PROJECTIVE DIFFERENTIAL GEOMETRY OF CURVED SURFACES*

(FIFTH MEMOIR)

BY

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Introduction.

On December 13, 1880, DARBOUX presented to the French Academy of Sciences a note on the contact between curves and surfaces, which contains some very important results.† One of these may be stated as follows: if we consider all of the sections of a surface made by planes which pass through the same tangent, the locus of the conics, which osculate these sections at their common point of contact with the tangent, is a quadric having contact of the second order with the given surface. Two of these osculating conics moreover hyperosculate the plane section to which they belong, i. e., have fifth order contact with it. Finally, there exist (in general) twenty-seven plane sections through a point of a given surface whose osculating conics have sixth order contact with them. DARBOUX also discussed an interesting problem of contact between a quadric and a given surface, and determined a STEINER surface which has fourth order contact with the given surface at a given point. He further proved that the quadric surfaces, the ruled surfaces of the third order and STEINER's Roman surface of the fourth order are the only ones through every point of which there passes an infinity of conics lying entirely on the surface. The other results in DARBOUX's paper are of a metrical character and do not concern us here.

Two weeks later MOUTARD‡ announced the fact that the same questions treated by DARBOUX had been discussed by him in two oral communications made to the *Société philomathique* about 1865. The second of these was concerned with metrical geometry and calls for no discussion in this place. Moreover no trace of it exists in print. The first however contained precisely the theorems mentioned above concerning the conics which have contact of the fourth, fifth and sixth order with a given surface at a given point,

* Presented to the Society (Chicago), January 1, 1909.

† Comptes Rendus, vol. 91 (1880), p. 969. The detailed discussion of these results is to be found in the Bulletin des Sciences Mathematiques et Astronomiques, 2d series, vol. 4 (1880), p. 348.

‡ Comptes Rendus, vol. 91 (1880), p. 1055.

together with some additional more special remarks which follow from an application of these results to surfaces of the third order. These theorems were communicated by MOUTARD to PONCELET in a letter, a part of which was reproduced by the latter in the second volume of his *Applications d'analyse et de géometrie.* A part of this again was used by CHASLES in 1870 in his *Rapport sur le progrès de la géometrie.* These important theorems discovered by MOUTARD in 1863, it seems, were practically lost until rediscovered independently by DARBOUX in 1880. A very careful search on my part has failed to show me any trace of these theorems in the standard treatises on the theory of surfaces, which have appeared since 1880, or in the "*Encyklopädie.*" Nor have I seen any mention of these results in the general literature of the subject. Consequently I thought them new when I rediscovered them in the spring of 1908, and announced them as such at the summer-meeting of the SOCIETY in Urbana. I shall henceforth speak of the first of the above mentioned results as the theorem of MOUTARD.

The method of proof which I have adopted is closely allied to that of DARBOUX. But there are several essential points of difference. The tetrahedron of reference chosen by me is one whose geometrical significance has been fully elucidated by my previous work, while that of DARBOUX, which is not the same, was introduced by him in a purely analytic fashion. Since, moreover, these theorems are necessary parts of any general treatment of the projective differential geometry of a curved surface, such as has been presented for the first time in these five memoirs, since other results closely related to them cannot be presented except in this connection, and since the modifications which DARBOUX's theorems undergo in certain important cases are also included in this treatment it becomes necessary, even at the expense of some repetition, to publish these new proofs. In this way the concepts of the MOUTARD-DARBOUX theory can be brought into organic connection with the theory developed by myself in the first four memoirs. Another method of establishing this connection would consist in obtaining the geometric characteristics of the DARBOUX tetrahedron of reference. This will be left for another paper, as will also the detailed discussion of the exceptional case when the point of the surface considered is parabolic. The case in which all points of the surface are parabolic, i. e., in which the surface is developable, shall however be included in this paper at least as far as the principal features of the theory are concerned. This enables us to apply our theorems in detail to all analytic surfaces, including developables, only special points of certain surfaces being excluded.

§ 1. *Darboux's tangents of quadric osculation.*

Let P be a point of a surface S, and let R be a non-vanishing portion of S, finite or infinite in extent, such that P is one of its interior points, and such that the homogeneous coördinates $x_1, \cdots, x_4$ of any point of R may be expressed as convergent series of positive integral powers of two parameters u and v. Moreover let the two asymptotic tangents of the surface at P be distinct. We shall then say that P is a *regular* point of the surface.

If we introduce non-homogeneous coördinates

$$x = \frac{x_1}{x_4}, \qquad y = \frac{x_2}{x_4}, \qquad z = \frac{x_3}{x_4},$$

these may be chosen in such a way that one of them, say z, assumes the form of a series proceeding according to positive integral powers of the other two, x and y. If, moreover, neither of the asymptotic tangents of P has more than three consecutive points in common with the surface at P, this series may by a projective transformation be put into the form

$$(1) \qquad z = xy + \tfrac{1}{6}(x^3 + y^3) + \tfrac{1}{24}(Ix^4 + Jy^4) + \cdots,\text{*}$$

where I, J, and all further coefficients are absolute invariants of the surface. Any non-ruled surface will admit of such a development in the vicinity of one of its regular points. DARBOUX † mentions this development but makes use of a different one, viz.:

$$(1a) \qquad z = xy + x^3 + y^3 + xy(ax^2 + by^2) + \cdots,$$

in the investigations mentioned in the introduction. The form (1) of the development is also derived by TRESSE ‡ but implicitly is already contained in a few remarks of HALPHEN's in his thesis on differential invariants.

The geometrical significance of the development (1), i. e., the geometrical definition of the tetrahedron of reference which gives rise to this development is explained in one of my previous papers.

