of odd order generate a group whose order is either twice or four times the order of their commutator.

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† Bulletin of the American Mathematical Society, vol. 5 (1899), p. 239.

The next case, in order of simplicity, is when one of the generating operators (s_1) is of order 2, the other (s_2) of order 3 and their commutator of order 2. These conditions may be expressed as follows: $s_1^2 = s_2^3 = (s_1 s_2^2 s_1 s_2)^2 = 1$. Since s_1 and $s_2^2 s_1 s_2$ are of order 2 and their product is also of this order, they are commutative and $1, s_1, s_2^2 s_1 s_2, s_1 s_2^2 s_1 s_2$ is the four-group. The other transform of s_1 with respect to the powers of s_2, viz., $s_2 s_1 s_2^2$, is commutative with each operator of this four-group since

$$s_1 s_2 s_1 s_2^2 = s_2 s_1 s_2^2 s_1\text{*} \text{ and } s_2 s_1 s_2^2 \cdot s_2^2 s_1 s_2 = s_2 s_1 s_2 s_1 s_2 = s_2^2 s_1 s_2^2 s_1 s_2^2 = s_2^2 s_1 s_2 \cdot s_2 s_1 s_2^2.$$

From this it follows that $s_1, s_2^2 s_1 s_2, s_2 s_1 s_2^2$ generate either the abelian group of order 8 and of type $(1, 1, 1)$ or a subgroup of order 4. As this group contains s_1 and is transformed into itself by s_2 it is invariant under the group $\{s_1, s_2\}$ generated by s_1 and s_2. If the order of this group is 8 one of the seven subgroups of order 4 which are contained in it is transformed into itself by s_2, and s_2 could not be commutative with the four operators of this subgroup,† hence it follows that $\{s_1, s_2\}$ contains the tetrahedral group. If it contains any additional operators it must include invariant operators, and, as its order cannot exceed 24, it must be the direct product of the tetrahedral group and the group of order 2. Hence the theorem: *If an operator of order 2 and an operator of order 3 have a commutator of order 2 they generate either the tetrahedral group or the direct product of this group and the group of order 2.*

It has now been proved that if s_1, s_2 satisfy the following conditions:

$$s_1^2 = 1, \qquad s_2^n = 1, \qquad (s_1 s_2^{-1} s_1 s_2)^2 = 1,$$

the group $\{s_1, s_2\}$ is entirely determined by the value of n when $n = 2$; and when $n = 3$, $\{s_1, s_2\}$ may be one of two groups. It will now be proved that $\{s_1, s_2\}$ may be one of an infinite system of groups whenever $n > 3$. For instance, when $n = 4$, we may let

$$s_1 = a_3 b_1 \cdot b_3 c_1 \cdot c_3 d_1 \cdots,$$

$$s_2 = a_1 a_2 a_3 a_4 \cdot b_1 b_2 b_3 b_4 \cdot c_1 c_2 c_3 c_4 \cdot d_1 d_2 d_3 d_4 \cdots,$$

$$s_2^{-1} s_1 s_2 = a_4 b_2 \cdot b_4 c_2 \cdot c_4 d_2 \cdots.$$

Since $s_1, s_2^{-1} s_1 s_2$ are commutative the commutator $s_1 s_2^{-1} s_1 s_2$ is of order 2. The group generated by s_1, s_2 is transitive and hence its order is a multiple of its degree. As this degree can be increased without limit, s_1, s_2 may be so selected that the order of $\{s_1, s_2\}$ exceeds any given number. It is evident that s_1 and s_2 can be chosen in a similar manner whenever $n > 3$, so that we have the

* These operators are of order 2 since they are the transforms of operators of order two.

† MOORE, Bulletin of the American Mathematical Society, vol. 1 (1894), p. 61.

result: *If an operator of order 2 and an operator of order n have a commutator of order 2, if $n > 3$, these operators may be so selected that they generate any one of an infinite system of groups of finite order.*

§ 2. *The groups generated by two operators of order three whose commutator is of order two.*

We shall suppose that s_1 and s_2 satisfy the following conditions:

$$s_1^3 = s_2^3 = 1, \qquad s_1^2 s_2^2 s_1 s_2 = s_2^2 s_1^2 s_2 s_1,$$

and shall first consider the group (H) generated by the four commutators

$$s_1^2 s_2^2 s_1 s_2, \qquad s_2^2 s_1 s_2 s_1^2, \qquad s_2 s_1^2 s_2^2 s_1, \qquad s_1 s_2 s_1^2 s_2^2.$$

