# ON THE INTEGRATION OF THE HOMOGENEOUS LINEAR DIFFERENCE EQUATION OF SECOND ORDER*

BY

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## I. *Introduction.*

1. It is proposed in the present paper to study the behavior for large positive integral values of $x$ of the general solution of the equation

$$(1) \qquad a_0(x)u(x+2)+a_1(x)u(x+1)+a_2(x)u(x)=0,$$

the coefficients $a_0(x)$, $a_1(x)$, $a_2(x)$ being given functions (real or complex) of $x$ and subject only to conditions relative to the form taken by the expression

$$4\,\frac{a_0(x)a_2(x+1)}{a_1(x)a_1(x+1)}-1$$

when $x$ is very large.† The general results are summarized in Theorem I, after which application is made to the following special equation considered by HORN: ‡

$$P_0(x)u(x+2)+x^kP_1(x)u(x+1)+x^{2k}P_2(x)u(x)=0,$$

where $k$ is any integer, positive, negative or zero and $P_0(x)$, $P_1(x)$, $P_2(x)$ are developable asymptotically (or as convergent series) in the form

$$P_\lambda(x)=a_\lambda+\frac{b_\lambda}{x}+\frac{c_\lambda}{x^2}+\cdots+\frac{p_\lambda+\omega_\lambda(x)}{x^n} \qquad (\lim_{x=\infty}\omega_\lambda(x)=0).$$

The results obtained for this type of equation are stated in Theorem II and are shown to be in accord with those obtained by HORN. Also certain results addi-

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† Papers relating to this subject but placing more restrictive conditions upon the coefficients have been published by HORN, Mathematische Annalen, vol. 53 (1900), pp. 117–192, and by the present author, Annali di Matematica, series 3, vol. 13 (1907), pp. 313-328. For a summary of literature concerning equations of form (1) see BARNES, Messenger of Mathematics, vol. 34 (1904), p. 53.

‡ *Loc. cit.*, p. 190. For simplicity we shall here confine ourselves to equations of order 2, whereas HORN considers similar equations of any order. A generalization of our results to any order is immediate, in view of the general character of the investigations contained in the above mentioned memoir in the Annali upon which the present paper depends.

tional to his are to be here found, and application of these is made in § 11 to the study of the behavior of Legendre's function of the first kind $P_n$ for large values of $n$.

## II. *Reduction of the Equation.*

2. We shall first show that in case $a_1(x) \neq 0$, $a_2(x) \neq 0$ for all $x \geqq \alpha =$ *constant*, * the study of equation (1) may be made to depend upon that of a certain linear difference equation of the type

$$\Delta^2 y(x) + a(x) y(x+2) = 0, \tag{2}$$

in which $a(x)$ depends only upon $a_0$, $a_1$ and $a_2$ and is defined for all $x \geqq \alpha$.

In fact, if one places $u(x) = t(x)y(x)$, equation (1) becomes

$$a_0(x)t(x+2)y(x+2) + a_1(x)t(x+1)y(x+1) + a_2(x)t(x)y(x) = 0$$

or, since

$$y(x+1) = y(x+2) - \Delta y(x+1),$$

$$y(x) = y(x+2) - 2\Delta y(x+1) + \Delta^2 y(x),$$

the same equation becomes

$$\begin{aligned} a_2(x)t(x)\Delta^2 y(x) - [a_1(x)t(x+1) + 2a_2(x)t(x)]\Delta y(x+1) \\ + [a_0(x)t(x+2) + a_1(x)t(x+1) + a_2(x)t(x)]y(x+2) = 0. \end{aligned} \tag{3}$$

Let us now choose the heretofore undetermined function $t(x)$ in such a manner that the second coefficient of (3) vanishes:

$$a_1(x)t(x+1) + 2a_2(x)t(x) = 0. \tag{4}$$

The function $t(x)$ thus becomes determined,† except for a constant factor, by the equation

$$t(x) = \prod_{x_1=\alpha}^{x_1=x-1} \left(\frac{-2a_2}{a_1}\right)_{x_1}, \tag{5}$$

since $a_1(x) \neq 0$ for $x \geqq \alpha$. Moreover, from the hypothesis $a_2(x) \neq 0$ it follows that $t(x) \neq 0$ and hence also $a_2(x)t(x) \neq 0$.

Consequently, equation (3) may be reduced to the form

$$\Delta^2 y(x) + [4(a_0/a_1)_x(a_2/a_1)_{x+1} - 1]\, y(x+2) = 0 \qquad (x \geqq \alpha), \tag{6}$$

i. e., to the desired form (2).