Obviously the most general quadric involving three arbitrary constants α, β, γ, which has contact of the second order with S at P is given by an equation of the form

$$(2) \qquad z = xy + z(\alpha x + \beta y + \gamma z),$$

which gives the development for z

$$(3) \qquad z = xy + xy(\alpha x + \beta y) + xy(\alpha x + \beta y)^2 + \gamma x^2y^2 + \cdots.$$

The projection of the curve of intersection of (1) and (2) upon the xy plane will be

$$\tfrac{1}{6}(x^3 + y^3) - xy(\alpha x + \beta y) + \tfrac{1}{24}(Ix^4 + Jy^4) - xy[(\alpha x + \beta y)^2 + \gamma xy] + \cdots = 0.$$

This curve has a triple point at P for all values of α and β. The three tangents of this triple point will coincide if and only if

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* Second memoir. These Transactions, vol. 9 (1908), p. 103.

† DARBOUX, l. c., p. 353.

‡ A. TRESSE. *Sur les invariants différentiels des groupes continus de transformations*, Paris, 1893.

$$\alpha = -\tfrac{1}{2}, \quad \beta = -\tfrac{1}{2}; \quad \alpha = -\tfrac{1}{2}\theta, \quad \beta = -\tfrac{1}{2}\theta; \quad \alpha = -\tfrac{1}{2}\theta^2, \quad \beta = -\tfrac{1}{2}\theta^2;$$

where θ is an imaginary third root of unity, while γ is not determined at all by this condition.

We have, therefore, the following results due to DARBOUX. *The most general quadric Q which has contact of the second order with S at P contains three arbitrary constants. It is not possible to dispose of these in such a way as to have the three tangents of the triple point of the curve of intersection of S and Q coincide with an arbitrary line, tangent to S at P. If the three taugents of the triple point coincide at all, they will coincide with one of the three tangents of S given by the equation,*

$$x^3 + y^3 = 0. \tag{4}$$

Moreover, to each of these three tangents there corresponds a single infinity of quadrics of this kind.

These three tangents are called tangents of quadric osculation. They have recently been obtained by SEGRE* from a different point of view.

Obviously we may obtain in this way a family of curves on the surface S, three of which pass through every regular point. These curves DARBOUX denotes as the *lines of quadric osculation.*

§ 2. *The osculating Steiner surface.*

Let

$$x = \frac{\phi_1(\lambda, \mu)}{\phi_4(\lambda, \mu)}, \qquad y = \frac{\phi_2(\lambda, \mu)}{\phi_4(\lambda, \mu)}, \qquad z = \frac{\phi_3(\lambda, \mu)}{\phi_4(\lambda, \mu)}. \tag{5}$$

where

$$\phi_k = a_k\lambda^2 + 2h_k\lambda\mu + b_k\mu^2 + 2g_k\lambda + 2f_k\mu + c_k, \qquad (k=1, 2, 3, 4), \tag{6}$$

be the equations of a Steiner surface, λ and μ being the parameters which determine one of its points. It is well known † that any Steiner surface may be so represented. Since, moreover, the most general Steiner surface depends upon fifteen arbitrary constants and since the requirement that one surface shall have contact of the fourth order with a given one is equivalent to fifteen conditions, we may expect to be able to determine one, or at most a finite number of Steiner surfaces which have contact of the fourth order with S at P. We shall find that a unique surface of this kind exists.

The equations (5) and (6) may be simplified somewhat without loss of generality. Let us assume first that the parameters of the point $x = y = z = 0$ are $\lambda = \mu = 0$. Then we may put $c_1 = c_2 = c_3 = 0$ and $c_4 = 1$. Moreover, let

$$g_1 f_2 - g_2 f_1 \neq 0. \tag{7}$$

* C. SEGRE. *Complementi alla teoria delle tangenti coniugate di una superficie.* Rendiconti della R. Accademia dei Lincei, vol. 17 (1908), series 5ª, 2° sem. fasc. 9°. Seduta dell' 8 novembre 1908.

† According to WEIERSTRASS; cf. CLEBSCH in Crelle's Journal, vol. 67 (1867).

Then the quantities

$$\alpha = 2g_1\lambda + 2f_1\mu, \qquad \beta = 2g_2\lambda + 2f_2\mu,$$

may be introduced as new parameters in place of λ and μ and it will still be true that for $\alpha = \beta = 0$, x, y and z will all vanish. By these assumptions equations (6) reduce to

$$(8)\qquad \begin{aligned} \phi_1 &= \alpha + A_1\alpha^2 + 2H_1\alpha\beta + B_1\beta^2, \\ \phi_2 &= \beta + A_2\alpha^2 + 2H_2\alpha\beta + B_2\beta^2, \\ \phi_3 &= 2G_3\alpha + 2F_3\beta + A_3\alpha^2 + 2H_3\alpha\beta + B_3\beta^2, \\ \phi_4 &= 1 + 2G_4\alpha + 2F_4\beta + A_4\alpha^2 + 2H_4\alpha\beta + B_4\beta^2, \end{aligned}$$

where G_4 and F_4 may moreover be equated to zero. For if they are not zero the transformation of the parameters given by

$$\alpha = \frac{\alpha'}{1 - G_4\alpha' - F_4\beta'}, \qquad \beta = \frac{\beta'}{1 - G_4\alpha' - F_4\beta'},$$

will give rise to a system of the same form as (8) but with zeros in the place of G_4 and F_4.