From the fact that two operators of order 3 whose product is of order 2 generate the tetrahedral group, it follows that each of these operators is commutative with two of the other three. For instance, $s_1^2 s_2^2 s_1 s_2$ is commutative with $s_2^2 s_1 s_2 s_1^2$ and also with $s_2 s_1^2 s_2^2 s_1$ since each of the two tetrahedral groups generated by $s_2^2 s_1 s_2$, s_1^2 and s_2, $s_1^2 s_2^2 s_1$, respectively, contains $s_1^2 s_2^2 s_1 s_2$ together with one or the other of the two commutators in question. Similarly, it may be observed that $s_2^2 s_1 s_2 s_1^2$ is commutative with $s_1^2 s_2^2 s_1 s_2$ and $s_1 s_2 s_1^2 s_2^2$; $s_2 s_1^2 s_2^2 s_1$ with $s_1 s_2 s_1^2 s_2^2$ and $s_1^2 s_2^2 s_1 s_2$; $s_1 s_2 s_1^2 s_2^2$ with $s_2^2 s_1 s_2 s_1^2$ and $s_2 s_1^2 s_2^2 s_1$. Hence the two dihedral groups $D_1 = \{ s_1^2 s_2^2 s_1 s_2,\ s_1 s_2 s_1^2 s_2^2 \}$ and $D_2 = \{ s_2^2 s_1 s_2 s_1^2,\ s_2 s_1^2 s_2^2 s_1 \}$ have each operator of the one commutative with every operator of the other. These two groups generate H.

The group H is an invariant subgroup of $G = \{ s_1, s_2 \}$ since H is transformed into itself by s_1 and s_2 by virtue of the following equations:

$$s_1 s_2^2 s_1 s_2 s_1 = (s_1^2 s_2^2 s_1 s_2 \cdot s_2^2 s_1 s_2 s_1^2)^{-1}, \qquad s_1^2 s_2 s_1^2 s_2^2 s_1^2 = s_1^2 s_2 s_1 s_2^2 \cdot s_2 s_1 s_2^2 s_1^2,$$

$$s_2^2 s_1^2 s_2^2 s_1 s_2^2 = s_2^2 s_1^2 s_2 s_1 \cdot s_1^2 s_2 s_1 s_2^2, \qquad s_2 s_1 s_2 s_1^2 s_2 = s_2 s_1 s_2^2 s_1^2 \cdot s_1 s_2^2 s_1^2 s_2.$$

As H is solvable, the commutator subgroup of G is solvable and hence G is solvable. If one of the two dihedral groups D_1, D_2 is abelian the other is also abelian, since the two generators of such an abelian group would be commutative with every operator of H and hence this would also be true of their transforms. As a generator of the other dihedral group would be such a transform, the statement is proven.

If H contains any operators of odd order the totality of these together with the identity must constitute an abelian characteristic subgroup of H. Hence neither s_1 nor s_2 can occur in H; and s_1, H generate a group whose order is three times that of H. This group is transformed into itself by s_2 since it involves $s_2^2 s_1 s_2$. The order of G is therefore the order of H multiplied by either

3 or 9, and the commutator quotient group* has one of these two numbers for its order. The smallest order that G can have is 12. If it has this order it is the tetrahedral group and we may let $s_1 = abc$, $s_2 = acd$ and then $s_1^2 s_2^2 s_1 s_2 = ab \cdot cd$. The order of H could not be less than 4, since a group whose order is twice an odd number contains a subgroup composed of all its operators of odd order together with the identity.