3. We shall now make the assumption that $4(a_0/a_1)_x(a_2/a_1)_{x+1} - 1$ for large values of $x$ has the form $-\nu^2 - \phi(x) - \theta(x)$, where $\nu$ is a constant (real or

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* Throughout the paper it will be understood, unless otherwise stated, that $x \geqq \alpha =$ a sufficiently large constant.

† See BOOLE's *Finite Differences*, Chap. IX, § 6.

complex) and $\nu(\nu^2-1)\neq 0$, where also $\phi(x)$ is a function of $x$ of the form $\mu/x^{\frac{1}{2}+\delta}$, in which $\mu$ and $\delta$ are constants with $0<\delta\leqq\frac{1}{2}$, and where $\theta(x)$ is any function of $x$ such that the series

$$\sum_{x_1=x}^{x_1=\infty}|\theta(x_1)|$$

converges. For example, the function $\theta(x)$ may be taken as any function of the form $g(x)\cdot\tau(x)/x$, where $|g(x)|<c=\text{const.}$ and $\tau(x)$ is one of the functions

$$\frac{1}{x^p},\ \frac{1}{(\log x)^{1+p}},\ \frac{1}{\log x(\log_2 x)^{1+p}},\ \cdots \qquad (p>0).$$

In particular, we observe that the above hypotheses are realized whenever the original equation (1) has coefficients $a_0(x)$, $a_1(x)$, $a_2(x)$ which are developable either in the form of convergent power series in $1/x$ ($x\geqq a$) or, more generally, as asymptotic series in $1/x$.

Equation (6) thus becomes

$$\Delta^2 y(x)-[\nu^2+\phi(x)+\theta(x)]y(x+2)=0. \tag{7}$$

In the study of this equation we shall now make use of Theorem II of the paper entitled *Sur les équations linéaires aux différences finies*,* using without further remark the results and notation there found.

In the present instance let us choose the auxiliary functions $z_1(x)$, $z_2(x)$ as follows, for reasons which will appear presently:

$$z_1(x)=\prod_{x_1=a}^{x_1=x-1}\big(1+\nu+\phi(x_1)/2\nu\big),\qquad z_2(x)=\prod_{x_1=a}^{x_1=x-1}\big(1-\nu-\phi(x_1)/2\nu\big). \tag{8}$$

Then

$$\Delta z_1(x)=\big(\nu+\phi(x)/2\nu\big)z_1(x),\qquad \Delta^2 z_1(x)=\big(\nu^2+\phi(x)+\xi_1(x)\big)z_1(x), \tag{9}$$

where

$$\xi_1(x)=\frac{1+\nu}{2\nu}\Delta\phi(x)+\frac{\phi(x+1)\phi(x)}{4\nu^2}.$$

From our hypothesis respecting $\phi(x)$ it appears directly that $\xi_1(x)$ has the character of one of the functions $\theta(x)$ mentioned above.

Likewise we have

$$\Delta z_2(x)=-\big(\nu+\phi(x)/2\nu\big)z_2(x),\qquad \Delta^2 z_2(x)=\big(\nu^2+\phi(x)+\xi_2(x)\big)z_2(x),$$

where $\xi_2(x)$ has the properties of $\xi_1(x)$ just mentioned and is obtained from it by changing $\nu$ into $-\nu$.

* Annali di Matematica, loc. cit., p. 301. The expression $\sum_{x_1=x+1}^{x_1=\infty}|u_m(x_1)|$ there occurring should be replaced by $\sum_{m=1}^{m=\infty}|u_m(x_1)|$.

Equations (8) and (9) together with those just noted give the following values for the expressions $A(x)$, $Q(x)$, $f_1(x)$, $f_2(x)$, $\bar{q}(x, x_1)$, $\Phi(x, x_1)$, $\Psi(x, x_1)$, and $Y(x)$ which occur in the statement of the above mentioned Theorem II:

$$(10) \qquad Q(x) = -2\left(\nu + \phi(x)/2\nu\right) z_1(x) z_2(x),$$

$$(11) \qquad A(x) = c_2 z_1(x) - c_1 z_2(x),$$

$$(12) \qquad \begin{aligned} f_1(x) &= [-\nu^2 - \phi(x) - \theta(x)] z_1(x) + \Delta^2 z_1(x) = [\xi_1(x) - \theta(x)] z_1(x), \\ f_2(x) &= [\xi_2(x) - \theta(x)] z_2(x), \end{aligned}$$

$$(13) \qquad \bar{q}(x, x_1) = \begin{vmatrix} z_1(x) & [\xi_1(x_1) - \theta(x_1)] z_1(x_1) \\ z_2(x) & [\xi_2(x_1) - \theta(x_1)] z_2(x_1) \end{vmatrix},$$

$$(14) \qquad \begin{aligned} \Phi(x, x_1) &= \frac{1}{2\left(\nu + \phi(x_1 + 1)/2\nu\right)} \\ &\times \left[\frac{[\xi_2(x_1) - \theta(x_1)] z_1(x+1)}{[1 - \nu - \phi(x_1)/2\nu] z_1(x_1 + 1)} - \frac{[\xi_1(x_1) - \theta(x_1)] z_2(x+1)}{[1 + \nu + \phi(x_1)/2\nu] z_2(x_1 + 1)}\right]. \end{aligned}$$