If these simplified expressions be substituted for x, y and z in (1), and if the coefficients of all of the powers of α and β up to and including those of the fourth order be equated to zero, the resulting Steiner surface will have contact of the fourth order with S at P. The resulting homogeneous equations are

$$(9)\qquad \begin{aligned} \rho x_1 &= 576\alpha - 72I\alpha^2 + 72J\alpha\beta - 96\beta^2, \\ \rho x_2 &= 576\beta - 96\alpha^2 + 72I\alpha\beta - 72J\beta^2, \\ \rho x_3 &= 576\alpha\beta, \\ \rho x_4 &= 576 + (24J - 9I^2)\alpha^2 + (18IJ - 80)\alpha\beta + (24I - 9J^2)\beta^2. \end{aligned}$$

In deducing these formulæ we assumed $g_1f_2 - g_2f_1 \neq 0$. Suppose

$$g_1f_2 - g_2f_1 = 0 \qquad \text{but} \qquad g_1f_3 - g_3f_1 \neq 0.$$

We may again assume $c_1 = c_2 = c_3 = 0$, $c_4 = 1$, $G_4 = F_4 = 0$. If we put

$$2g_1\lambda + 2f_1\mu = \alpha, \qquad 2g_3\lambda + 2f_3\mu = \beta,$$

we shall have, instead of (8), the system

$$(8a)\qquad \begin{aligned} \phi_1 &= \alpha + A_1\alpha^2 + 2H_1\alpha\beta + B_1\beta^2, \\ \phi_2 &= k\alpha + A_2\alpha^2 + 2H_2\alpha\beta + B_2\beta^2, \\ \phi_3 &= \beta + A_3\alpha^2 + 2H_3\alpha\beta + B_3\beta^2, \\ \phi_4 &= 1 + A_4\alpha^2 + 2H_4\alpha\beta + B_4\beta^2. \end{aligned}$$

But it is easy to see that a surface whose equations are of this form cannot have contact even of the first order with S at P. In fact, its tangent plane at the point $P(\alpha=\beta=0)$ would be

$$kx - y = 0$$

instead of $z=0$. Finally, both of the determinants $g_1 f_2 - g_2 f_1$ and $g_1 f_3 - g_3 f_1$ cannot vanish simultaneously since the point P is supposed to be a regular point for the surface S and therefore also for the osculating Steiner surface. We may therefore state the following theorem: *A unique Roman surface of Steiner can be determined which has contact of the fourth order with a given non-ruled analytic surface at one of its regular points. Referred to the canonical tetrahedron the parametric equations of this Steiner surface are given by equations* (9).

We shall refrain from developing in detail the geometric relations between this Steiner surface and the elements of the canonical tetrahedron, all of which may be expressed in terms of the invariants I and J.

§ 3. *Osculating conics of plane sections.*

Let the behavior of the surface S in the vicinity of the regular point P be expressed by its canonical development

$$\begin{aligned} z = xy + \tfrac{1}{6}(x^3+y^3) + \tfrac{1}{24}(Ix^4+Jy^4) + \frac{1}{5!}(K_0x^5 + 5K_1x^4y + 10K_2x^3y^2 \\ + 10K_3x^2y^3 + 5K_4xy^4 + K_5y^5) + \frac{1}{6!}(L_0x^6 + 6L_1x^5y + 15L_2x^4y^2 \\ + 20L_3x^3y^3 + 15L_4x^2y^4 + 6L_5xy^5 + L_6y^6) + \cdots, \end{aligned} \tag{10}$$

the terms up to those of the sixth order being explicitly written down. Consider the intersection of this surface with the plane

$$z = \lambda x + \mu y, \qquad \lambda = \frac{\lambda_1}{\lambda_3}, \qquad \mu = \frac{\lambda_2}{\lambda_3}. \tag{11}$$

The projection of this plane section upon the xy plane will be given by the development

$$\begin{aligned} \lambda x + \mu y - xy - \frac{1}{3!}(x^3+y^3) - \frac{1}{4!}(Ix^4+Jy^4) - \frac{1}{5!}(K_0x^5 + 5K_1x^4y \\ + 10K_2x^3y^2 + 10K_3x^2y^3 + 5K_4xy^4 + K_5y^5) - \frac{1}{6!}(L_0x^6 + 6L_1x^5y \\ + 15L_2x^4y^2 + 20L_3x^3y^3 + 15L_4x^2y^4 + 6L_5xy^5 + L_6y^6) - \cdots = 0. \end{aligned} \tag{12}$$

If the secant plane does not contain the z axis, the projection of the osculating conic of the plane section will be the osculating conic of the curve (12) in the xy plane: and conversely, the quadric cone whose vertex is the center of projection (the point at infinity of the z axis) and whose directrix is the conic osculating the curve (12) at the origin, will determine as its intersection with the plane (11) the osculating conic of the plane section considered. To be sure, the osculating conics of a plane section whose plane contains the z axis escapes this method of treatment. It may be obtained by choosing a different center of projection, or more simply by considering it as the limit of a sequence of conics determined by a sequence of planes which approach coincidence with the plane considered.

Let

$$ax^2 + 2hxy + by^2 + 2gx + 2fy = 0 \tag{13}$$

be the osculating conic of (12). Then this equation (13) must be satisfied to terms of the fourth order inclusive if the value of y as function of x be substituted into it from (12). If $\mu \neq 0$ we may always assume that (12) is capable of a solution of the form

$$y = kx + lx^2 + mx^3 + nx^4 + px^5 + qx^6 + \cdots. \tag{14}$$

If μ vanishes, a similar expression may be found for x in terms of y, unless λ also is equal to zero. But there is no need of considering these cases separately as it will become evident how the formulae will have to be modified for $\mu = 0$. Finally if $\lambda = \mu = 0$, i. e., if the secant plane is the tangent plane, the osculating conic obviously degenerates and becomes indeterminate. Let us assume $\mu \neq 0$. The substitution of (14) in (12) will give

$$k = -\frac{\lambda}{\mu}, \quad l = -\frac{\lambda}{\mu^2}, \quad m = \frac{1}{6\mu^4}(-\lambda^3 - 6\lambda\mu + \mu^3),$$

$$n = \frac{1}{24\mu^5}[J\lambda^4 - 16\lambda^3 - 24\lambda\mu + 4\mu^3 + I\mu^4], \tag{15}$$

$$\mu p = n + \tfrac{1}{2}k^2m + \tfrac{1}{2}kl^2 + \tfrac{1}{6}Jk^3l + \tfrac{1}{120}(K, k),$$

$$\mu q = p + klm + \tfrac{1}{2}k^2n + \tfrac{1}{6}l^3 + \tfrac{1}{24}J(4k^3m + 6k^2l^2) + \tfrac{1}{120}(K, k)'l + \tfrac{1}{720}(L, k),$$