When H is abelian its order cannot exceed 16 and hence the order of G cannot exceed 144 in this case. A group of order 144 in which H is abelian may be generated as follows: Let

$$s_1 = abc \cdot def \cdot ghi \cdot jkl \quad \text{and} \quad s_2 = adg \cdot ejh \cdot fki.$$

The four given commutators will then be in order:

$$ad \cdot be \cdot gl \cdot hj, \qquad ad \cdot cf \cdot gl \cdot ik, \qquad ag \cdot bh \cdot dl \cdot ej, \qquad ag \cdot ci \cdot dl \cdot fk.$$

These four substitutions illustrate that the four commutators given in our first theorem may be distinct and they may be used as the four independent generators of the abelian group of order 16 and of type (1, 1, 1, 1). As s_1 and H generate a transitive group of order 48 while H involves three systems of intransitivity, and hence the average number of letters in all its substitutions is 9, all the substitutions of this group of order 48 which are not in H must be of degree 12. Otherwise the average number of letters in all its substitutions would not be 11. Since s_2 is not in this group of order 48 the order of G is actually 144. That is, there is a G in which H has its maximal order as an abelian group and the order of G is 144.

When H is an abelian group of order 16 the order of G cannot be 48, since any two non-commutative substitutions (t_1, t_2) of order 3 in such a group would have the property that either $t_1 t_2$ or $t_1 t_2^2$ would be of order 2, and hence they would generate the tetrahedral group. Moreover H cannot be abelian and of order 8, since either s_1 or s_2 and this group of order 8 and of type (1, 1, 1) would generate the direct product of the tetrahedral group and the group of order 2. As this group of order 24 contains a characteristic tetrahedral subgroup involving all its operators of order 3, it is not generated by two operators of order 3, nor is a group of order 72 which contains it invariantly generated by one of its operators of order 3 and an operator of order 3 which is not found in the invariant subgroup. This proves the theorem: *If H is abelian the order of G is* 12, 36, *or* 144. We gave an example where the order of G is 12 and also one where this order is 144. It is easy to see that the following substitutions generate such a group of order 36:

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* The commutator quotient group is the quotient group with respect to the commutator subgroup.

$s_1 = abc \cdot def \cdot ghi \cdot jkl$, $s_2 = adg \cdot ble \cdot cij$, $s_1^2 s_2^2 s_1 s_2 = ak \cdot bh \cdot cf \cdot dg \cdot el \cdot ij$,

$s_2 s_1^2 s_2^2 s_1 = ad \cdot ci \cdot eh \cdot bl \cdot gk \cdot fj$, $s_2^2 s_1 s_2 s_1^2 = ag \cdot be \cdot cj \cdot dk \cdot fi \cdot hl$, $s_1 s_2 s_1^2 s_2^2 = s_1^2 s_2^2 s_1 s_2$.

This group of order 36 contains $abc \cdot dli \cdot ejg \cdot fkh$ invariantly and permutes its cycles according to the alternating group of degree 4. Hence G is the direct product of the tetrahedral group and the group of order 3. It is not difficult to see that G is always such a direct product when its order is 36, since it must contain a tetrahedral group invariantly. That is, when H is of order 4, G is either the tetrahedral group or it is the direct product of this group and the group of order 3. In other words, there is one and only one group of each of the orders 12 and 36 which is generated by two operators of order 3 whose commutator is of order 2.

When H is non-abelian both of the dihedral subgroups D_1, D_2 are non-abelian. We shall now prove that the order of D_1 cannot be twice an odd number. If this were the case, D_1 would involve no invariant operator besides the identity and hence H would be the direct product of D_1 and D_2. The odd operators of H (together with the invariant operator of order 2 in D_2 if such an operator is present) would generate an invariant subgroup of G. The corresponding quotient group would be abelian since its operators of order 2 are invariant under it. This is impossible since H was supposed to be generated by the commutators of G. Hence *the order of each of the subgroups D_1, D_2 is divisible by 4 whenever H is non-abelian.*

It will now be proved that when H is non-abelian D_1 and D_2 have the same order. This follows directly from the facts that the order of each of these dihedral groups is divisible by 4, that one of the two generators of order 2 of one of them can be transformed into one of the two generators of the other, and that each of the operators of D_1 is commutative with every operator of D_2. It is now easy to see that H is not the direct product of D_1 and D_2. Such a direct product would contain the direct product of the maximal cyclic subgroups in D_1 and D_2 as a characteristic subgroup, and the quotient group of G with respect to this characteristic subgroup would be abelian. The latter direct product is a characteristic subgroup of G, since it is generated by all the operators of highest order in H which are commutative with half the operators of H.