Moreover, since $\lim_{x_1=\infty} \phi(x_1) = 0$, $\nu(1 - \nu^2) \neq 0$, while $\theta(x_1)$, $\xi_1(x_1)$, $\xi_2(x_1)$ have the properties above mentioned, we now see that

$$(15) \qquad \Phi(x, x_1) = \frac{z_1(x+1)}{z_1(x_1+1)} \left[ s_1(x_1) + s_2(x_1) \frac{z_1(x_1+1) z_2(x+1)}{z_1(x+1) z_2(x_1+1)} \right] |\theta(x_1)|,$$

where $|s_1(x_1)|$ and $|s_2(x_1)|$ are less than some assignable constant, and $\theta(x)$, though not identical with the $\theta(x)$ of (7), has the properties before described of that function.

Finally, we have

$$(16) \qquad \Psi(x, x_1) = A(x_1 + 1) \Phi(x, x_1)$$

and

$$(17) \qquad Y(x) = \frac{-1}{Q(x+1)} [A(x+1) + u_1(x) + u_2(x) + \cdots + u_n(x) + \cdots].$$

## III. *The special case* $|1 + \nu| = |1 - \nu|$.

4. Let us suppose in the first place that $|1 + \nu| = |1 - \nu|$ and that $|1 + \nu + \phi(x)/2\nu| = |1 - \nu - \phi(x)/2\nu| = \rho_x$, $(x \geqq a)$. For this case $|z_1(x)| = |z_2(x)|$. Whence if we place

$$(18) \qquad G(x) = \prod_{n=a}^{n=x} \rho_n$$

we may write

$$(19) \qquad \Phi(x, x_1) = \omega_1(x, x_1) |\theta(x_1)| \frac{G(x)}{G(x_1)} \qquad (x \geqq a,\ x_1 > x),$$

where $|\omega_1(x, x_1)| < \Omega_1 = \text{const.}$

Similarly we have

$$(20)\qquad A(x+1)=G(x)\left[c_1 r_1(x)+c_2 r_2(x)\right]\qquad (x \geqq \alpha),$$

where $|r_1(x)|=|r_2(x)|=1$. Whence, by (16), it follows that

$$(21)\qquad \Psi(x, x_1)=\omega_2(x, x_1)|\theta(x_1)|G(x)\qquad (x \geqq \alpha,\ x_1>\alpha),$$

where $|\omega_2(x, x_1)|<\Omega_2=\text{const.}$

Equations (19) and (21) having been obtained, we turn to consider the series

$$(22)\qquad |u_1(x)|+|u_2(x)|+\cdots+|u_m(x)|+\cdots.$$

Referring to the definition of the term $u_m(x)$ given in the before mentioned Theorem II, we have in the present instance

$$(23)\quad \Phi(x,x_1)\Phi(x_1,x_2)\cdots\Phi(x_{m-2},x_{m-1})\Psi(x_{m-1},x_m)=G(x)P(x,x_1,x_2,\cdots,x_m),$$

where for $x_m \geqq x_{m-1} \geqq x_{m-2} \cdots \geqq x_1 \geqq x \geqq \alpha$ we may write

$$(24)\qquad |P(x, x_1, x_2, \cdots, x_m)|<\Omega_1^{m-1}\Omega_2|\theta(x_1)\theta(x_2)\cdots\theta(x_m)|.$$

Relations (23) and (24) now enable us to show that the three conditions $(a)$, $(b)$, $(c)$ of the theorem are here fulfilled. Condition $(a)$ is fulfilled inasmuch as we have

$$\sum_{x_1=x+1}^{x_1=\infty}\sum_{x_2=x_1+1}^{x_2=\infty}\cdots\sum_{x_m=x_{m-1}+1}^{x_m=\infty}|\Phi(x, x_1)\Phi(x_1, x_2)\cdots\Phi(x_{m-2}, x_{m-1})\Psi(x_{m-1}, x_m)|$$
$$<G(x)\Omega_1^{m-1}\Omega_2 W(x),$$

where

$$W(x)=\sum_{x_1=x+1}^{x_1=\infty}|\theta(x_1)|\sum_{x_2=x_1+1}^{x_2=\infty}|\theta(x_2)|\cdots\sum_{x_m=x_{m-1}+1}^{x_m=\infty}|\theta(x_m)|,$$

and this expression has a meaning by virtue of our hypotheses concerning the function $\theta(x)$.