where

$$(K, k) = K_0 + 5K_1k + 10K_2k^2 + 10K_3k^3 + 5K_4k^6 + K_5k^5,$$

$$(K, k)' = 5K_1 + 20K_2k + 30K_3k^2 + 20K_4k^3 + 5K_5k^4,$$

$$(L, k) = L_0 + 6L_1k + 15L_2k^2 + 20L_3k^3 + 15L_4k^4 + 6L_5k^5 + L_6k^6.$$

If we substitute (14) into equation (13) and equate to zero the coefficients of x, x^2, x^3, x^4 we obtain the conditions necessary for the determination of the osculating conic. Upon solution these give

$$
(17)\quad
\begin{aligned}
a &= 2(-l^4 + kml^2 - k^2 ln + k^2 m^2) = \lambda^2(\alpha - 12\beta - 12\lambda\mu^4),\\
b &= 2(m^2 - ln) = \mu^2(\alpha - 12\beta - 12\lambda^4\mu),\\
h &= -ml^2 + 2kln - 2km^2 = \lambda\mu(\alpha - 18\beta + 36\lambda^2\mu^2),\\
f &= l^3 = -36\lambda^3\mu^4, \qquad g = -kl^3 = -36\lambda^4\mu^3,
\end{aligned}
$$

where

$$
(18)\quad
\begin{aligned}
\alpha &= 2(\lambda^3 - \mu^3)^2 + 3\lambda\mu(J\lambda^4 + I\mu^4),\\
\beta &= \lambda\mu(\lambda^3 + \mu^3).
\end{aligned}
$$

If $\mu = 0$, these formulæ give as the osculating conic of a plane section through one of the asymptotic tangents that tangent counted twice. This same result is obtained if the development in powers of y be employed which exists in place of (14) when $\mu = 0$, $\lambda \neq 0$. If both λ and μ vanish, the secant plane is tangent to the surface and the osculating conic becomes indeterminate. It consists of one of the asymptotic tangents and any other tangent of the surface at the given point. The pair of asymptotic tangents in that case constitutes a degenerate hyperosculating conic.

Let us examine the locus of the osculating conics corresponding to a fixed tangent, i. e., corresponding to those secant planes for which

$$
(19)\qquad \frac{\lambda}{\mu} = c
$$

is a constant. Put

$$
(20)\qquad A = 2(c^3 - 1)^2 + 3c(Jc^4 + I), \qquad B = c(c^3 + 1).
$$

If we divide by the common factor μ^6, the coefficients of the equation of the projection of the osculating conic become

$$
(21)\quad
\begin{aligned}
a &= c^2\mu[A\mu - 12(B + c)], \qquad b = \mu[A\mu - 12(B + c^4)],\\
h &= c(A\mu^2 - 18B\mu + 36c^2), \qquad f = -36c^3\mu, \qquad g = -36c^4\mu.
\end{aligned}
$$

The locus of these conics is found by eliminating μ between the equations

$$
(22)\qquad ax^2 + 2hxy + by^2 + 2gx + 2fy = 0, \qquad z = \mu(cx + y),
$$

where, a, h, b, g, f are given by (21). The result is

$$
(23)\qquad Az^2 - 72c^3z - 12(c^2 + Bc)xz - 12(c^4 + B)yz + 72c^3xy = 0,
$$

so that the required locus is a quadric surface, which, as may be easily verified, is never a cone unless $c = 0$ or ∞, in which cases it degenerates into the tangent plane counted twice. We have thus proved the theorem of Moutard for the case of a non-ruled surface. Since (23) is of the sixth degree in c, we see that six quadrics of this kind pass through an arbitrary point of space which is not in the tangent plane. All of these quadrics have contact of the second-order with S at the origin, as they are included in the form (2). We may express these results as follows. *Consider the section of a non-ruled surface S made by a plane passing through one of its regular points P. The conic which has fourth-order contact with this plane section at P generates a quadric surface if the plane turns about a fixed tangent of the surface at P as axis. Thus there corresponds a unique quadric surface to every tangent of the surface at P. Each of these quadrics has second order contact with S, and six of them pass through an arbitrary point of space which is not in the tangent plane. Leaving aside the case when one of the quadrics degenerates into the tangent plane counted twice, there exists no cone among them.*

We proceed to complete these considerations by studying the plane section of the surface S made by a pencil of planes whose axis passes through P, but is not a tangent of the surface.

Let

$$z = \lambda x + \mu y, \qquad \text{or} \qquad \lambda_3 z = \lambda_1 x + \lambda_2 y \tag{24}$$

be the equation of a plane which contains the point whose coördinates in the non-homogeneous canonical system are α, β, γ. Then we shall have the relation

$$\gamma = \lambda\alpha + \mu\beta, \qquad \gamma \neq 0, \tag{24a}$$

between λ and μ, where of course only the ratios of $\alpha : \beta : \gamma$ are of importance. This plane intersects the tangent plane in the straight line

$$z = 0, \qquad \lambda x + \mu y = 0, \qquad \text{where} \qquad \lambda\alpha + \mu\beta = \gamma, \tag{25}$$

and the osculating conic of its intersection with S will be the intersection of the plane (24) with the Moutard quadric which belongs to the tangent (25). We may write the equation of the plane (24) as follows

$$z = \lambda x + \frac{\gamma - \lambda\alpha}{\beta} y. \tag{26}$$

The Moutard quadric of the tangent of S in which (26) intersects the tangent plane is given by (23) if the value of c which occurs in that equation be equated to

$$c = \frac{\lambda}{\mu} = \frac{\beta\lambda}{\gamma - \lambda\alpha}. \tag{27}$$

The intersection of (26) and this quadric is the osculating conic of the plane section considered. The elimination of λ between (23), (26), and (27) will give the equation of the locus of the osculating conics of the plane sections of S determined by the pencil of planes considered.