The case which remains to be considered is the one in which D_1 and D_2 are non-abelian and have two common operators when the order of D_1 exceeds 8. The operators of highest order in these subgroups would again generate a characteristic subgroup of G since they include all of the operators of H which have this order and are commutative with half of its operators. As the quotient group with respect to this characteristic subgroup would be abelian, we have the important result: If two operators of order 3 have a commutator of order 2 they generate a group whose order is one of the five numbers, 12, 36, 144, 96, 288. This order, however, could not be 96. If G were of order 96 it would be isomorphic with the given group of order 144. Since not more than three operators of this G_{144} could correspond to the identity in G_{96} and as the isomorphism could not be $(\alpha, 1)$, it follows that G_{144} would contain an invariant subgroup of order 3. As the corresponding quotient group could not be generated by two operators of order 3, whose commutator is of order 2, this is impossible. That is, it is not possible to generate a group of order 96 by two operators of order 3 whose commutator is of order 2.

In the preceding paragraph we proved that the substitution group of order 144 which was constructed above does not contain an invariant subgroup of order 3. It is possible to prove a much more general theorem, viz.: *If a group of order* 144 *is generated by two operators of order* 3 *whose commutator is of order* 2, *it is the direct product of two tetrahedral groups.* Since such a group contains the abelian subgroup of order 16 and of type (1, 1, 1, 1) invariantly and this subgroup involves 35 subgroups of order 4, the group contains at least two invariant subgroups of order 4. If these had a common operator of order 2, G would contain an invariant subgroup of order 8. This is impossible since the corresponding quotient group could not be generated by two operators of order 3. Hence every such group of order 144 contains two invariant subgroups of order 4 and these two subgroups have only the identity in common. With respect to these subgroups the quotient group of G is the direct product of the tetrahedral group and the group of order 3, since this quotient is generated by two operators of order 3 whose commutator is of order 2. Hence G contains two invariant subgroups of order 12 which have at most three common operators. If they had three such operators these would constitute an invariant subgroup of G. This is impossible since the corresponding quotient group could not be generated by two operators of order 3 having a commutator of order 2. Hence G is the direct product of two alternating groups of order 12.

It has now been proved that there is one and only one group of each of the orders 12, 36 and 144 which is generated by two operators of order 3 having a commutator of order 2. It has also been proved that if another such group exists its order is 288 and it must have a $(2, 1)$ isomorphism with the direct product of two alternating groups. Moreover, its H is generated by two octic groups having each operator of the one commutative with every operator of the other and having two operators in common. As a transitive group, H may therefore be generated by the regular octic and its associate,* or by the regular quaternion group and its associate, since this group may be generated in either of these two ways. This suggests that G may be generated by the regular

*Quarterly Journal of Mathematics, vol. 28 (1896), p. 249.

group of order 24 which does not involve a subgroup of order 12 and its associate. From this fact we readily obtain the following relations:

$$s_1 = ark \cdot bxn \cdot cwl \cdot dqm \cdot evo \cdot ftj \cdot gsp \cdot hui,$$

$$s_2 = adc \cdot ehg \cdot ijp \cdot lmn \cdot qwv \cdot rus,$$

$$s_1^2 s_2^2 s_1 s_2 = ad \cdot bc \cdot eh \cdot fg \cdot ik \cdot jp \cdot ln \cdot mo \cdot qu \cdot rv,$$

$$s_2^2 s_1 s_2 s_1^2 = ae \cdot dh \cdot io \cdot jp \cdot km \cdot ln \cdot qv \cdot ru \cdot st \cdot wx,$$

$$s_2 s_1^2 s_2^2 s_1 = ac \cdot bd \cdot eg \cdot fh \cdot ij \cdot kp \cdot lo \cdot mn \cdot rv \cdot sw,$$

$$s_1 s_2 s_1^2 s_2^2 = ae \cdot cg \cdot ij \cdot kl \cdot mn \cdot op \cdot qx \cdot rs \cdot tu \cdot vw.$$

Having proved the existence of such a group of order 288, we have further to prove that there is only one of this order. This fact follows almost directly from the isomorphism between G and the direct product of two tetrahedral groups. To each of these two tetrahedral groups there corresponds a group of order 24 in G. As this group of order 24 cannot contain a subgroup of order 12 the theorem is proved and we have as a final result the following theorem: *There are four and only four groups that can be generated by two operators of order* 3 *whose commutator is of order* 2. *The orders of these are respectively* 12, 36, 144 *and* 288. The orders of their commutator subgroups are 4, 4, 16 and 32 respectively. As only the last of these is non-abelian, we note that the given group of order 288 is the only group which can be generated by two operators of order 3 whose commutator is of order 2, and which does not have an abelian commutator subgroup.