As to condition $(b)$, let us put

$$(25)\qquad \theta_1(x)=\sum_{x_1=x+1}^{x_1=\infty}|\theta(x_1)|.$$

Since $\theta_1(x)$ becomes arbitrarily small for all values of $x$ sufficiently large, the term $u_m(x)$ takes the form

$$(26)\qquad u_m(x)=G(x)V_m(x),$$

where

$$(27)\qquad |V_m(x)|<\Omega_1^{m-1}\Omega_2|\theta_1(x)|^m.$$

Whence, if $\alpha$ be sufficiently large, we shall have

$$(28)\qquad |u_m(x)|<G(x)k^m,\qquad k=\text{const.}<1\qquad (x \geqq \alpha).$$

Thus the series (22) converges for $x \geqq \alpha$ to a value $U(x)$ such that

$$(29) \qquad U(x) = \gamma(x)\, G(x), \qquad \gamma(x) < \frac{k}{1-k}.$$

By virtue of (19) and (20) we may now write

$$|\Phi(x, x_1)|\, U(x_1) = \beta_1(x, x_1)\, G(x),$$

where

$$\beta_1(x, x_1) < \Omega_3 |\theta(x_1)| \qquad (\Omega_3 = \text{const.},\ x \geqq \alpha,\ x_1 \geqq \alpha).$$

Thus the series

$$\sum_{x_1=x+1}^{x_1=\infty} |\Phi(x, x_1)|\, U(x_1) \qquad (x \geqq \alpha)$$

converges.

Finally, the expressions $Y(x_1) f_r(x_1)$, $(r = 1, 2)$, of condition $(c)$ are to be considered. We obtain in the first place from (17) and (29)

$$(30) \qquad Y(x) = -\left(\frac{A(x+1)}{Q(x+1)} + \frac{\beta_2(x) G(x)}{Q(x+1)}\right) \quad \left(|\beta_2(x)| < \gamma(x) < \frac{k}{1-k}\right).$$

But from (10) and (11) we have

$$(31) \quad \frac{A(x+1)}{Q(x+1)} = -\frac{\nu}{2\nu^2 + \phi(x+1)}\left(\frac{c_2}{z_2(x+1)} - \frac{c_1}{z_1(x+1)}\right) = \frac{\beta_3(x)}{G(x)} \quad (x \geqq \alpha),$$

where $|\beta_3(x)| < \beta_3 = \text{const.}$

Similarly we obtain

$$(32) \qquad \frac{G(x)}{Q(x+1)} = \frac{\beta_4(x)}{G(x)} \qquad (|\beta_4(x)| < \beta_4 = \text{const.}).$$

Whence follows

$$Y(x) = \frac{\beta_5(x)}{G(x)} \qquad (|\beta_5(x)| \leqq \beta_5 = \text{const.}).$$

Moreover, we have from (12),

$$f_r(x) = \beta_6(x, r)\, G(x)\, |\theta(x)| \qquad (r = 1, 2),$$

where $|\beta_6(x, r)| < \beta_6 = \text{const.}$

Thence, noting that $\rho_a \neq 0$, we have

$$Y(x) f_r(x) = \beta_7(x, r)\, |\theta(x)| \quad (|\beta_7(x, r)| < \beta_7 = \text{const.}),$$

and, therefore, the expressions

$$\sum_{x_1=x+1}^{x_1=\infty} Y(x_1) f_r(x_1) \qquad (r = 1, 2;\ x \geqq \alpha),$$

have a meaning. Thus all the conditions of Theorem II become fulfilled.

Moreover, the function $\gamma(x)$ of (29) has the properties of the function $\theta_1(x)$

defined in (25); i. e., $\lim_{x=\infty} \gamma(x) = 0$. Consequently from (30), (31) and (32) we have

$$(33) \qquad Y(x) = \frac{\nu}{2\nu^2 + \phi(x+1)}\left(\frac{c_2}{z_2(x+1)} - \frac{c_1}{z_1(x+1)} + \frac{\delta(x)}{G(x)}\right)$$

where $x \geqq \alpha$, $\lim_{x=\infty} \delta(x) = 0$.