We find

$$c = \frac{\gamma y - \beta z}{-\gamma x + \alpha z} \tag{28}$$

which may be substituted in (23) to obtain the required locus, which is of the eighth order. It will be convenient however to introduce a new system of non-homogeneous coördinates by putting

$$\xi = -\gamma x + \alpha z, \qquad \eta = \gamma y - \beta z, \qquad \zeta = z, \tag{29a}$$

or

$$x = \frac{1}{\gamma}(-\xi + \alpha\zeta), \qquad y = \frac{1}{\gamma}(\eta + \beta\zeta), \qquad z = \zeta, \tag{29b}$$

a transformation which is equivalent to choosing as ζ axis the axis of the given pencil of planes without changing the x or y axes.

If this be done, the equation of the required locus becomes

$$\begin{aligned} &\zeta^2 [2\gamma^2\xi^6 + 3\gamma(\gamma J - 4\alpha)\xi^5\eta - 24\beta\gamma\xi^4\eta^2 + 4(18\alpha\beta - \gamma^2)\xi^3\eta^3 \\ &- 24\alpha\gamma\xi^2\eta^4 + 3\gamma(\gamma I - 4\beta)\xi\eta^5 + 2\gamma^2\eta^6] + \zeta[-72\gamma^2\xi^3\eta^3 \\ &+ 12\gamma\xi^6\eta - 24(\gamma + 3\beta)\xi^4\eta^3 + 24(\gamma + 3\alpha)\xi^3\eta^4 - 12\gamma\xi\eta^6] \\ &- 27\xi^4\eta^4 = 0. \end{aligned} \tag{30}$$

It may be verified that this surface of the eighth order, S_8, has a septuple point at the origin and is therefore rational. The ζ axis is a sextuple line, while the ξ and η axes are both double lines of the surface. The two planes that are tangent to the two sheets of S_8 which intersect along the ξ axis are fixed, i. e., they do not vary with the point of contact. The same is true of the two tangent planes which belong to any point of the η axis. One of these tangent planes moreover is the plane $\zeta = 0$, tangent to S at P. The other two planes containing the ξ and η axis respectively are

$$\gamma\zeta + 6\eta = 0 \qquad \text{and} \qquad \gamma\zeta - 6\xi = 0, \tag{31a}$$

or returning to the canonical system of coördinates

$$6\gamma y + (\gamma - 6\beta)z = 0, \qquad 6\gamma x + (\gamma - 6\alpha)z = 0. \tag{31b}$$

Since, moreover, the intersection of the $\xi\eta$ plane with S_8 consists of the ξ and η axis each counted four times, the $\xi\eta$ plane is a stationary tangent plane of S_8 along the entire ξ and η axes. The surface S_8 has no other nodal points than those indicated.

Referred to the canonical tetrahedron, (30) is homogeneous with respect to α, β, γ, as it should be.

We are in a position to express more clearly than formerly the significance of the canonical system of coördinates. The tetrahedron of reference requires no further discussion, but the unit point of the system was defined in a somewhat indirect way.* Let us now consider the surface S_8 whose axis is the z axis. Its equation is obtained from (30) if we put the ratios of $\alpha:\gamma$ and $\beta:\gamma$ equal to zero. The two planes (31b) then reduce to

$$6y+z=0, \qquad 6x+z=0. \tag{32}$$

Their line of intersection intersects the canonical quadric of the point P of the surface S,

$$z=xy$$

in the origin and the further point

$$x=-6, \qquad y=-6, \qquad z=36.$$

Let us speak of the surface S_8 which is determined by any line l through P as a surface of osculating conics; let l be called its axis and let the two planes (31b) be called its singular tangent planes. We find the following result.

The unit point of the canonical system of coördinates is chosen in such a way that the second point which the singular tangent planes of that surface of osculating conics, whose axis is the directrix of the second kind, have in common with the canonical quadric shall have as its coördinates the values

$$x=-6, \qquad y=-6, \qquad z=36.$$

If (13) is the equation of the projection of the osculating conic of one of the plane sections of the surface S, its coefficients are given by (17) or (21). If the development (14) for y be substituted in the left member of (18), the resulting power-series will have zero coefficients for its first five terms. Let C_5 and C_6 denote the coefficients of x^5 and x^6 in this power series. Then

$$\begin{aligned} C_5&=2fp+2hn+2(kn+lm)b, \\ C_6&=2fq+2hp+(2kp+2ln+m^2)b. \end{aligned} \tag{33}$$

We find from (15)

$$p=\frac{1}{120\mu^7}[(K,k)\mu^6-10\lambda^5+25J\lambda^4\mu+10\lambda^2\mu^3+5I\mu^5-200\lambda^3\mu+20\mu^4-120\lambda\mu^2],$$

$$\begin{aligned} q=\frac{1}{720\mu^8}[&(L,k)\mu^7+35J\lambda^6-20J\lambda^3\mu^3+15I\lambda^2\mu^4-6(K,k)'\lambda\mu^5+6(K,k)\mu^6 \\ &-420\lambda^5+450J\lambda^4\mu+240\lambda^2\mu^3+30I\mu^5-2400\lambda^3\mu+120\mu^4-720\lambda\mu^2]. \end{aligned}$$

Consequently we shall have

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* Second memoir. These Transactions, vol. 9 (1908), p. 110.

$$k = -c, \qquad l = -\frac{c}{\mu}, \qquad m = \frac{1}{6\mu^2}[(1-c^3)\mu - 6c],$$

$$n = \frac{1}{24\mu^3}[(Jc^4 + I)\mu^2 + 4(1-4c^3)\mu - 24c],$$

$$(35)\quad p = \frac{1}{120\mu^4}[(K,k)\mu^3 + (-10c^5 + 25Jc^4 + 10c^2 + 5I)\mu^2 + 5(-40c^3+4)\mu - 120c],$$

$$q = \frac{1}{720\mu^5}[(L,k)\mu^4 + \{35Jc^6 - 20Jc^3 + 15Ic^2 - 6(K,k)'c + 6(K,k)\}\mu^3$$
$$+(-420c^5 + 450Jc^4 + 240c^2 + 30I)\mu^2 + (-2400c^3 + 120)\mu - 720c].$$

It will be convenient to make use of the relation

$$(36)\qquad h + kb = 6(c^5 - c^2)\mu + 36c^3$$

in computing C_6.

