THE SECOND DERIVATIVES OF THE EXTREMAL-INTEGRAL*

BY

ARNOLD DRESDEN

Introduction.

Suppose that an extremal has been found for the problem of minimizing the integral †

(1) $$I = \int F(x, y, x', y')\, dt,$$

which passes through the points $A_1(a_1, b_1)$ and $A_2(a_2, b_2)$, along which $F_1 > 0$, and for which A_1 and A_2 are not conjugate points. Then it may be shown as in § 1 below that if $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are taken in a sufficiently small vicinity of A_1 and A_2 respectively, a unique extremal $\mathfrak{C}$ can be constructed passing through P_1 and P_2, ‡ along which $F_1 > 0$ and for which P_1 and P_2 are not conjugate points. The integral (1) taken along $P_1 P_2$ becomes a single-valued function of x_1, y_1, x_2 and y_2, uniquely defined for sufficiently small values of $|x_1 - a_1|$, $|y_1 - b_1|$, $|x_2 - a_2|$ and $|y_2 - b_2|$, which we denote by

(2) $$\mathfrak{J}(x_1, y_1, x_2, y_2).$$

This function, commonly called the "*extremal-integral*," is identical with Hamilton's principal function. If the original extremal $A_1 A_2$ actually furnishes a minimum for (1), then (2) must be a minimum in the ordinary sense for $x_1 = a_1$, $y_1 = b_1$, $x_2 = a_2$, $y_2 = b_2$. We are thus enabled to derive necessary conditions for a minimum of (1) by a discussion of (2) and its derivatives with respect to x_1, y_1, x_2 and y_2. §

The first derivatives of the function $\mathfrak{J}$ were given by HAMILTON in 1835. ||

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* Presented to the Society September 11, 1908. Received for publication June 15, 1908.

† The function F and the extremals are restricted by homogeneity and continuity conditions, for an explicit statement of which we refer the reader to BOLZA, *Vorlesungen über Variationsrechnung* (Leipzig, Teubner, 1908), pp. 193, 194. For terms and notations current in the Calculus of Variations and used here without explanation we refer to the same source.

‡ Compare BOLZA, loc. cit., § 37*a*.

§ This method of the Calculus of Variations, frequently called the "method of differentiation," was suggested by DIENGER in 1867. For further bibliographical reference see BOLZA, loc. cit., § 38.

|| Philosophical Transactions of the Royal Society of London, 1835, part I, p. 99.

The object of the present paper is to obtain *explicit expressions for the second derivatives of the extremal-integral** (§ 1–§ 4), by means of which a simple determination of conjugate and focal points will be possible. In § 5 we treat by this method the problem of minimizing the integral (1) when *one end-point is movable along a fixed curve*, and in § 6 the same problem when *both end-points are movable.* Thus new proofs are given of the theorems first proved by BLISS.† In § 7 the same method is applied to the discussion of *conjugate points on discontinuous solutions*, previously investigated by CARATHEODORY ‡ and BOLZA. § The results arrived at are in appearance in *direct contradiction* with theirs. The discussion of this contradiction appears in § 8, where it is shown by means of a relation between the Weierstrass E-function and Caratheodory's invariant Ω_0, that the case in which the contradiction occurred cannot arise.

§ 1. *Construction of an extremal through two given points.*||

We take Euler's differential equation, written by BLISS ¶ in the following form:

$$\frac{dx}{ds} = \cos\theta, \qquad \frac{dy}{ds} = \sin\theta, \qquad \frac{d\theta}{ds} = \frac{F_{x'y} - F_{xy'}}{F_1\left(\sqrt{x'^2 + y'^2}\right)^3}. \tag{3}$$

Denoting by

$$\begin{aligned} x &= \mathfrak{X}(s - s_i;\ x_i, y_i, \theta_i), \\ y &= \mathfrak{Y}(s - s_i;\ x_i, y_i, \theta_i), \\ \theta &= \Theta(s - s_i;\ x_i, y_i, \theta_i), \end{aligned} \tag{4}$$

that particular solution of (3) which satisfies the initial conditions

$$x = x_i,\ y = y_i,\ \theta = \theta_i$$

at $s = s_i$, we solve the system

$$x_2 = \mathfrak{X}(s_2 - s_1;\ x_1, y_1, \theta_1), \qquad y_2 = \mathfrak{Y}(s_2 - s_1;\ x_1, y_1, \theta_1) \tag{5}$$

for s_2 and θ_1 as functions of x_1, y_1, x_2 and y_2, the value s_1 being chosen arbitrarily. This is always possible, according to the well-known theory of implicit

*In the Sächsische Berichte, 1883–1884, part II, p. 99, A. MAYER has used a similar method for the problem of variable end-points.

†See Transactions of the American Mathematical Society, vol. 3 (1902), p. 136, and Mathematische Annalen, vol. 58 (1903), p. 70.

‡ *Dissertation*, Göttingen, 1904, p. 31, and Mathematische Annalen, vol. 62 (1906), p. 449.

§ *Vorlesungen*, chap. VIII, and American Journal of Mathematics, vol. 30 (1908), p. 209.

|| Compare BOLZA, *Vorlesungen*, § 37*a*). For notation, ibid., § 27*b*).

¶ Transactions of the American Mathematical Society, vol. 7 (1906), p. 188.

functions, if A_1 and A_2 are not conjugate.* When s_2 and θ_1 are so determined, *the functions*

$$(6)\qquad \begin{aligned} x &= \mathfrak{X}(s - s_1;\ x_1, y_1, \theta_1) \equiv x(s), \\ y &= \mathfrak{Y}(s - s_1;\ x_1, y_1, \theta_1) \equiv y(s), \\ \theta &= \Theta(s - s_1;\ x_1, y_1, \theta_1) \end{aligned}$$

represent the required extremal $\mathfrak{E}$, and we have †

$$(7)\qquad \begin{aligned} &\mathfrak{X}(0;\ x_1, y_1, \theta_1) = x_1, \quad \mathfrak{Y}(0;\ x_1, y_1, \theta_1) = y_1, \quad \Theta(0;\ x_1, y_1, \theta_1) = \theta_1, \\ &\mathfrak{X}(s_2 - s_1;\ x_1, y_1, \theta_1) = x_2, \quad \mathfrak{Y}(s_2 - s_1;\ x_1, y_1, \theta_1) = y_2, \quad \Theta(s_2 - s_1;\ x_1, y_1, \theta_1) = \theta_2. \end{aligned}$$

Further, we get from (3) and (7) the following useful identities:

$$(8)\qquad \begin{aligned} \mathfrak{X}_s(0;\ x_1, y_1, \theta_1) &= \cos\theta_1, & \mathfrak{Y}_s(0;\ x_1, y_1, \theta_1) &= \sin\theta_1, \\ \mathfrak{X}_s(s_2 - s_1;\ x_1, y_1, \theta_1) &= \cos\theta_2, & \mathfrak{Y}_s(s_2 - s_1;\ x_1, y_1, \theta_1) &= \sin\theta_2, \\ \mathfrak{X}_{x_1}(0;\ x_1, y_1, \theta_1) &= 1, & \mathfrak{Y}_{x_1}(0;\ x_1, y_1, \theta_1) &= 0, \\ \mathfrak{X}_{y_1}(0;\ x_1, y_1, \theta_1) &= 0, & \mathfrak{Y}_{y_1}(0;\ x_1, y_1, \theta_1) &= 1, \\ \mathfrak{X}_{\theta_1}(0;\ x_1, y_1, \theta_1) &= 0, & \mathfrak{Y}_{\theta_1}(0;\ x_1, y_1, \theta_1) &= 0. \end{aligned}$$

It follows from (4) that *the extremal* $\mathfrak{E}$ *may also be represented in the form*

$$(9)\qquad \begin{aligned} x &= \mathfrak{X}(s - s_2;\ x_2, y_2, \theta_2) \equiv x(s), \\ y &= \mathfrak{Y}(s - s_2;\ x_2, y_2, \theta_2) \equiv y(s), \\ \theta &= \Theta(s - s_2;\ x_2, y_2, \theta_2), \end{aligned}$$

from which we can derive formulas analogous to (7) and (8) and obtainable from them by interchange of the subscripts 1 and 2.‡ It is evident that the functions $x(s)$ and $y(s)$ as defined by (6) on the one hand, and by (9) on the other hand, are identical *in the variable* s for the range $s_1 \leqq s \leqq s_2$.