Upon applying now the result of Theorem II we find that the general solution of (7), under the present hypotheses respecting $\nu$, $\phi(x)$, and $\theta(x)$, will have the following form when $x \geqq \alpha$ for sufficiently large $\alpha$:

$$y(x) = \frac{k_1(1+\epsilon_1(x))}{z_1(x-1)} + \frac{k_2(1+\epsilon_2(x))}{z_2(x-1)} \qquad (\lim_{x=\infty} \epsilon_1(x) = \lim_{x=\infty} \epsilon_2(x) = 0),$$

$k_1$, $k_2$ being arbitrary constants. Since also

$$\frac{z_1(x)}{z_1(x-1)} = 1 + \eta_1(x), \qquad \frac{z_2(x)}{z_2(x-1)} = 1 + \eta_2(x) \qquad (\lim_{x=\infty} \eta_1(x) = \lim_{x=\infty} \eta_2(x) = 0),$$

this may be thrown into the form

$$(34) \qquad y(x) = \frac{k_1(1+\epsilon_1(x))}{z_1(x)} + \frac{k_2(1+\epsilon_2(x))}{z_2(x)} \qquad (\lim_{x=\infty} \epsilon_1(x) = \lim_{x=\infty} \epsilon_2(x) = 0).$$

## IV. *Derivation of a first integral for the general case.*

5. We turn now to consider the cases in which $|1+\nu| \neq |1-\nu|$. Placing $\rho_x = |1 + \nu + \phi(x)/2\nu|$ as before and also placing $\sigma_x = |1 - \nu - \phi(x)/2\nu|$ and

$$(35) \qquad H(x) = \prod_{n=a}^{n=x} \sigma_n,$$

we shall have

$$\frac{z_2(x+1)}{z_1(x+1)} = \lambda_3(x, x_1)\frac{H(x)}{H(x_1)} \qquad (|\lambda_3(x, x_1)| = 1).$$

Thus, from (15) we obtain

$$\Phi(x, x_1) = \frac{H(x)}{H(x_1)}\left(s_1(x_1)\frac{G(x)H(x_1)}{G(x_1)H(x)} + s_2(x_1)\right)|\theta(x_1)|.$$

Then

$$\frac{G(x)H(x_1)}{G(x_1)H(x)} = \left(\frac{\sigma_{x_1}}{\rho_{x_1}}\right)\left(\frac{\sigma_{x_1-1}}{\rho_{x_1-1}}\right)\cdots\left(\frac{\sigma_{x+1}}{\rho_{x+1}}\right).$$

Let us now suppose $x_1 > x \geqq \alpha$ and let us consider $\nu$ to be that square root of $\nu^2$ in (7) for which $|1+\nu| > |1-\nu|$. Then the factors in the right hand member of our last equation will each be less than 1 and we shall be able to write

$$(36) \qquad \Phi(x, x_1) = \omega_3(x, x_1)|\theta(x_1)|\frac{H(x)}{H(x_1)} \qquad (|\omega_3(x, x_1)| < \Omega_3 = \text{const.}).$$

As regards the function

$$\Psi(x, x_1) = A(x_1+1)\Phi(x, x_1) = [c_2 z_1(x_1+1) - c_1 z_2(x_1+1)]\,\Phi(x, x_1),$$

let us take in the present instance $c_2 = 0$. Then

$$A(x+1) = c_1 r_1(x) H(x) \qquad (|r(x)| = 1),$$

so that

$$\Psi(x, x_1) = \omega_4(x, x_1) H(x) |\theta(x_1)| \qquad (|\omega_4(x, x_1)| < \Omega_4 = \text{const.}). \tag{37}$$

We may now proceed as in the former case. Thus we have in the first place

$$u_m(x) = H(x) V_m(x) \qquad (x \geqq a), \tag{38}$$

where $V_m(x)$ has the properties indicated in (27), whence also

$$|u_m(x)| < H(x) k^m \qquad (k = \text{const.} < 1;\ x \geqq a), \tag{39}$$

and

$$Y(x) = \frac{\nu}{2\nu^2 + \phi(x+1)}\left(\frac{-c_1}{z_1(x+1)} + \frac{\delta_1(x)}{G(x)}\right) \qquad (\lim_{x=\infty} \delta_1(x) = 0).$$

Thus in place of (34) we now have the solution

$$y_1(x) = \frac{k_1(1+\epsilon_1(x))}{z_1(x)} \qquad (x \geqq a,\ \lim_{x=\infty} \epsilon_1(x) = 0). \tag{40}$$

## V. *Derivation of the second integral.*

6. Having obtained but a particular solution $y_1(x)$ of (7) when $|1+\nu| \neq |1-\nu|$, we proceed in the present section to obtain a second solution and therewith the general solution.