We find

$$\Lambda = \frac{30\mu^2}{c}C_5 = [20(c^3-1)^3 - 18c^2(K, -c) + 45c(c^3-1)(I + Jc^4)]\mu^2$$
$$-180c[c^6 - 1 + c(Jc^4 - I)]\mu + 360c^2(c^3-1),$$

$$(37)\quad \mathrm{M} = 720\mu^4 C_6 = [-72(L, -c)c^3 + 72(K, -c)(c^3-1)c^2 - 60c(Jc^4+I)A$$
$$+20(c^3-1)^2A]\mu^3 + [432(K, -c)'c^4 + 4320c^5(Jc^4+I) - 1080c^2(Jc^4+I)$$
$$+1200c(c^9-1) - 3600c^4(c^3-1)]\mu^2 + [-15120Jc^7 + 6480Ic^3 - 12960c^8$$
$$-8640c^2(c^3-1)]\mu + 25920c^6 - 17280c^3.$$

For a given value of c, there are two values of μ for which Λ becomes equal to zero. In other words, *through every tangent of the surface there can be passed two planes in general different from the tangent plane, such that their intersections with the surface shall be hyperosculated by their osculating conics.* It is easy to verify that through every line which contains the given point P of the surface S, and which is not tangent to S at P, there pass nine such planes. Consequently, all of the planes whose intersections with the surface S are hyperosculated by their osculating conics at the point P, generate a cone of the ninth class, which has the tangent plane as a septuple plane. The elements, along which this cone touches the tangent plane, are the asymptotic tangents, each of which counts as two, and the three Darboux tangents of quadric osculation.

Among the plane sections just considered, for which $C_5 = 0$, there will be some for which C_6 also vanishes. Their osculating conics will have seven consecutive points in common with them. They are determined by the simultaneous equations $\Lambda = \mathrm{M} = 0$. It seems desirable, however, to replace the equation $M = 0$ which is not symmetrical with respect to I and J, by the equation

$$N = M - 60c\Lambda = 0$$

which is symmetrical. Put

$$(38)\quad \begin{aligned} D &= 20(c^3-1)^3 - 18c^2(K, -c) + 45c(c^3-1)(I+Jc^4), \\ E &= -180\,[c^6 - 1 + c(Jc^4 - I)], \qquad F = 360(c^3-1), \\ D' &= -72(L, -c)c^3 + 72(K, -c)c^2(c^3-1) - 60c(Jc^4+I)A + 20(c^3-1)^2A, \\ E' &= 432(K, -c)'c^3 + 1080(K, -c)c^2 + 1620c(c^3+1)(Jc^4+I), \\ F' &= -4320c(Jc^4+I) - 2160c^6 - 8640c^3 - 2160, \\ G' &= 4320(c^3+1). \end{aligned}$$

Then the two conditions for seven-pointic conics become

$$(39)\quad \begin{aligned} D\mu^2 + Ec\mu + Fc^2 &= 0, \\ D'\mu^3 + E'c\mu^2 + F'c^2\mu + G'c^3 &= 0. \end{aligned}$$

Let us put

$$(40)\quad \mu = \frac{1}{m}$$

so that (39) become

$$(41)\quad \begin{aligned} D + Ecm + Fc^2m^2 &= 0, \\ D' + E'cm + F'c^2m^2 + G'c^3m^3 &= 0. \end{aligned}$$

We may eliminate cm between these two equations by SYLVESTER's dialytic method. The resultant is

$$(42)\quad \Delta = \begin{vmatrix} D & E & F & 0 & 0 \\ 0 & D & E & F & 0 \\ 0 & 0 & D & E & F \\ D' & E' & F' & G' & 0 \\ 0 & D' & E' & F' & G' \end{vmatrix} = 0,$$

which is certainly not of degree higher than 33 in c, as may be seen at once by noting that D, E, F are of degrees 9, 6, 3 and D', E', F', G' of degrees 12, 9, 6, 3 respectively. We shall see, however, that Δ is really only of the thirtieth degree in c, and that it has c^3 as a factor. So that after division by this factor, there will remain an irreducible equation of the twenty-seventh degree.

We shall show first that Δ contains c^3 as a factor. Let us put $cm = u$ in (41) and interpret u and c as cartesian coördinates. The two curves (41) will have the point $c = 0$, $u = \frac{1}{6}$ in common, as is easily verified. In fact for $c = 0$, both equations are satisfied by $u = \frac{1}{6}$ and by no other common value of u. Let us put

$$u - \tfrac{1}{6} = v.$$

The two equations become

$$(43) \qquad d + ev + fv^2 = 0, \qquad d' + e'v + f'v^2 + g'v^3 = 0,$$

where

$$d = D + \tfrac{1}{6}E + \tfrac{1}{36}F, \qquad e = E + \tfrac{1}{3}F, \qquad f = F,$$

$$(44) \qquad d' = D' + \tfrac{1}{6}E' + \tfrac{1}{36}F' + \tfrac{1}{216}G', \qquad e' = E' + \tfrac{1}{3}F' + \tfrac{1}{12}G',$$

$$f' = F' + \tfrac{1}{2}G', \qquad g' = G'.$$

If these coefficients be developed with respect to powers of c, we shall find

$$d = -15Ic - 18K_0c^2 + (70 + 90K_1)c^3 + \cdots,$$

$$e = 60 + 180Ic + 120c^3 + \cdots, \qquad f = -360 + 360c^3,$$

$$(45) \qquad d' = 90Ic + (108K_0 - 180I^2)c^2 - (72L_0 + 180K_1 + 300)c^3 + \cdots,$$