Whenever a distinction between these two forms of the extremal shall be necessary in the sequel, we shall use the following abbreviated notations:

$$(10)\qquad \begin{aligned} \mathfrak{X}(s - s_i;\ x_i, y_i, \theta_i) &= \mathfrak{X}^i(s), \qquad (i = 1, 2). \\ \mathfrak{Y}(s - s_i;\ x_i, y_i, \theta_i) &= \mathfrak{Y}^i(s), \\ \cos\theta_i = p_i, \qquad \sin\theta_i &= q_i. \end{aligned}$$

* Compare C. JORDAN, *Cours d'Analyse*, vol. 1, 2d ed., § 92. The theorem is applicable because equations (5) are satisfied by the tangential angle of $A_1 A_2$ at A_1 and by the parameter value of A_2 on $A_1 A_2$, if we substitute for x_1, y_1, and x_2, y_2 the coördinates of A_1 and A_2, and because furthermore the Jacobian does not vanish if A_1 and A_2 are not conjugate (see also § 2 and §4, and BOLZA, loc. cit., p. 234).

† The last of these equations is to be considered as defining θ_2.

‡ It is to be observed that s_2 and θ_2, used in (9), are not arbitrary, but are quantities defined by (5) and (7) respectively, whereas s_1 and θ_1 following from them are identical with the quantities defined by these same symbols in (5).

§ 2. *The extremal-integral.*

Along this extremal $\mathfrak{E}$ we compute now the integral (1), which furnishes us the function (2) in two forms:

$$\mathfrak{J}(x_1, y_1, x_2, y_2) = \int_{s_1}^{s_2} F[\mathfrak{X}^1(s), \mathfrak{Y}^1(s), \mathfrak{X}^1_s(s), \mathfrak{Y}^1_s(s)]ds$$

$$= \int_{s_1}^{s_2} F[\mathfrak{X}^2(s), \mathfrak{Y}^2(s), \mathfrak{X}^2_s(s), \mathfrak{Y}^2_s(s)]ds.$$

We find then the *first derivatives of the extremal-integral*,

$$(11) \qquad \begin{aligned} \frac{\partial \mathfrak{J}}{\partial x_1} &= -F_{x'}(x_1, y_1, p_1, q_1), & \frac{\partial \mathfrak{J}}{\partial x_2} &= F_{x'}(x_2, y_2, p_2, q_2), \\ \frac{\partial \mathfrak{J}}{\partial y_1} &= -F_{y'}(x_1, y_1, p_1, q_1), & \frac{\partial \mathfrak{J}}{\partial y_2} &= F_{y'}(x_2, y_2, p_2, q_2), \end{aligned}$$

which formulæ correspond to *Hamilton's first derivatives of the principal function.** For the determination of the second derivatives of (2) we have first to determine $\partial \theta_i / \partial z$, in which $i = 1, 2$, and z is any one of the 4 variables x_1, y_1, x_2, y_2. Differentiating (5) with respect to x_1, we obtain

$$0 = \mathfrak{X}^1_s(s_2)\frac{\partial s_2}{\partial x_1} + \mathfrak{X}^1_{x_1}(s_2) + \mathfrak{X}^1_{\theta_1}(s_2)\frac{\partial \theta_1}{\partial x_1},$$

$$0 = \mathfrak{Y}^1_s(s_2)\frac{\partial s_2}{\partial x_1} + \mathfrak{Y}^1_{x_1}(s_2) + \mathfrak{Y}^1_{\theta_1}(s_2)\frac{\partial \theta_1}{\partial x_1}.$$

Similar equations are obtained by differentiating with respect to x_2, y_1 and y_2, and four more by interchange of subscripts and superscripts 1 and 2. All these equations are uniquely solvable † for $\partial \theta_i / \partial z$, so that *we obtain the following results:*

$$(12) \qquad \begin{aligned} \frac{\partial \theta_1}{\partial x_1} &= -\frac{\xi_1(s_2)}{u_1(s_2)}, & \frac{\partial \theta_2}{\partial x_1} &= -\frac{q_1}{u_2(s_1)}, \\ \frac{\partial \theta_1}{\partial y_1} &= -\frac{\eta_1(s_2)}{u_1(s_2)}, & \frac{\partial \theta_2}{\partial y_1} &= \frac{p_1}{u_2(s_1)}, \\ \frac{\partial \theta_1}{\partial x_2} &= -\frac{q_2}{u_1(s_2)}, & \frac{\partial \theta_2}{\partial x_2} &= -\frac{\xi_2(s_1)}{u_2(s_1)}, \\ \frac{\partial \theta_1}{\partial y_2} &= \frac{p_2}{u_1(s_2)}, & \frac{\partial \theta_2}{\partial y_2} &= -\frac{\eta_2(s_1)}{u_2(s_1)}, \end{aligned}$$

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* HAMILTON, loc. cit.; compare also BOLZA, loc. cit., § 37*b*).

† Compare with the first footnote on page 469.

where

$$
(13)\qquad
\begin{aligned}
\xi_i(s) &= \mathfrak{X}^i_s(s)\,\mathfrak{Y}^i_{x_i}(s) - \mathfrak{X}^i_{x_i}(s)\,\mathfrak{Y}^i_s(s),\\
\eta_i(s) &= \mathfrak{X}^i_s(s)\,\mathfrak{Y}^i_{y_i}(s) - \mathfrak{X}^i_{y_i}(s)\,\mathfrak{Y}^i_s(s),\\
u_i(s) &= \mathfrak{X}^i_s(s)\,\mathfrak{Y}^i_{\theta_i}(s) - \mathfrak{X}^i_{\theta_i}(s)\,\mathfrak{Y}^i_s(s).
\end{aligned}
\qquad (i=1,2),
$$

§ 3. *Particular solutions of Jacobi's differential equation.*

The functions ξ_i, η_i and u_i have the following properties:

1) They are particular solutions of Jacobi's differential equation for the extremal $\mathfrak{E}$,

$$
F_2\omega - \frac{d}{ds}\left(F_1\frac{d\omega}{ds}\right) = 0,
$$

the arguments of F_1 and F_2 being $x(s)$, $y(s)$, $x'(s)$, $y'(s)$. The proof of this statement can be given in precisely the same way as is usually followed for the proof of Jacobi's theorem:*

2) They satisfy the following conditions:

$$
(14)\qquad
\begin{aligned}
&\xi_i(s_i) = -q_i, &&\eta_i(s_i) = p_i, &&u_i(s_i) = 0, \quad (i=1,2),\\
&\xi'_i(s_i) = -y''(s_i), &&\eta'_i(s_i) = x''(s_i), &&u'_i(s_i) = 1,
\end{aligned}
$$

which follow from (13) by means of (3), (8) and interchange of subscripts 1 and 2.

For our further work *we introduce now also those particular solutions* $v_i(s)$ *of Jacobi's equation which satisfy the conditions*

$$
(15)\qquad v_i(s_i) = 1, \qquad v'_i(s_i) = 0.
$$

It is clear that $u_1(s)$, $v_1(s)$ and $u_2(s)$, $v_2(s)$ are linearly independent solutions of that equation,† so that we can express $\xi_1(s)$, $\eta_1(s)$ and $\xi_2(s)$, $\eta_2(s)$ linearly in terms of $u_1(s)$, $v_1(s)$ and $u_2(s)$, $v_2(s)$ respectively.‡ We find, using (14) and (15), that

$$
(16)\quad \xi_i(s) = -y''(s_i)u_i(s) - q_iv_i(s), \qquad \eta_i(s) = x''(s_i)u_i(s) + p_iv_i(s).
$$

Using (16), we can now transform (12) and we obtain *the following formulæ which express the partial derivatives of the tangential angles of the extremal* $\mathfrak{E}$ *at* P_1 *and* P_2 *in terms of two sets of two linearly independent integrals of Jacobi's differential equation for that extremal:*

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* Compare BOLZA, loc. cit., § 12*b*.