For this purpose equation (7) may be written in the form

$$(1 - \nu^2 - \phi(x) - \theta(x))\,y(x+2) - 2y(x+1) + y(x) = 0. \tag{41}$$

If now $u(x)$ and $v(x)$ be any two linearly independent solutions of the equation

$$a_0(x)u(x+2) + a_1(x)u(x+1) + a_2(x)u(x) = 0, \tag{42}$$

we have, after placing for brevity $a_0(x) = a_0$, etc., and $u(x+1) = u_1$, $u(x+2) = u_2$, etc.:

$$a_0(u_2 v_1 - u_1 v_2) + a_2(u_0 v_1 - u_1 v_0) = 0$$

and hence

$$\frac{u_2 v_1 - u_1 v_2}{u_1 v_0 - u_0 v_1} = \frac{a_2}{a_0}.$$

Therefore, for any fixed integer $a$ we obtain

$$u(\alpha+n)v(\alpha+n-1)-u(\alpha+n-1)v(\alpha+n)=c\prod_{p=\alpha}^{p=\alpha+n-2}\frac{a_2(p)}{a_0(p)}\qquad (n=2,3,\cdots),$$

where $c$ is a constant as regards $n$. Whence

$$\frac{u(\alpha+n)}{v(\alpha+n)}-\frac{u(\alpha+1)}{v(\alpha+1)}=c\sum_{m=2}^{m=n}\frac{1}{v(\alpha+m)v(\alpha+m-1)}\prod_{p=\alpha}^{p=\alpha+m-2}\frac{a_2(p)}{a_0(p)}.$$

Thus, if $v(x)$ be looked upon as a *known* solution of (42), the solution $u(x)$ may be expressed in the following form for all values of $x=\alpha+n$, $\alpha$ being a constant:

(43) $$u(x)=c_1v(x)+c_2v(x)S_n$$

where

$$S_n=\sum_{m=1}^{m=n}\frac{1}{v(\alpha+m)v(\alpha+m-1)}\prod_{p=\alpha}^{p=\alpha+m-2}\frac{a_2(p)}{a_0(p)}.$$

We proceed to study the properties of $S_n$ for large values of $n$ in the case of the special equation (41), for which one solution $v(x)\equiv y_1(x)$, given by (40), is known.

Thus we have in the present instance

(44) $$\frac{1}{v(\alpha+m)}=k_1(1+\epsilon_m)z_1(\alpha+m)=k_1(1+\epsilon_m)\prod_{p=\alpha}^{p=\alpha+m-1}\left(1+\nu+\frac{\phi(p)}{2\nu}\right)\quad (\lim_{m=\infty}\epsilon_m=0),$$

and from (41)

$$\frac{a_2(p)}{a_0(p)}=\frac{1+\eta_p}{\left(1+\nu+\frac{\phi(p)}{2\nu}\right)\left(1-\nu-\frac{\phi(p)}{2\nu}\right)},$$

where

$$\eta_p=\frac{\phi(p)-\left[\frac{\phi(p)}{2\nu}\right]^2}{1-\nu^2-\phi(p)-\theta(p)}.$$

Thence,

(45) $$\prod_{p=\alpha}^{p=\alpha+m-2}\frac{a_2(p)}{a_0(p)}=\frac{\prod_{p=\alpha}^{p=\alpha+m-2}(1+\eta_p)}{\prod_{p=\alpha}^{p=\alpha+m-2}\left(1+\nu+\frac{\phi(p)}{2\nu}\right)\left(1-\nu-\frac{\phi(p)}{2\nu}\right)}.$$

Moreover, by virtue of our hypotheses upon $\phi(x)$ and $\theta(x)$, the series

$$\sum_{p=\alpha}^{p=\infty}|\eta_p|$$

is convergent, and hence it follows that

(46) $$\prod_{p=\alpha}^{p=\alpha+m-2}(1+\eta_p)=k_2(1+\epsilon_m)\qquad (k_2=\text{const.},\ \lim_{m=\infty}\epsilon_m=0).$$

Upon making use of (44), (45) and (46) we find that the $m$th term of $S_n$ is of the form

$$k_2(1+\epsilon_m)\frac{\prod_{p=a}^{p=a+m-1}\left(1+\nu+\frac{\phi(p)}{2\nu}\right)}{\prod_{p=a}^{p=a+m-2}\left(1-\nu-\frac{\phi(p)}{2\nu}\right)}=\frac{k_2'(1+\epsilon_m')z_1(\alpha+m)}{z_2(\alpha+m)} \qquad (\lim_{m=\infty}\epsilon_m'=0).$$