§ 3. *The groups generated by two operators of orders two and three respectively whose commutator is of order three.*

Let s_1 and s_2 be any two operators which satisfy the following conditions:

$$s_1^2 = 1, \qquad s_2^3 = 1, \qquad (s_1 s_2^2 s_1 s_2)^3 = 1.$$

The existence of such operators is proved by the fact that two generators of the symmetric group of order 6 satisfy these conditions. Moreover, the two substitutions ab, bcd may be substituted for s_1 and s_2 respectively since $ab \cdot bdc \cdot ab \cdot bcd = abc$. Hence it follows that each of the two smallest symmetric groups may be generated by two substitutions which satisfy the conditions imposed on s_1 and s_2. It will soon appear that these are the only symmetric groups which can be generated by two substitutions satisfying the given conditions. In fact, all such groups will be proved to be solvable and to constitute only a small class of the solvable groups.

The three conjugates of $s_1 s_2^2 s_1 s_2$ with respect to s_2 are

$$s_1 s_2^2 s_1 s_2, \qquad s_2^2 s_1 s_2^2 s_1 s_2^2, \qquad s_2 s_1 s_2^2 s_1.$$

Their continued product, in the given order, is the identity. Since the transforms with respect to s_1 of the first and third of these conjugates are respectively the inverses of the operators, it follows that the group generated by any two of them is invariant under the group (G) generated by s_1 and s_2. The group generated by these conjugate operators is the commutator subgroup of G since the conjugates of the commutator of s_1 and s_2 under s_1, s_2 must generate the commutator subgroup of G. We have then as a first result that *the commutator subgroup of G is generated by two operators of order* 3 *whose product is also of this order.*

The groups which can be generated by two operators of order three whose product is of order three have been studied and it has been proved that the commutator subgroup of such a group is always abelian, and it is either cyclic or it may be generated by two independent operators.* Moreover the quotient group of such a group with respect to the commutator subgroup (i. e., the commutator quotient group) is either of order 3 or of order 9. Hence G is one of those groups in which one arrives at the identity by forming the successive commutator subgroups, or the successive derivatives according to LIE's notation. This proves that G is solvable.†

It will be convenient to represent the commutator subgroup of G by K and the commutator subgroup of K by K_1. It has been observed that K_1 is abelian, that the order of K is either 3 times or 9 times that of K_1 and that K is generated by two operators ($s_2 s_1 s_2^2 s_1$, $s_1 s_2^2 s_1 s_2$) of order 3 whose product is of this order. It is of interest to observe that some of the groups which satisfy the last condition cannot be used for K in constructing the groups under consideration. Suppose K_1 is cyclic and hence has an abelian group of isomorphisms. The totality of the operators of G which are commutative with a generator of K_1 constitute an invariant subgroup of G. As the corresponding quotient group is abelian this invariant subgroup includes K, and hence the order of K_1 cannot exceed 3. This proves that the order of K is 3^α, $\alpha < 4$, whenever K_1 is cyclic, and that K cannot be any arbitrary group which may be generated by two operators of order 3 whose product is of order 3. ‡

The quotient group of G with respect to K_1 is generated by two operators of orders 2 and 3 respectively whose commutator is of order 3. The order of this quotient group could not be 18, since the commutator subgroup of the group of order 18 which is generated by two such operators is of order 3, and hence this

*Annals of Mathematics, vol. 3 (1901), p. 40.

†American Journal of Mathematics, vol. 20 (1898), p. 277.

‡Annals of Mathematics, loc. cit.

commutator subgroup could not correspond to K. That is, *when* K_1 *is cyclic its order is* 3 *and the order of* G *is* 162. The existence of this G is proved by the following substitutions:

$$s_1 = af \cdot bd \cdot ce \cdot jn \cdot ko \cdot lm \cdot xa \cdot yv \cdot wz,$$

$$s_2 = ajs \cdot bkt \cdot clu \cdot dmv \cdot enw \cdot fox \cdot gpy \cdot hqz \cdot ira,$$

$$s_1 s_2^2 s_1 s_2 = aif \cdot bgd \cdot che \cdot jlk \cdot mno \cdot swz \cdot txa \cdot uvy,$$

$$s_2 s_1 s_2^2 s_1 = acb \cdot def \cdot jnq \cdot kor \cdot lmp \cdot sax \cdot tyv \cdot uzw.$$