† Compare C. JORDAN, *Cours d'Analyse*, 2d ed., vol. 3, § 122.

‡ Ibid., § 119.

$$
(17) \qquad
\begin{aligned}
\frac{\partial \theta_1}{\partial x_1} &= y''(s_1) + \frac{q_1 v_1(s_2)}{u_1(s_2)}, & \frac{\partial \theta_2}{\partial x_1} &= -\frac{q_1}{u_2(s_1)}, \\
\frac{\partial \theta_1}{\partial y_1} &= -x''(s_1) - \frac{p_1 v_1(s_2)}{u_1(s_2)}, & \frac{\partial \theta_2}{\partial y_1} &= \frac{p_1}{u_2(s_1)}, \\
\frac{\partial \theta_1}{\partial x_2} &= -\frac{q_2}{u_1(s_2)}, & \frac{\partial \theta_2}{\partial x_2} &= y''(s_2) + \frac{q_2 v_2(s_1)}{u_2(s_1)}, \\
\frac{\partial \theta_1}{\partial y_2} &= \frac{p_2}{u_1(s_2)}, & \frac{\partial \theta_2}{\partial y_2} &= -x''(s_2) - \frac{p_2 v_2(s_1)}{u_2(s_1)}.
\end{aligned}
$$

§ 4. *The second derivatives of the extremal-integral.*

We can now at once determine the second derivatives of (2) with respect to any two of the variables x_1, y_1, x_2, y_2. Differentiating (11) and remembering that

$$\frac{\partial p_i}{\partial z} = -q_i \frac{\partial \theta_i}{\partial z}, \qquad \frac{\partial q_i}{\partial z} = p_i \frac{\partial \theta_i}{\partial z},$$

we obtain the following

THEOREM. *The second derivatives of the extremal integral are given by the table*

(18)

	$\frac{\partial \mathfrak{J}}{\partial x_1}$	$\frac{\partial \mathfrak{J}}{\partial y_1}$	$\frac{\partial \mathfrak{J}}{\partial x_2}$	$\frac{\partial \mathfrak{J}}{\partial y_2}$
$\frac{\partial}{\partial x_1}$	$-L(s_1) + F_1(s)\frac{q_1^2 v_1(s_2)}{u_1(s_2)}$	$-M(s_1) - F_1(s_1)\frac{p_1 q_1 v_1(s_2)}{u_1(s_2)}$	$-F_1(s_1)\frac{q_1 q_2}{u_1(s_2)}$	$F_1(s_1)\frac{q_1 p_2}{u_1(s_2)}$
$\frac{\partial}{\partial y_1}$	$-M(s_1) - F_1(s_1)\frac{p_1 q_1 v_1(s_2)}{u_1(s_2)}$	$-N(s_1) + F_1(s_1)\frac{p_1^2 v_1(s_2)}{u_1(s_2)}$	$F_1(s_1)\frac{p_1 q_2}{u_1(s_2)}$	$-F_1(s_1)\frac{p_1 p_2}{u_1(s_2)}$
$\frac{\partial}{\partial x_2}$	$F_1(s_2)\frac{q_1 q_2}{u_2(s_1)}$	$-F_1(s_2)\frac{p_1 q_2}{u_2(s_1)}$	$L(s_2) - F_1(s_2)\frac{q_2^2 v_2(s_1)}{u_2(s_1)}$	$M(s_2) + F_1(s_2)\frac{p_2 q_2 v_2(s_1)}{u_2(s_1)}$
$\frac{\partial}{\partial y_2}$	$-F_1(s_2)\frac{q_1 p_2}{u_2(s_1)}$	$F_1(s_2)\frac{p_1 p_2}{u_2(s_1)}$	$M(s_2) + F_1(s_2)\frac{p_2 q_2 v_2(s_1)}{u_2(s_1)}$	$N(s_2) - F_1(s_2)\frac{p_2^2 v_2(s_1)}{u_2(s_1)}$

in which the functions F_1, L, M, N have the same meaning as in the Weierstrass theory, and in which the functions $u_1(s)$, $v_1(s)$, $u_2(s)$ and $v_2(s)$ are defined by (14) *and* (15), *and p_1, q_1, p_2, q_2 by* (10).

In order to show that these formulæ are independent of the order in which the two differentiations are performed, it is sufficient to prove

$$(19) \qquad \frac{F_1(s_1)}{u_1(s_2)} + \frac{F_1(s_2)}{u_2(s_1)} = 0.$$

We know that $u_2(s_2) = 0$, and since P_2 and P_1 are not conjugate, that $u_1(s_2) \neq 0$. Accordingly $u_1(s)$ and $u_2(s)$ are linearly independent solutions of Jacobi's equation, which we now write in the form:

$$\omega'' - \frac{F_1'}{F_1}\omega' + \frac{F_2}{F_1}\omega = 0.$$

Hence, by Abel's theorem,*

$$u_1(s)u_2'(s) - u_2(s)u_1'(s) = \frac{C}{F_1(s)}.$$

By applying (14) for $i = 1$ and $i = 2$, we obtain

$$-u_2(s_1) = \frac{C}{F_1(s_1)}, \qquad u_1(s_2) = \frac{C}{F_1(s_2)},$$

from which (19) follows immediately.

We conclude this paragraph by *establishing a relation between the functions* $u_i(s)$, $v_i(s)$ *and the Weierstrassian function* $\Theta(s, s_i)$ which is in current use in the literature. If $\vartheta_1(s)$ and $\vartheta_2(s)$ are any two linearly independent solutions of Jacobi's equation, we have†

$$\Theta(s, s_i) = \vartheta_1(s)\vartheta_2(s_i) - \vartheta_1(s_i)\vartheta_2(s).$$

Hence

$$\Theta(s_i, s_i) = 0,$$

$$\frac{\partial}{\partial s}\Theta(s, s_i)\Big|_{s=s_i} = \vartheta_1'(s_i)\vartheta_2(s_i) - \vartheta_1(s_i)\vartheta_2'(s_i),$$

$$\frac{\partial}{\partial s_i}\Theta(s, s_i)\Big|_{s=s_i} = \vartheta_1(s_i)\vartheta_2'(s_i) - \vartheta_1'(s_i)\vartheta_2(s_i),$$

$$\frac{\partial^2}{\partial s\,\partial s_i}\Theta(s, s_i)\Big|_{s=s_i} = \vartheta_1'(s_i)\vartheta_2'(s_i) - \vartheta_1'(s_i)\vartheta_2'(s_i).$$

Writing now

$$D(s) = \vartheta_1(s)\vartheta_2'(s) - \vartheta_1'(s)\vartheta_2(s),\text{‡}$$

$$(20) \qquad u_i(s) = -\frac{\Theta(s, s_i)}{D(s_i)}, \qquad v_i(s) = \frac{\frac{\partial}{\partial s_i}\Theta(s, s_i)}{D(s_i)},$$

we have

$$u_i(s_i) = 0, \qquad v_i(s_i) = 1,$$

$$u_i'(s_i) = 1, \qquad v_i'(s_i) = 0.$$

The functions $u_i(s)$ and $v_i(s)$ being uniquely determined by (14) and (15), it follows that those defined in (20) are identical with them.

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* Compare STURM, *Cours d' Analyse*, 12th edition, vol. 2, § 609.

† Compare BOLZA, loc. cit., p. 233.

‡ Ibid.