The expression $S_n$ may therefore be put into the form

$$S_n = k_2'\frac{z_1(\alpha+n)}{z_2(\alpha+n)}S_n', \tag{47}$$

where

$$S_n' = \sum_{m=1}^{m=n}\frac{z_1(\alpha+m)z_2(\alpha+n)}{z_1(\alpha+n)z_2(\alpha+m)}(1+\epsilon_m'),$$

which we may now show directly to be of the form $k_3+\epsilon_n$ where $k_3=$ const. and $\lim_{n=\infty}\epsilon_n = 0$. In fact, we may write

$$S_n' = \left(\sum_{m=1}^{m=n-p-1}+\sum_{m=n-p}^{m=n}\right)\frac{z_1(\alpha+m)z_2(\alpha+n)}{z_1(\alpha+n)z_2(\alpha+m)}(1+\epsilon_m'),$$

where the first sum appearing on the right evidently approaches a limit by virtue of (8) when $n=\infty$, while by taking $p$ sufficiently large the second sum may be made less in absolute value than any pre-assigned positive quantity $\epsilon$, whatever $n$ may be $(n>p+1)$, since

$$\left|\frac{z_1(\alpha+m)z_2(\alpha+n)}{z_1(\alpha+n)z_2(\alpha+m)}\right| = \frac{\sigma_{a+n}\sigma_{a+n-1}\cdots\sigma_{a+m-1}}{\rho_{a+n}\rho_{a+n-1}\cdots\rho_{a+m-1}} < q^{n-m+2} \qquad (q=\text{const.}<1).$$

Availing ourselves of the form just established for $S_n'$ and recalling the definition of $v(\alpha+n)$, we see from (47) that

$$S_n = \frac{k(1+\epsilon_n)}{v(\alpha+n)z_2(\alpha+n)} = \frac{k\left(1+\epsilon(x)\right)}{v(x)z_2(x)} \qquad (\lim_{n=\infty}\epsilon_n = \lim_{x=\infty}\epsilon(x)=0).$$

Placing this value in (43) and recalling that the expression $c_1v(x)=c_1y_1(x)$ there appearing is itself a solution of (41), we obtain the following second solution linearly independent of $y_1(x)$:

$$y_2(x) = \frac{k_2\left(1+\epsilon_2(x)\right)}{z_2(x)} \qquad (x\geqq a,\ \lim_{x=\infty}\epsilon_2(x)=0).$$

Our results may now be summarized in the following general theorem:

*Theorem* I: *Given the equation*

$$a_0(x)u(x+2)+a_1(x)u(x+1)+a_2(x)u(x)=0$$

*whose coefficients (real or complex) are defined for all positive integral values of $x$ sufficiently large $(x\geqq a)$, and satisfy for such values the conditions:*

(1) $a_0(x)$, $a_1(x)$, $a_2(x)$ *never vanish.*

(2) *The expression*

$$4\frac{a_0(x)a_2(x+1)}{a_1(x)a_1(x+1)}-1$$

*has the form* $-\nu^2-\mu/x^{\frac{1}{2}+\delta}-\theta(x)$ *where* $\mu$, $\nu$, $\delta$ *are constants such that* $0<\delta\leqq\frac{1}{2}$ *and* $\nu(1-\nu^2)\neq 0$ *and where* $\theta(x)$ *is any function of* $x$ *such that the series*

$$\sum_{x_1=x}^{x_1=\infty}|\theta(x_1)| \qquad (x\geqq\alpha)$$

*converges.*

*Then if* $|1+\nu|\neq|1-\nu|$, *there are two particular solutions of the equation having the forms*

$$u_1(x)=\frac{t(x)\left(1+\epsilon_1(x)\right)}{z_1(x)},\quad u_2(x)=\frac{t(x)\left(1+\epsilon_2(x)\right)}{z_2(x)} \quad (\lim_{x=\infty}\epsilon_1(x)=\lim_{x=\infty}\epsilon_2(x)=0),$$

*in which*

$$t(x)=\prod_{x_1=\alpha}^{x_1=x-1}\left(\frac{-2a_2}{a_1}\right)_{x_1},\quad z_1(x)=\prod_{x_1=\alpha}^{x_1=x-1}\left\{1+\nu+\frac{\phi(x_1)}{2\nu}\right\},\quad z_2(x)=\prod_{x_1=\alpha}^{x_1=x-1}\left\{1-\nu-\frac{\phi(x_1)}{2\nu}\right\}.$$

*When* $|1+\nu|=|1-\nu|$ *the result continues to hold true provided that*

$$\left|1+\nu+\frac{\phi(x)}{2\nu}\right|=\left|1-\nu-\frac{\phi(x)}{2\nu}\right| \quad \textit{for } x\geqq\alpha.$$

## VI. *Application to the equation considered by Horn.**

7. We proceed to apply the above theorem to the study of the solutions $u(x)$ of the equation,

$$(48)\qquad P_0(x)u(x+2)+x^kP_1(x)u(x+1)+x^{2k}P_2(x)u(x)=0,$$

where $k$ is any integer, positive, negative or zero and $P_0(x)$, $P_1(x)$ and $P_2(x)$ are either convergent series for all sufficiently large values of $x$ or are developable asymptotically in the form

$$P_\lambda(x)=a_\lambda+\frac{b_\lambda}{x}+\frac{c_\lambda}{x^2}+\cdots+\frac{p_\lambda+\omega_\lambda(x)}{x^n} \qquad (\lim_{x=\infty}\omega_\lambda(x)=0).$$