$$e' = 360 + 180Ic + 1080K_0c^2 - (3240K_1 - 2520)c^3 + \cdots,$$

$$f' = -4320Ic - 6480c^3, \qquad g' = 4320 + 4320c^3.$$

We may now obtain from either equation of (43) a development for v in powers of c, and we shall find that these two developments agree up to the terms of the second order inclusive. These common terms of the developments are

$$v = \tfrac{1}{4}Ic + \tfrac{1}{120}(36K_0 - 45I^2)c^2 \cdots;$$

the terms of the third order in the two developments do not agree in general. We see, therefore, that the two curves (41) have contact of second order at the point $c = 0$, $u = \frac{1}{6}$, consequently this point counts as three among the intersections of the two curves, and Δ must contain c^3 as a factor. In the same way, it may be shown, and moreover it can be seen without further computation, that $c = \infty$ must be a triple root of the equation $\Delta = 0$, i. e., the coefficients of the three highest powers of c in Δ must vanish. The eliminant of the equations (41), therefore, reduces to the twenty-seventh degree. The three roots $c = 0$ and $c = \infty$, which have just been removed from (42), obviously do not give proper solutions of the problem under consideration. The other twenty-seven do. In fact, if the surface considered were a cubic, a conic having seven-pointic contact with it would have to be entirely upon the surface. But owing to the existence of twenty-seven straight lines upon a cubic surface, there exists upon it a family of conics, of which twenty-seven pass through every point. Since in this special case twenty-seven seven-pointic conics actually exist, we see that our eliminant of the twenty-seventh order contains no factors strange to the problem. We conclude therefore that the following theorem is true: *At a regular point of a non-ruled surface twenty-seven conics may be constructed having contact of the sixth order with it.*

The discriminant of the equation $\Lambda = 0$, after removing the extraneous factor c^2, is of the twelfth degree in c. *Consequently there are twelve tangents through a regular point of a non-ruled surface for which the two hyperosculating conics coincide.*

§ 4. *Osculating and hyperosculating conics at a regular point of a ruled surface.*

If the surface S under consideration is a ruled surface the development (10) breaks down. We may write in that case

$$(46) \qquad z = xy + x^3 + K_0 x^5 + K_1 x^4 y + K_2 x^3 y^2 + \frac{1}{6!} L(x, y) + \cdots ,^*$$

where

$$(47) \quad L(x, y) = L_0 x^6 + 6L_1 x^5 y + 15 L_2 x^4 y^2 + 20 L_3 x^3 y^3 + 15 L_4 x^2 y^4 ,$$

the terms involving y^5 and y^6 being necessarily absent. More generally, if we denote by u_n the terms of the nth degree in the development (46), the coefficients of y^n and y^{n-1} in u_n must be equal to zero. In fact, if the coefficients of y^n were not zero, the line $x = z = 0$ which is a generator of the ruled surface could not be upon it. If the coefficient of y^{n-1} were not equal to zero, the planes which contain this generator g, each of which is tangent to the surface at one of the points of g, could not be projectively related to their points of contact. (CHASLES's correlation.)

As before, let us cut the surface by a plane

$$z = \lambda x + \mu y ,$$

and consider the projection of the curve of intersection

$$\lambda x + \mu y - xy - x^3 - K_0 x^5 - K_1 x^4 y - K_2 x^3 y^2 - \frac{1}{6!} L(x, y) - \cdots = 0 .$$

We shall find

$$y = kx + lx^2 + mx^3 + nx^4 + px^5 + qx^6 + \cdots ,$$

where

$$k = -\frac{\lambda}{\mu}, \qquad l = \frac{\lambda}{\mu^2}, \qquad m = \frac{\mu^2 - \lambda}{\mu^3}, \qquad n = \frac{\mu^2 - \lambda}{\mu^4},$$

$$(48) \quad p = \frac{1}{\mu^5}(K_0 \mu^4 + K_2 \lambda^2 \mu^2 - K_1 \lambda \mu^3 + \mu^2 - \lambda),$$

$$q = \frac{1}{\mu^6}\left[\frac{1}{720}\mu^5\left(L, -\frac{\lambda}{\mu}\right) + K_0 \mu^4 - 2K_1 \lambda \mu^3 + 3K_2 \lambda^2 \mu^2 + \mu^2 - \lambda\right],$$

* Third Memoir. These Transactions, vol. 9 (1908), p. 299.

and where $(L, -\lambda/\mu)$ denotes $L(1, -\lambda/\mu)$. The projection of the osculating conic becomes

$$(49) \qquad ax^2 + 2hxy + by^2 + 2gx + 2fy = 0$$

where

$$(50) \qquad \begin{aligned} &a = 2c^2\mu(\mu - 2c), \quad b = 2\mu(\mu - c), \quad h = c(\mu - c)(2\mu - c), \\ &f = -c^3\mu, \qquad\qquad g = -c^4\mu, \qquad\qquad c = \lambda/\mu. \end{aligned}$$

The locus of the osculating conics of all the sections whose planes pass through a fixed tangent, $(\lambda/\mu = c)$, *is again a quadric surface, viz.*;

$$(51) \qquad z^2 - (2c^2x + cy + c^3)z + c^3xy = 0.$$

We see that *three of these quadrics pass through any point of space which is not in the tangent plane,* while in the case of a non-ruled surface there were six. As in the general case, none of these quadrics is a cone except in the case when it degenerates into the tangent plane counted twice.