§ 5. *The case of one variable end-point.*

We suppose that we have found an extremal $\mathfrak{E}$ which actually furnishes a minimum for (1) when the first end-point P_1 is movable along the fixed curve $\mathfrak{C}$ defined by the equations

$$\mathfrak{C}: \qquad x = \tilde{x}(a), \qquad y = \tilde{y}(a) \qquad\qquad (a_0 \leqq a \leqq a_2),$$

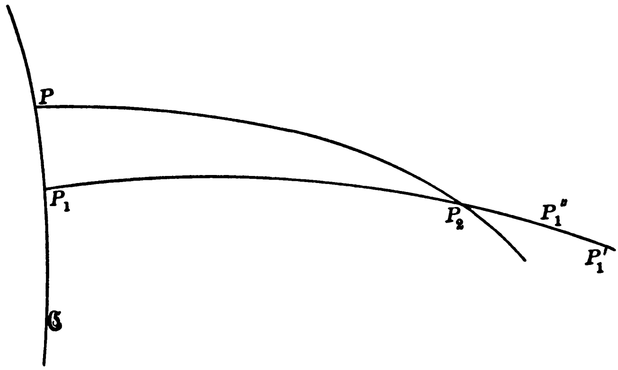

FIG. 1.

the second end-point P_2 being fixed. If a point P is taken on $\mathfrak{C}$ sufficiently near P_1, the construction of the unique extremal PP_2 can be carried out as described in § 1 and the extremal-integral can be computed along PP_2. *This extremal-integral becomes now a function of the parameter a of the point P,*

$$\mathfrak{J}\,[\tilde{x}(a), \tilde{y}(a), x_2, y_2] = \mathfrak{J}[a],$$

and must be a minimum for $a = a_1$, if P_1P_2 actually minimizes (1). Hence the necessary conditions for a minimum of (1) are in this case

$$\mathfrak{J}'[a_1] = 0 \tag{21}$$

and

$$\mathfrak{J}''[a_1] \geqq 0. \tag{22}$$

Making use of (11) and writing

$$\tilde{p}_1 = \frac{\tilde{x}'(a_1)}{\sqrt{\tilde{x}'^2(a_1) + \tilde{y}'^2(a_1)}}, \qquad \tilde{q}_1 = \frac{\tilde{y}'(a_1)}{\sqrt{\tilde{x}'^2(a_1) + \tilde{y}'^2(a_1)}}, \tag{23}$$

we find *

$$\mathfrak{J}'[a_1] = -F_{x'}(x_1, y_1, p_1, q_1)\tilde{p}_1 - F_{y'}(x_1, y_1, p_1, q_1)\tilde{q}_1;$$

this shows that (21) is nothing but the well-known *transversality-condition.* †

* BOLZA, loc. cit., § 38.

† Ibid, § 36.

Further, with the help of (18),

$$\mathfrak{J}''[a_1] = \tilde{p}_1^2[-L(s_1)+q_1^2F_1(s_1)Z_1(s_2)] + 2\tilde{p}_1\tilde{q}_1[-M(s_1)-p_1q_1F_1(s_1)Z_1(s_2)]$$
$$+\tilde{q}_1^2[-N(s_1)+p_1^2F_1(s_1)Z_1(s_2)] - \tilde{x}_1''F_{x'}(s_1) - \tilde{y}_1''F_{y'}(s_1),$$

where

$$\tilde{x}_1'' = \tilde{x}''(s_1), \qquad \tilde{y}_1'' = \tilde{y}''(s_1)$$

and

$$Z_i(s) = \frac{v_i(s)}{u_i(s)}, \qquad (i=1,2) \tag{24}$$

Introducing further the abbreviations

$$(25)\quad \begin{aligned} A_1 &= \tilde{p}_1^2L(s_1) + 2\tilde{p}_1\tilde{q}_1M(s_1) + \tilde{q}_1^2N(s_1) + \tilde{x}_1''F_{x'}(s_1) + \tilde{y}_1''F_{y'}(s_1) \\ B_1 &= F_1(s_1)(\tilde{p}_1q_1 - p_1\tilde{q}_1)^2, \end{aligned}$$

both A_1 and B_1 being constants depending upon the curve $\mathfrak{C}$ and the point P_1, we obtain the following formula:*

$$\mathfrak{J}''[a_1] = -A_1 + B_1Z_1(s_2).$$

A further necessary condition for a minimum is therefore

$$-A_1 + B_1Z_1(s_2) \geqq 0,$$

or, since $B_1 > 0$,†

$$Z_1(s_2) \geqq \frac{A_1}{B_1}.$$

We have defined by equation (24),

$$Z_1(s) = \frac{v_1(s)}{u_1(s)}.$$

Hence

$$Z_1'(s) = \frac{u_1(s)v_1'(s) - v_1(s)u_1'(s)}{u_1^2(s)}.$$

But $u_1(s)$ and $v_1(s)$ being linearly independent solutions of Jacobi's equation,‡ we have by Abel's theorem,§

$$u_1(s)v_1'(s) - v_1(s)u_1'(s) = \frac{k}{F_1(s)}.$$

By means of (14) and (15) we find

$$u_1(s_1)v_1'(s_1) - v_1(s_1)u_1'(s_1) = -1.$$

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* Compare Bolza, loc. cit., § 39; Bliss, Transactions of the American Mathematical Society, vol. 3 (1902), p. 136.

† Leaving aside the case that $\mathfrak{C}$ and $\mathfrak{C}$ are tangent at P_1.

‡ See § 3.

§ See the first footnote on p. 473.

Therefore

$$k = -F_1(s_1),$$

and

$$u_1(s)v_1'(s) - v_1(s)u_1'(s) = -\frac{F_1(s_1)}{F_1(s)}. \tag{26}$$

Hence

$$Z_1'(s) = -\frac{F_1(s_1)}{F_1(s)u_1^2(s)} < 0. \tag{27}$$

Further, it is evident that

$$\lim_{s \doteq s_1+0} Z_1(s) = +\infty,$$

and hence by Sturm's theorem, *

$$\lim_{s \doteq s_1'-0} Z_1(s) = -\infty,$$

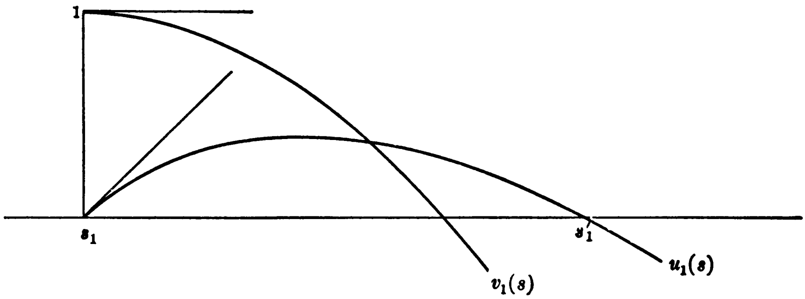

FIG. 2.

s_1' being the parameter value of the conjugate point of P_1 on $\mathfrak{C}$. We conclude that $Z_1(s)$ *is a monotonic decreasing function, taking every real value once and but once, as s increases from* s_1 *to* s_1'. Consequently there must be one real value of s between s_1 and s_1' for which

$$Z_1(s) = \frac{A_1}{B_1}.$$

Denoting this value by s_1'', it follows from (27) that *we must have*

$$s_2 \leqq s_1'', \tag{28}$$

in order that condition (22) *may be fulfilled.*

Thus we have given a new proof for Bliss's condition. BLISS † has investigated the geometrical meaning of the point determined on $\mathfrak{C}$ by s_1''. The properties of this so-called focal point have also been discussed by BOLZA.‡

* See STURM, *Cours d'Analyse*, 12th ed., vol. 2, no. 609.

† Transactions of the American Mathematical Society, vol. 3 (1902), p. 136.

‡ Loc. cit., § 39c.