It is not difficult to determine all the groups which may be generated by two operators satisfying the conditions imposed on s_1 and s_2, and which also involve an abelian commutator subgroup. In fact, it follows from the preceding paragraph that the order of K cannot exceed 9 and if this order is 9 the order of G is 54 and G may be represented as a transitive substitution group of degree 9. Moreover, s_1 transforms each operator of K into its inverse since it transforms each of the operators $s_1 s_2^2 s_1 s_2$, $s_2 s_1 s_2^2 s_1$ into its inverse. If such a group exists it must therefore be one of the two simply isomorphic groups of degree 9 known as 54_1 and 54_3.* The existence of this G is proved by the following representations:

$$s_1 = ab \cdot de \cdot gh, \quad s_2 = adg \cdot bfi \cdot ceh, \quad s_1 s_2^2 s_1 s_2 = abc \cdot def, \quad s_2 s_1 s_2^2 s_1 = acb \cdot gih.$$

When the order of K is 3 the order of G is either 6 or 18. In the former case G is evidently the symmetric group of order 6. In the latter case, the group generated by s_2 is not invariant under G since it does not include $s_1 s_2^2 s_1 s_2$. Hence G is the transitive group of degree 6 and order 18; that is, G is the direct product of the symmetric group of order 6 and the group of order 3. We have now proved that *there are three and only three groups which have an abelian commutator subgroup and may be generated by two operators of orders* 2 *and* 3 *respectively whose commutator is of order* 3. The orders of these groups are 6, 18 and 54 respectively.

When the commutator subgroup of G is non-abelian and K_1 is non-cyclic its order cannot be less than 12. When this order is 12 K is the tetrahedral group and the order of G is either 24 or 72. In the former case K includes s_2 and G is the symmetric group of order 24, since this is the only group of this order which has the tetrahedral group for its commutator subgroup.† We have thus found another characteristic property of the symmetric group of order 24 which may be expressed as follows: The octahedral group is the smallest group that can be generated by two operators of orders 2 and 3 respectively having a com-

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*Cole, Bulletin of the New York Mathematical Society, vol. 2 (1893), p. 252.

†Quarterly Journal of Mathematics, vol. 28 (1896), p. 282.

mutator of order 3 and has a non-abelian commutator subgroup. When the order of K is 12 and K does not include s_2, G contains the octahedral group generated by s_1 and K as an invariant subgroup. It also contains an invariant subgroup of order 3, since s_2 transforms K in the same manner as one of its own operators. Hence we have proved that *G is either the octahedral group or the direct product of this group and a group of order 3 whenever the commutator subgroup of G is of order 12.*

It may be observed from the results of the second paragraph that the commutator quotient group G/K is either of order 2 or the cyclic group of order 6. In the latter case the order of G is divisible by 9 since K does not include s_2. In the former case s_2 is found in K. We proceed to prove that the necessary and sufficient condition that the order of G be divisible by 9 is that its commutator quotient group shall be of order 6. This will be done by proving that we arrive at a contradiction when we assume that the commutator quotient group of G is of order 2 and that the order of G is divisible by 9. When this quotient group is of order 2 the order of K/K_1 cannot be 9, since G/K_1 would then be of order 18 and a group of this order which is generated by an operator of order 3 and an operator of order 2 cannot have a commutator subgroup of order 9. Hence we may assume that the orders of K/K_1 and G/K are 3 and 2 respectively.

All the operators whose orders are not divisible by 3 in K generate an invariant subgroup of G, and the corresponding quotient group is of order 2.3^α. This quotient group is generated by two operators of orders 2 and 3 respectively and its commutator subgroup is of order 3^α. Hence its subgroup of order 3^α cannot be abelian when $\alpha > 1$. If this group of order 3^α were non-abelian its commutator subgroup would be a characteristic subgroup. From this it follows that G must be isomorphic with a group of order 2.3^β, $\beta > 1$, in which the subgroup of order 3^β is abelian, whenever $\alpha > 1$. As this group would be generated by an operator of order 2 and an operator of order 3, and as its commutator subgroup would be of order 3^β, we have arrived at a contradiction; hence it is proved that *the necessary and sufficient condition that the order of G be divisible by 9 is that its commutator quotient group shall be of order 6.*