Consider the plane sections of S whose planes have in common a fixed line l through the given point P when l is *not* a tangent of the surface. As in the general case, we denote by α, β, γ the coördinates of any point of this line l and put

$$\xi = -\gamma x + \alpha z, \qquad \eta = \gamma y - \beta z, \qquad \zeta = z, \qquad \gamma \neq 0.$$

The locus of the osculating conics of these plane sections is the quintic surface

$$(52) \qquad \zeta^2[\gamma^2\xi^3 - 2\alpha\gamma\xi\eta^2 - \beta\gamma\xi^2\eta + \alpha\beta\eta^3] + \zeta[\gamma\xi^2\eta^2 - \gamma^2\eta^3 - \beta\xi\eta^3 + \alpha\eta^4] - \xi\eta^4 = 0,$$

which has the origin as a quadruple point and is therefore rational, which has the given line l as a triple line, and which has the asymptotic tangent t of S at P as a double line. Moreover, the two sheets of the quintic (52) *which have the points of t in common are tangent to each other and to the tangent plane of S throughout the entire extent of t.*

We find in this case

$$(53) \qquad \begin{aligned} C_5 &= \frac{2c}{\mu^2}\left[-(2 + K_0c^2 - K_1c^3 + K_2c^4)\mu^2 + 3c\mu - c^2\right], \\ C_6 &= \frac{1}{\mu^3}\left[\{2 - 2K_0c^2 + 2K_1c^3 - 2K_2c^4 - \tfrac{1}{360}c^3(L, -c)\}\mu^3 \right. \\ &\qquad \left. + (-10c + 2K_1c^4 - 4K_2c^5)\mu^2 + 12c^2\mu - 4c^3\right]. \end{aligned}$$

Consequently *there exist again two sections through every tangent, whose osculating conics hyperosculate them. There are four tangents through P such that these hyperosculating conics coincide.* The conics which have seven-point contact with the ruled surface at P are obtained by equating C_5 and C_6 simultaneously to zero. Put again

$$\mu = \frac{c}{u},$$

so that the conditions $C_5 = C_6 = 0$ become

$$(54)\qquad \begin{gathered} D - 3u + u^2 = 0, \\ D' + E'u + 12u^2 - 4u^3 = 0, \end{gathered}$$

where

$$(55)\qquad \begin{aligned} D &= 2 + K_0 c^2 - K_1 c^3 + K_2 c^4, \\ D' &= 2 - 2K_0 c^2 + 2K_1 c^3 - 2K_2 c^4 - \frac{c^3}{360}(L, -c), \\ E' &= -10 + 2K_1 c^3 - 4K_2 c^4. \end{aligned}$$

The resultant of (54) is

$$(56)\qquad D(E' + 4D)^2 + 3D'(E' + 4D) + D'^2 = 0,$$

which is of the fourteenth degree, but may be easily shown to contain c^2 and no higher power of c as a factor if

$$(57)\qquad 720K_1 - L_0 \neq 0.$$

We may therefore state our result as follows: *At a regular point of a ruled surface there can be constructed, in general, twelve conics having contact of the sixth order with the surface.*

If the surface is a developable, the expansion of its equation in the vicinity of an ordinary point may be written in the form *

$$(58)\quad z = x^2 + Kx^3y + \frac{1}{6!}(L_0x^6 + 6L_1x^5y + 15L_2x^4y^2 + 20L_3x^3y^3 + 15L_4x^2y^4) + \cdots,$$

where K may be supposed to be equal to unity or zero, according as a certain differential invariant of the fourth order is or is not different from zero. In the latter case however there will, in general, be present in the development also one of the terms of the fifth order. We shall find incidentally the significance of the vanishing of this differential invariant of the fourth order as a consequence of the considerations which follow.

If we maintain the same notations as in the other cases, we shall find, assuming that the secant plane does not pass through the generator, $(\mu \neq 0)$,

$$(59)\qquad \begin{gathered} k = -\frac{\lambda}{\mu}, \qquad l = \frac{1}{\mu}, \qquad m = 0, \qquad n = -\frac{\lambda K}{\mu^2}, \\ p = \frac{K}{\mu^2}, \qquad q = \frac{1}{720\mu}(L, k), \end{gathered}$$

whence we obtain

$$(60)\quad a = 2(Kc^3\mu^2 - 1), \quad b = 2Kc\mu^2, \quad h = 2Kc^2\mu^2, \quad f = \mu, \quad g = c\mu$$

* This is essentially the form of development given by DARBOUX. But this may be further simplified in the terms of the sixth order and the geometry of the transformation suggests a number of questions which will be left for a future occasion.

as the coefficients of the equation of the projection of the osculating conic of the section of the developable made by the plane

$$z = \lambda x + \mu y,$$

if λ/μ be denoted by c. *The locus of the osculating conics, as the secant plane turns around a fixed tangent t determined by the ratio $\lambda/\mu = c$, is again a quadric surface*

$$cKz^2 - x^2 + z = 0, \tag{61}$$

which, however, in this case, is a cone, and its vertex is a point on the y axis, which is independent of the value of c. The geometrical significance of the y axis in the development (58) is therefore apparent. It is of interest to notice that we obtain a single infinity of curves on the developable defined projectively, if we associate with each of the points of the surface as direction of progress the line which joins it to the vertex of the associated cones.

If we consider the osculating conics of the sections of the developable S that are made by a plane turning around a fixed line l which passes through the point P of the surface, but which is not tangent to S at P, we shall find as its locus a ruled surface of the third order (a Cayley cubic scroll) if K is different from zero.

If K vanishes, this surface degenerates into a plane and a quadric cone. This, then, is the significance of the vanishing of the differential invariant of the fourth order which was mentioned above.

We find

$$\tfrac{1}{2} C_5 = \frac{K}{\mu}(1 - 4kc^3\mu^2), \qquad C_6 = \tfrac{1}{360}(L, -c) - 4Kc^2. \tag{62}$$

Consequently *there will be*, as in the general case, *two plane sections through a fixed tangent whose osculating conics hyperosculate it, if $K \neq 0$.* Since the condition $C_6 = 0$ is of the fourth degree in c, and to each of these values the condition $C_5 = 0$ makes correspond two values of μ, we see that *there will be eight conics which have contact of the sixth order with a developable at any one of its ordinary points.*

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