§ 6. *The case of two variable end-points.*

In the same manner we treat now the case in which both end-points P_1 and P_2 are movable along two fixed curves, $\overline{\mathfrak{C}}$ and $\overline{\overline{\mathfrak{C}}}$ respectively. Let these curves be represented by

$$\overline{\mathfrak{C}}: \qquad x = \tilde{x}(a), \qquad y = \tilde{y}(a) \qquad\qquad (a_0 \leqq a \leqq a_2),$$

and

$$\overline{\overline{\mathfrak{C}}}: \qquad x = \breve{x}(b), \qquad y = \breve{y}(b) \qquad\qquad (b_0 \leqq b \leqq b_3),$$

and let us suppose that we had found an extremal $\mathfrak{E}$ actually furnishing a minimum for (1). Then if $P(a)$ and $Q(b)$ are taken sufficiently near P_1 and P_2

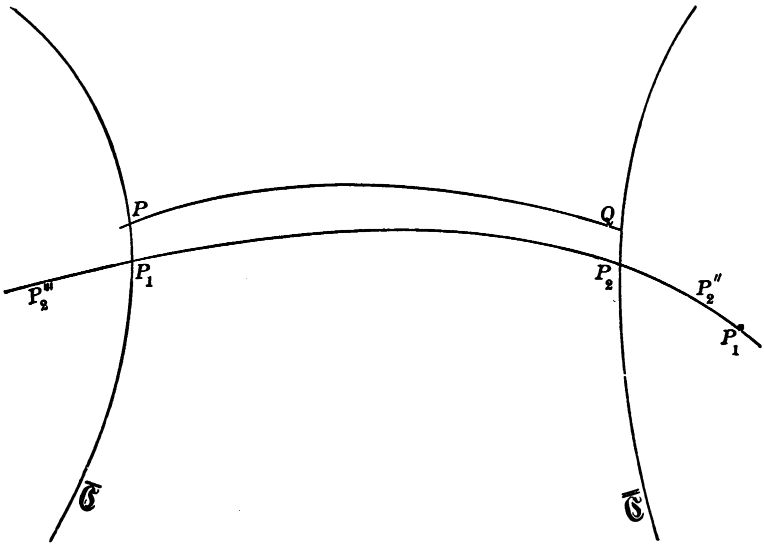

FIG. 3.

on $\overline{\mathfrak{C}}$ and $\overline{\overline{\mathfrak{C}}}$ respectively, the unique extremal PQ can be constructed and *the extremal-integral obtained now is a function of a and b*,

$$\mathfrak{J}\,[\tilde{x}(a), \tilde{y}(a), \breve{x}(b), \breve{y}(b)] = \mathfrak{J}\,[a, b].$$

This function must be a minimum for $a = a_1$, $b = b_2$, if $P_1 P_2$ actually minimizes (1).

The necessary conditions for $\mathfrak{J}(a_2, b_2)$ *to be a minimum are*

$$(29) \qquad \frac{\partial \mathfrak{J}[a, b]}{\partial a} = 0, \qquad \frac{\partial \mathfrak{J}[a, b]}{\partial b} = 0,$$

$$(30) \qquad \frac{\partial^2 \mathfrak{J}[a, b]}{\partial a^2}\,\xi^2 + 2\,\frac{\partial^2 \mathfrak{J}[a, b]}{\partial a\,\partial b}\,\xi\eta + \frac{\partial^2 \mathfrak{J}[a, b]}{\partial b^2}\,\eta^2 \geqq 0,$$

for all real values of ξ *and* η.*

* See C. JORDAN, *Cours d'Analyse*, 2d ed., vol. 1, §§ 395–401.

As in § 5, equations (29) lead to the *transversality-conditions*

$$F_{x'}(x_1, y_1, p_1, q_1)\breve{p}_1 + F_{y'}(x_1, y_1, p_1, q_1)\breve{q}_1 = 0,$$
$$F_{x'}(x_2, y_2, p_2, q_2)\breve{p}_2 + F_{y'}(x_2, y_2, p_2, q_2)\breve{q}_2 = 0,$$

$\breve{p}_1$ and $\breve{q}_2$ being defined by formulæ analogous to (23).

Further, proceeding as in § 5, we find

$$\frac{\partial^2 \mathfrak{J}}{\partial a^2} = -A_1 + B_1 Z_1(s_2),$$

$$\frac{\partial^2 \mathfrak{J}}{\partial b^2} = A_2 - B_2 Z_2(s_1),$$

(31)
$$\frac{\partial^2 \mathfrak{J}}{\partial a\, \partial b} = F_1(s_2)\frac{(p_1\breve{q}_1 - \breve{p}_1 q_1)(p_2\breve{q}_2 - \breve{p}_2 q_2)}{u_2(s_1)}$$

$$= -F_1(s_1)\frac{(p_1\breve{q}_1 - \breve{p}_1 q_1)(p_2\breve{q}_2 - \breve{p}_2 q_2)}{u_1(s_2)},$$

where the last two expressions are equivalent on account of (19); where also A_1, B_1, $Z_1(s)$ are defined by (25) and (24), and A_2, B_2, $Z_2(s)$ by analogous formulæ, obtained from them by a change of index.

Condition (30) will be fulfilled if *

(32)
$$\frac{\partial^2 \mathfrak{J}}{\partial a^2} \geqq 0$$

and

(33)
$$\frac{\partial^2 \mathfrak{J}}{\partial a^2}\frac{\partial^2 \mathfrak{J}}{\partial b^2} - \left(\frac{\partial^2 \mathfrak{J}}{\partial a\, \partial b}\right)^2 \geqq 0,$$

from which follows

(32a)
$$\frac{\partial^2 \mathfrak{J}}{\partial b^2} \geqq 0.$$

As in § 5, the relation (32) leads at once to the inequality

(28)
$$s_2 \leqq s_1''.$$

In the same manner we could show that (32*a*) leads to the relation

$$s_2''' \leqq s_1,$$

s_2''' being defined by the equation

(34)
$$Z_2(s) = \frac{A_2}{B_2} \qquad (s < s_1).$$

It can furthermore easily be shown that there is also a value of s beyond s_2, for which (34) is fulfilled. This value we denote by s_2''.

For the further discussion of (33) we introduce now s_1'' and s_2'' by means of the relations

$$A_1 = B_1 Z_1(s_1''), \qquad A_2 = B_2 Z_2(s_2'').$$

* See C. JORDAN, ibid.

Then (31) becomes

$$\frac{\partial^2 \mathfrak{J}}{\partial a^2} = B_1 \{ Z_1(s_2) - Z_1(s_1'') \},$$

(35)
$$\frac{\partial^2 \mathfrak{J}}{\partial b^2} = B_2 \{ Z_2(s_2'') - Z_2(s_1) \},$$

$$\left(\frac{\partial^2 \mathfrak{J}}{\partial a \partial b}\right)^2 = - \frac{B_1 B_2}{u_1(s_2) u_2(s_1)}.$$

We desire to express the functions of s with subscript 2 in terms of those with subscript 1. For this purpose we write

$$u_2(s) = \alpha_{11} u_1(s) + \alpha_{12} v_1(s), \qquad v_2(s) = \alpha_{21} u_1(s) + \alpha_{22} v_1(s).$$

Then using (14) and (15) we can determine α_{ij}, and we find by making use of (26),

(36)
$$u_2(s) = \frac{F_1(s_2)}{F_1(s_1)} \{ u_1(s) v_1(s_2) - u_1(s_2) v_1(s) \},$$

$$v_2(s) = - \frac{F_1(s_2)}{F_1(s_1)} \{ u_1(s) v_1'(s_2) - u_1'(s_2) v_1(s) \}.$$

Consequently

$$Z_2(s) = - \frac{u_1(s) v_1'(s_2) - u_1'(s_2) v_1(s)}{u_1(s) v_1(s_2) - u_1(s_2) v_1(s)},$$

and in particular

$$Z_2(s_1) = - \frac{u_1'(s_2)}{u_1(s_2)}, \qquad Z_2(s_2'') = - \frac{v_1'(s_2) - u_1'(s_2) Z_1(s_2'')}{v_1(s_2) - u_1(s_2) Z_1(s_2'')}.$$

From (36) or from (19) it follows that

$$u_2(s_1) = - \frac{F_1(s_2)}{F_1(s_1)} u_1(s_2).$$

We can also write

$$\frac{\partial^2 \mathfrak{J}}{\partial b^2} = B_2 \frac{F_1(s_1)}{F_1(s_2) u_1(s_2) \{ v_1(s_2) - u_1(s_2) Z_1(s_2'') \}}$$

Hence, since B_1, B_2, $F_1(s_1)$, $F_1(s_2)$ are all > 0, the relation (33) becomes

$$\frac{Z_1(s_2) - Z_1(s_1'')}{u_1(s_2) \{ v_1(s_2) - u_1(s_2) Z_1(s_2'') \}} - \frac{1}{u_1^2(s_2)} \geqq 0$$

or

$$\frac{Z_1(s_2'') - Z_1(s_1'')}{Z_1(s_2) - Z_1(s_2'')} \geqq 0.$$

Since s_2'' was by definition beyond s_2, *this leads, in view of the relation* (27), *to the condition*

$$s_2'' < s_1'',$$

a result which has previously been obtained by BLISS,* who also pointed out its geometrical interpretation.