It is not difficult to prove that the number of these groups whose commutator quotient group is of order 2 is infinite. Let p represent one of the infinite system of primes which satisfy the condition $p \equiv 1 \bmod 3$, and consider the holomorph of the group (H) of order p^2 and of type (1, 1). As H contains $p+1$ subgroups of order p and as its group of isomorphisms (I) contains only $p-1$ operators which transform each one of these subgroups into itself,* it follows that I has a $(p-1, 1)$ isomorphism with the triply transitive group of degree $p+1$ and of order $p(p^2-1)$. Each of the operators of the regular cyclic subgroup of order $p-1$ is transformed into its inverse under this triply

*Transactions of the American Mathematical Society vol. 2 (1901), p. 260.

transitive group and hence this group includes the symmetric group of order 6.

This symmetric group is simply isomorphic with a subgroup of I, hence the holomorph of H contains a subgroup of order $6p^2$ in which all the subgroups of order 3 are conjugate. To obtain two operators which may be used for s_1 and s_2 we need only select an operator of order 3 and an operator of order 2 which does not transform this operator of order 3 into its inverse. As such operators always exist, it follows that *there is an infinite system of groups which are generated respectively by two operators of orders* 2 *and* 3 *having a commutator of order* 3.

§ 4. *Conclusion.*

The preceding results are analogous to those which relate to the groups of genus zero. As the latter are defined by the orders of two generators and the order of their product and on account of this simple definition are extremely useful, it seems desirable to inquire into those groups which are either completely defined by the orders of two operators and the order of their commutator or are almost completely determined by these properties. The main results which have been obtained may be stated as follows: *Any dihedral groups whose order is divisible by* 8 *may be defined by the orders of two generators and the order of the commutator of these generators*, and it appears very probable that no other non-abelian group can be defined in this manner. When the commutator of two generators is of order 2 and the orders of these generators are either 2, 3 or 3, 3 the groups are almost completely determined by these conditions. In the former case there are two possible groups while in the latter case there are four. When two generators are of order 2 and their commutator is of any odd order there are two possible groups.

There is an infinite system of groups such that each group is generated by two operators whose commutator is of order 2 *while the order of one of the generators is a given number which exceeds* 3 *and the order of the other is any given even number* > 0. This general theorem follows directly from the proof given above that there is such an infinite system when the order of one of the generators is 2 while the order of the other is any number larger that 3. There is also an infinite system of groups each of which may be generated by two operators of orders 3 and 4 respectively whose commutator is of order 2. This fact follows directly from the following substitutions which have a commutator of order 2:

$$s_1 = abc \cdot def \cdot ghi \cdot jkl \cdot mno \cdot pqr,$$

$$s_2 = c\alpha f\beta \cdot i\gamma l\delta \cdot o\epsilon r\theta \cdot ah \cdot bg \cdot de \cdot jq \cdot kp \cdot mn.$$

These substitutions can readily be constructed if it is observed that a cycle of order 3 is transformed into a cycle having two common letters but in the reverse order by any substitution whose commutator with the cycle is of order 2. It is clear that s_1 and s_2 generate a transitive group and that the degree of this transitive group can be increased beyond any given number by increasing the degrees of s_1, s_2 according to the law which is exhibited by the given substitutions. That there is an infinite system of groups such that each one may be generated by two operators of orders 3 and 5 respectively whose commutator is of order 2 may be deduced in a similar manner from the following substitutions:

$$s_1 = abc \cdot def \cdot ghi \cdot jkl \cdot mno,$$

$$s_2 = bdgmj \cdot cfh\beta l \cdot a\alpha ok\gamma \cdot e\delta ine.$$

If it is desired to double the degree of the transitive group generated by s_1, s_2 the cycle $a\alpha ok\gamma$ may be divided into the two parts $a\alpha \cdots$, $ok\gamma \cdots$ and one of these cycles may be extended by a letter from an additional cycle of s_1 while the remaining four letters necessary to complete these cycles do not belong to s_1. The rest of s_2 may be constructed as before. In this manner we may evidently increase the degree of $\{s_1, s_2\}$ without limit and the commutator of s_1, s_2 will always be of order 2.

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