§ 7. *Discontinuous solutions.*

We propose next to investigate under what conditions a curve which has a finite discontinuity in its slope, a so-called discontinuous solution, may minimize the integral (1). We suppose that we have a broken curve $P_1P_0P_2$ actually minimizing (1). We know then from the current theory that each one of its branches must be an extremal along which Legendre's and Jacobi's conditions for ordinary extremals must be satisfied.† Taking now a point P sufficiently near P_0, we can construct uniquely the extremals P_1P and PP_2, the first one by identifying P with the point P_2 of § 1, the second one by identifying

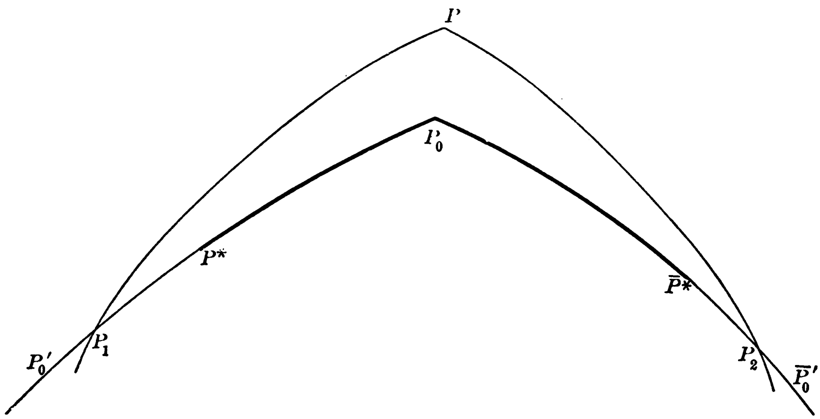

FIG. 4.

P with the point P_1 of § 1. Computing (1) along each of these extremals, we obtain the extremal-integral along the broken curve P_1PP_2 as a function of the coördinates x, y of P:

$$\mathfrak{J}(x_1, y_1, x, y) + \mathfrak{J}(x, y, x_2, y_2) = \mathfrak{J}\{x, y\}.$$

This function is to be a minimum for $x = x_0$, $y = y_0$, if $P_1P_0P_2$ is actually to furnish a minimum for (1).

We suppose that $P_1P_0P_2$ is represented in the form

$$x = x(s), \qquad y = y(s), \qquad s_1 \leqq s \leqq s_0,$$

$$x = \bar{x}(s), \qquad y = \bar{y}(s), \qquad s_0 \leqq s \leqq s_2,$$

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* Mathematische Annalen, vol. 58 (1904), p. 74.

† Compare BOLZA, loc. cit., § 48*a*.

and introduce the notation

$$\phi[x(s), y(s), x'(s), y'(s)] \equiv \phi(s), \quad \phi[\bar{x}(s), \bar{y}(s), \bar{x}'(s), \bar{y}'(s)] \equiv \bar{\phi}(s),$$

ϕ being any function of x, y, x', y', subject to the ordinary continuity-restrictions.

In order to be able to apply the results of §§ 1–4, we must identify, throughout the discussion, P_0 with P_2 of § 1 when we consider P_0 as a point of P_1P_0, and P_0 with P_1 of § 1 when we consider P_0 as a point of P_0P_2. So, for instance, $u_2(s)$ goes over into $u_0(s)$, whereas $u_1(s)$ becomes $\bar{u}_0(s)$, etc.

With this agreement, we proceed to establish necessary conditions for a minimum of $\mathfrak{J}\{x, y\}$. *These necessary conditions are derived from the well-known relations*

$$(37) \qquad \frac{\partial \mathfrak{J}\{x_0, y_0\}}{\partial x} = 0, \qquad \frac{\partial \mathfrak{J}\{x_0, y_0\}}{\partial y} = 0,$$

and

$$(38) \qquad \frac{\partial^2 \mathfrak{J}\{x_0, y_0\}}{\partial x^2}\xi^2 + 2\frac{\partial^2 \mathfrak{J}\{x_0, y_0\}}{\partial x \partial y}\xi\eta + \frac{\partial^2 \mathfrak{J}\{x_0, y_0\}}{\partial y^2}\eta^2 \leqq 0.$$

which must hold for all real values of ξ and η.

We obtain, by using (11),

$$\frac{\partial \mathfrak{J}\{x_0, y_0\}}{\partial y} = F_{x'}(s_0) - \bar{F}_{x'}(s_0), \qquad \frac{\partial \mathfrak{J}\{x_0, y_0\}}{\partial y} = F_{y'}(s_0) - \bar{F}_{y'}(s_0),$$

whence we conclude that (37) *is identical with the Erdmann-Weierstrass corner condition.* *

Further, by using (18), the condition (38) becomes

$$- (A_0\xi^2 + 2B_0\xi\eta + C_0\eta^2) + \bar{F}_1(s_0)\bar{Z}_0(s_2)(\xi\bar{q}_0 - \eta\bar{p}_0)^2$$
$$- F_1(s_0)Z_0(s_1)(\xi q_0 - \eta p_0)^2 \geqq 0,$$

where

$$A_0 = \bar{L}(s_0) - L(s_0), \qquad B_0 = \bar{M}(s_0) - M(s_0), \qquad C_0 = \bar{N}(s_0) - N(s_0).$$

By means of the transformation

$$(39) \qquad \xi = p_0\xi' - \bar{p}_0\eta', \qquad \eta = q_0\xi' - \bar{q}_0\eta',$$

the above homogeneous quadratic form goes over into

$$P_0(s_2)\xi'^2 + 2\Omega_0\xi'\eta' + R_0(s_1)\eta'^2,$$

where

$$P_0(s_2) = -T_0 + \bar{F}_1(s_0)\bar{Z}_0(s_2)(p_0\bar{q}_0 - \bar{p}_0q_0)^2,$$

$$R_0(s_1) = -\bar{T}_0 - F_1(s_0)Z_0(s_1)(p_0\bar{q}_0 - \bar{p}_0q_0)^2.$$

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* Compare BOLZA, loc. cit., § 48*b*.

$$\begin{aligned}(40)\qquad \Omega_0 &= A_0 p_0 \bar{p}_0 + 2B_0(p_0 \bar{q}_0 + \bar{p}_0 q_0) + C_0 q_0 \bar{q}_0,^* \\ T_0 &= A_0 p_0^2 + 2B_0 p_0 q_0 + C_0 q_0^2, \\ \bar{T}_0 &= A_0 \bar{p}_0^2 + 2B_0 \bar{p}_0 \bar{q}_0 + C_0 \bar{q}_0^2.\end{aligned}$$

In passing, we notice the relation

$$(41)\qquad \Omega_0^2 = T_0 \bar{T}_0 - (A_0 C_0 - B_0^2)(p_0 \bar{q}_0 - \bar{p}_0 q_0)^2.$$

Moreover the determinant $p_0 \bar{q}_0 - \bar{p}_0 q_0$ of the system (39) is not zero if

$$\bar{\theta}_0 - \theta_0 \not\equiv 0 \pmod{2\pi},$$

i. e., if there is a corner at P_0. *The conditions for a minimum may therefore be stated as follows:* †

I. *If* $\Omega_0 = 0$, *necessary conditions are*

$$(42)\qquad P_0(s_2) \geqq 0,$$

$$(43)\qquad R_0(s_1) \geqq 0.$$

II. *If* $\Omega_0 \neq 0$, *we have as necessary conditions*

$$(42a)\qquad P_0(s_2) > 0,$$

$$(44)\qquad P_0(s_2) R_0(s_1) - \Omega_0^2 \geqq 0.$$

Conditions (42a) and (44) have as a consequence

$$(43a)\qquad R_0(s_1) > 0.$$

I. Since $\bar{Z}_0(s)$ is identical with $Z_1(s)$ of § 5, we conclude immediately that *there exists between* s_0 *and* $\bar{s}_0'$ *one and only one value* $\bar{s}^*$ *for which* $P_0(s) = 0$, ‡ *and that a necessary condition for a minimum is*

$$(45)\qquad s_2 \leqq \bar{s}^*.$$

The equation $P_0(s) = 0$ *satisfied by* $\bar{s}^*$ *can be written in the form*

$$(46)\qquad \bar{Z}_0(\bar{s}^*) = \frac{T_0}{\bar{F}_1(s_0)(p_0 \bar{q}_0 - \bar{p}_0 q_0)^2}.$$

In the same manner, *the relation* (43) *leads to the necessary condition*

$$(47)\qquad s_1 \geqq s^*,$$

* This function Ω_0 is the invariant, introduced by CARATHEODORY in a different form (see *Dissertation*, p. 31 and Mathematische Annalen, vol. 62 (1906), p. 473), to which the present one can be easily reduced by means of the relations between L, M, N and the derivatives of F, as well as the homogeneity properties of F. Compare BOLZA, loc. cit., § 49, and American Journal of Mathematics, vol. 30 (1908), pp. 212 and 214.

† Compare footnote on page 477.

‡ The symbol $\bar{s}_0'$ denotes the parameter value conjugate to s_0 on P_0P_2, i. e., the root of $\bar{u}_0(s) = 0$ which follows next after s_0.

s^* being defined by the equation

$$Z_0(s^*) = \frac{-\overline{T}_0}{F_1(s_0)(p_0\bar{q}_0 - \bar{p}_0 q_0)^2}. \tag{48}$$

The parameter values s^* and $\bar{s}^*$ define points P^* and $\bar{P}^*$ on P_1P_0 and P_0P_2 respectively (see Fig. 4). Our result is, then, *that the end-points of a minimizing "broken extremal" must lie on the arc bounded by these two critical points.**

II. In the first place, *conditions* (45) *and* (47) *must be fulfilled in the stronger forms*

$$s_2 < \bar{s}^*, \tag{45a}$$

and

$$s_1 > s^*. \tag{47a}$$

Secondly, from the properties of $Z_0(s)$ and $\overline{Z}_0(s)$ it follows that if we consider P_1 as fixed, a point $\overline{P}_1$ is uniquely determined on P_0P_2; or if we consider P_2 as fixed, a point $\overline{P}_2$ is uniquely determined on P_1P_0, by the relation

$$P_0(s_2)R_0(s_1) - \Omega_0^2 = 0. \tag{49}$$

Furthermore, in order to have (44) *fulfilled, we must have*

$$s_2 \leqq \bar{s}_1 (\text{or } s_1 \geqq \bar{s}_2), \tag{50}$$

where $\bar{s}_1$, and $\bar{s}_2$ are the parameter values of $\overline{P}_1$ and $\overline{P}_2$ respectively.†

Summarizing, we obtain for this case the following

THEOREM. *If the inequality*

$$\Omega_0 = A_0p_0\bar{p}_0 + B_0(p_0\bar{q}_0 + \bar{p}_0q_0) + C_0q_0\bar{q}_0 \neq 0$$

holds, then necessary conditions for a minimum of the integral (1) *are*

$$s_1 > s^* \qquad \text{and} \qquad s_0 \leqq s_2 < \bar{s}_1;$$

if $\Omega_0 = 0$, it is necessary that

$$s_1 \geqq s^* \qquad \text{and} \qquad s_2 \leqq \bar{s}^*.$$

The relation (49) connecting s_1 with $\bar{s}_1$ (and s_2 with $\bar{s}_2$) may be written in explicit form by means of (40) and (41). We find that

$$\begin{aligned} A_0C_0 - B_0^2 &+ F_1(s_0)(A_0p_0^2 + 2B_0p_0q_0 + C_0q_0^2)Z_0(s_1) \\ &- \overline{F}_1(s_0)(A_0\bar{p}_0^2 + 2B_0\bar{p}_0\bar{q}_0 + C_0\bar{q}_0^2)\overline{Z}_0(\bar{s}_1) \\ &- F_1(s_0)\overline{F}_1(s_0)(p_0\bar{q}_0 - \bar{p}_0q_0)^2 Z_0(s_1)\overline{Z}_0(\bar{s}_1) = 0. \end{aligned} \tag{49c}$$

The relation occurs in this form in Bolza's work.‡

* For the geometrical interpretation of these points, see BOLZA, loc. cit., § 49*a*, and American Journal of Mathematics, vol. 30 (1908), p. 217; also CARATHEODORY, *Dissertation*, p. 31.

† For the geometrical interpretation of $\overline{P}_1$ and $\overline{P}_2$, see ibid.

‡ See the first footnote on p. 482.

We see that the point $\overline{P}_1$ plays the rôle in the theory of discontinuous extremals which the conjugate point plays in the theory of continuous extremals. For this reason $\overline{P}_1$ is called the *conjugate point* of P_1 on P_0P_2.

In order to show that conditions (47a) and (50) have (45a) as their consequence, we must prove that

$$(51) \qquad s_0 < \bar{s}_1 < \bar{s}^*, \qquad \text{if} \qquad s_0 > s_1 > s^*.$$

Indicating the functional dependence of $\bar{s}_1$ on s_1 by means of the equation

$$\bar{s}_1 = s_2(s_1),$$

which is implicitly contained in (49a), we can easily show by means of (46) and (48) that

$$s_2(s^*) = s_0, \qquad s_2(s_0) = \bar{s}^*,$$

and furthermore, by using (41) and (27), that

$$s_2'(s_1) = \frac{F_1(s_0)Z_0'(s_1)\Omega_0^2}{\overline{F}(s_0)\overline{Z}_0'[s_2(s_1)]R_c^2(s_1)} > 0,$$

from which the inequalities (51) follow immediately.

§ 8. *Contradiction with previous results.*

The theorem stated in the preceding paragraph is in direct contradiction with results previously obtained by CARATHEODORY * and BOLZA,† who give *sufficient conditions for a minimum, less restricting than the necessary conditions arrived at here.* We shall briefly state the contradiction.

According to BOLZA any one of the combinations (AIa, B), (AIb, B), (AII, B) from the following set are sufficient conditions for a minimum:

$$\begin{aligned} A.\ \text{I}a:&\ \Omega_0 > 0, \qquad P_0' < P_1 < P^*, \qquad P_0 < P_2 < \overline{P}_1;\\ \text{I}b:&\ \Omega_0 > 0, \qquad P^* < P_1 < P_0, \qquad P_0 < P_2 < \overline{P}_0';\\ \text{II}:&\ \Omega_0 < 0, \qquad P^* < P_1 < P_0, \qquad P_0 < P_2 < \overline{P}_1. \end{aligned}$$

$$B. \begin{cases} E(x, y;\, x', y';\, \tilde{x}', \tilde{y}') > 0 \text{ on } \mathfrak{E} \text{ for } s_1 \leqq s \leqq s_0,\ \tilde{\theta} \neq \theta, \\ \qquad\qquad \text{except possibly for } s = s_0,\ \tilde{\theta} = \bar{\theta}_0; \\ E(\bar{x}, \bar{y};\, \bar{x}', \bar{y}';\, \tilde{x}', \tilde{y}') > 0 \text{ on } \overline{\mathfrak{E}} \text{ for } s_0 \leqq s \leqq s_2,\ \tilde{\theta} \neq \bar{\theta}, \\ \qquad\qquad \text{except possibly for } s = s_0,\ \tilde{\theta} = \theta_0. \end{cases}$$

On the other hand, we have found above the following necessary conditions:

$$\Omega_0 \neq 0, \qquad P^* < P_1 \leqq P_0, \qquad P_0 < P_2 \leqq \overline{P}_1.$$

* *Dissertation*, pp. 31 and 32, where no explicit conditions are stated, but the implication is made that $\overline{P}_1$ need not always be the bound for minimizing extremals.

† *Vorlesungen*, § 50.

It is evident that there is *accord for* $\Omega_0 < 0$, *but contradiction for* $\Omega_0 > 0$.

We investigate now the behavior of the E-function in the neighborhood of P_0. We know that

$$E(x, y;\, x', y';\, \tilde{x}', \tilde{y}') = F(x, y, \tilde{x}', \tilde{y}') - \tilde{x}' F_{x'}(x, y, x', y') - \tilde{y}' F_{y'}(x, y, x', y').^{*}$$

Writing

$$E[x(s), y(s);\, x'(s), y'(s);\, \cos\tilde{\theta}, \sin\tilde{\theta}] \equiv E(s;\, \tilde{\theta}),$$

$$E[\bar{x}(s), \bar{y}(s);\, \bar{x}'(s), \bar{y}'(s);\, \cos\tilde{\theta}, \sin\tilde{\theta}] \equiv \bar{E}(s;\, \tilde{\theta}),$$

$$\phi[x(s), y(s), \tilde{x}', \tilde{y}'] \equiv \tilde{\phi}(s),$$

$$\phi[\bar{x}(s), \bar{y}(s), \tilde{x}', \tilde{y}'] \equiv \bar{\tilde{\phi}}(s),$$

we derive the equations

$$E_s(s;\, \tilde{\theta}) = x'\tilde{F}_x(s) + y'\tilde{F}_y(s) - \tilde{x}'\frac{d}{ds}F_{x'} - \tilde{y}'\frac{d}{ds}F_{y'},$$

$$\bar{E}_s(s;\, \tilde{\theta}) = \bar{x}'\bar{\tilde{F}}_x(s) + \bar{y}'\bar{\tilde{F}}_y(s) - \tilde{x}'\frac{d}{ds}\bar{F}_{x'} - \tilde{y}'\frac{d}{ds}\bar{F}_{y'}.$$

But $x(s)$, $y(s)$ and $\bar{x}(s)$, $\bar{y}(s)$ representing extremals, we have

$$\frac{d}{ds}F_{x'} = F_x, \qquad \frac{d}{ds}F_{y'} = F_y, \qquad \frac{d}{ds}\bar{F}_{x'} = \bar{F}_x, \qquad \frac{d}{ds}\bar{F}_{y'} = \bar{F}_y.$$

Hence,

$$E_s(s;\, \tilde{\theta}) = x'\tilde{F}_x(s) + y'\tilde{F}_y(s) - \tilde{x}'F_x(s) - \tilde{y}'F_y(s),$$

$$\bar{E}_s(s;\, \tilde{\theta}) = \bar{x}'\bar{\tilde{F}}_x(s) + \bar{y}'\bar{\tilde{F}}_y(s) - \tilde{x}'\bar{F}_x(s) - \tilde{y}'\bar{F}_y(s),$$

and therefore

$$E_s(s_0;\, \bar{\theta}_0) = \Omega_0, \qquad \bar{E}(s_0;\, \theta_0) = -\,\Omega_0.\dagger \tag{52}$$

Further, CARATHEODORY ‡ has shown that the Erdmann-Weierstrass corner conditions are equivalent to

$$\left\{\begin{array}{l} E(s_0;\, \bar{\theta}_0) = 0, \\ \dfrac{\partial}{\partial\tilde{\theta}}E(s_0;\, \tilde{\theta}_0) = 0, \end{array}\right. \qquad \text{or} \qquad \left\{\begin{array}{l} \bar{E}(s_0;\, \theta_0) = 0, \\ \dfrac{\partial}{\partial\tilde{\theta}}\bar{E}(s_0;\, \theta_0) = 0. \end{array}\right. \tag{53}$$

We expand now $E(s;\, \tilde{\theta})$ by Taylor's expansion at $s = s_0$, $\tilde{\theta} = \bar{\theta}_0$, and find

$$\begin{aligned} E(s;\, \tilde{\theta}) &= E(s_0;\, \tilde{\theta}_0) + \frac{s - s_0}{1}\Omega_0 + \frac{\tilde{\theta} - \theta_0}{1}E_{\tilde{\theta}}(s_0;\, \bar{\theta}_0) + \cdots, \\ \bar{E}(s;\, \tilde{\theta}) &= \bar{E}(s_0;\, \theta_0) - \frac{s - s_0}{1}\Omega_0 + \frac{\tilde{\theta} - \theta_0}{1}E_{\tilde{\theta}}(s_0;\, \theta_0) + \cdots. \end{aligned} \tag{54}$$

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* BOLZA, loc. cit., p. 243.

† This is the second form of Ω_0, referred to in the first footnote on p. 482.

‡ *Dissertation*, p. 8.

It follows that *if the corner conditions are fulfilled and* $\Omega_0 > 0$ *at the point* $s = s_0$, *then* $E(s; \tilde{\theta})$ *takes the sign of* $s - s_0$, *while* $\bar{E}(s; \tilde{\theta})$ *takes the sign of* $s_0 - s$, *for small values of* $|s - s_0|$.

Consequently, if $\Omega_0 > 0$, we have $E < 0$ in a vicinity of P_0 on both P_1P_0 and P_0P_2. This shows that *the sufficient conditions* AIa *and* AIb *are incompatible with* B. This *removes the contradiction referred to above.*

By means of (54) we can give furthermore a simple proof of a theorem previously proved by CARATHEODORY *:

"*If one follows a strong extremal* $\mathfrak{C}$ *up to a point* P_0 *where it ceases to be strong, and if at* P_0 *the invariant* Ω_0 *does not vanish* (*for some value* $\theta = \bar{\theta}_0$),† *then there is another extremal* $\bar{\mathfrak{C}}$ *passing through* P_0, *which begins to be strong at* P_0 *and which forms with* $\mathfrak{C}$ *a discontinuous solution of the problem.*"

If $\mathfrak{C}$ ceases to be strong at $P_0(s_0)$, then we must have the two relations

$$(55)\qquad \begin{aligned} &E(s; \tilde{\theta}) > 0 \qquad \text{for} \qquad s < s_0 \qquad (0 \leqq \tilde{\theta} \leqq 2\pi,\ \theta \neq \tilde{\theta}), \\ &E(s; \tilde{\theta}) < 0 \qquad \text{for} \qquad s > s_0, \end{aligned}$$

for at least one value of $\tilde{\theta}$, say $\bar{\theta}_0$, different from θ. From the continuity of E follows then

$$E(s_0; \bar{\theta}_0) = 0.$$

Hence by (54),

$$E(s; \tilde{\theta}) = \frac{s - s_0}{1}\Omega_0 + \frac{\tilde{\theta} - \theta_0}{1} E_{\tilde{\theta}}(s_0; \bar{\theta}_0) + \cdots.$$

Supposing for the moment $\Omega_0 < 0$, we see that if (55) is to be fulfilled, E must be of constant sign whenever $s - s_0$ keeps its sign.‡ But putting $\tilde{\theta} - \theta_0 = \lambda(s - s_0)$, we find

$$E(s; \tilde{\theta}) = (s - s_0)[\Omega_0 + \lambda E_{\tilde{\theta}}(s_0; \tilde{\theta}_0)] + \cdots,$$

from which it is evident that after the sign of $s - s_0$ is once fixed, E can be made positive as well as negative by a proper choice of λ, unless $E_{\tilde{\theta}}(s_0; \bar{\theta}_0) = 0$. Consequently, if all the hypotheses of the theorem are fulfilled, we may conclude that

$$E(s_0; \bar{\theta}_0) = 0, \qquad \frac{\partial}{\partial \tilde{\theta}} E(s_0; \bar{\theta}_0) = 0.$$

This shows that the corner condition must be fulfilled at the point P_0 by the direction $\bar{\theta}_0$. By repeating with respect to $\bar{E}$ the above argument the second part of Caratheodory's theorem may easily be proved.

We shall defer to a later paper an example showing the application of the results of the last two sections.

THE UNIVERSITY OF CHICAGO,
June, 1908.

* Mathematische Annalen, vol. 62 (1906), p. 473.

† The statement in parenthesis is mine.

‡ This discussion is valid only for a neighborhood of P_0, a limitation which does not interfere however with our argument.