For this purpose we begin by making the transformation $u(x)=[\Gamma(x)]^ky(x)$. Thus equation (48) takes the form

$$(49)\qquad \begin{aligned}&\left(a_0+\frac{b_0}{x}+\frac{c_0}{x^2}+\cdots\right)y(x+2)\\ &\quad+\left(a_1'+\frac{b_1'}{x}+\frac{c_1'}{x^2}+\cdots\right)y(x+1)+\left(a_2'+\frac{b_2'}{x}+\cdots\right)y(x)=0,\end{aligned}$$

* Loc. cit., p. 190.

in which the coefficients of $y(x+1)$ and $y(x)$ are of the form just indicated for $P_\lambda(x)$. In particular,

$$(50) \qquad a_1' = a_1, \qquad b_1' = b_1 - a_1 k, \qquad a_2' = a_2, \qquad b_2' = b_2 - a_2 k.$$

For convenience, let us drop the primes in (49) and take as object of study the equation

$$(51) \quad \left(a_0+\frac{b_0}{x}+\cdots\right)y(x+2)+\left(a_1+\frac{b_1}{x}+\cdots\right)y(x+1)+\left(a_2+\frac{b_2}{x}+\cdots\right)y(x)=0,$$

in which the coefficients are of the type $P_\lambda(x)$.

Equation (4) becomes in the present instance

$$(52) \quad \left(a_1+\frac{b_1}{x}+\frac{c_1}{x^2}+\cdots\right)t(x+1)+2\left(a_2+\frac{b_2}{x}+\frac{c_2}{x^2}+\cdots\right)t(x)=0.$$

Place $t(x)=x^g(1+h/x)v(x)$ where $g$ and $h$ are constants yet to be determined. Equation (52) then takes the form

$$\left(1+\frac{h}{x+1}\right)\left(a_1+\frac{b_1}{x}+\cdots\right)v(x+1)+2\left(1+\frac{1}{x}\right)^{-g}\left(1+\frac{h}{x}\right)\left(a_2+\frac{b_2}{x}+\cdots\right)v(x)=0,$$

or, upon developing $(1+1/x)^{-g}$ by the binomial theorem,

$$(53) \quad \begin{aligned}&\left(a_1+(b_1+ha_1)\frac{1}{x}+(c_1+hb_1-ha_1)\frac{1}{x^2}+\cdots\right)v(x+1)+2\Big(a_2+(b_2-ga_2+ha_2)\frac{1}{x}\\ &\qquad +(c_2-gb_2+hb_2+\tfrac{1}{2}g(g+1)a_2-gha_2)\frac{1}{x^2}+\cdots\Big)v(x)=0.\end{aligned}$$

Let us now choose the undetermined constants $g$, $h$ so that the term in $1/x$ in each coefficient of (53) vanishes. In case $a_1 \neq 0$, $a_2 \neq 0$, these two conditions determine $g$ and $h$ as follows:

$$g=\frac{a_1b_2-a_2b_1}{a_1a_2}, \qquad h=-\frac{b_1}{a_1}.$$

The constants $g$, $h$ having been thus determined, we may now apply directly to equation (53) the results embodied in Theorem III of my previous paper* and write for sufficiently large $\alpha$

$$v(x)=c_1\left(-\frac{2a_2}{a_1}\right)^x\left(1+\epsilon_1(x)\right) \qquad (c_1=\text{const. } x\geqq\alpha,\ \lim_{x=\infty}\epsilon_1(x)=0),$$

and therefore also

$$(54) \qquad t(x)=c_1x^{\frac{a_1b_2-a_2b_1}{a_1a_2}}\left(-\frac{2a_2}{a_1}\right)^x\left(1+\epsilon_2(x)\right) \qquad (\lim_{x=\infty}\epsilon_2(x)=0).$$

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* Loc. cit., p. 313.

We turn now to consider the forms taken by $z_1(x)$, $z_2(x)$ for the equation (51). In the first place, let us construct the expression

$$\frac{4a_0(x)a_2(x+1)}{a_1(x)a_1(x+1)}-1$$

referred to in condition ($b$) of the preceding theorem. Since

$$\frac{a_0(x)}{a_1(x)}=\frac{a_0}{a_1}+\frac{a_1b_0-a_0b_1}{a_1^2}\left(\frac{1}{x}\right)+\cdots,$$

$$\frac{a_2(x+1)}{a_1(x+1)}=\frac{a_2}{a_1}+\frac{a_1b_2-a_2b_1}{a_1^2}\left(\frac{1}{x+1}\right)+\cdots,$$

we obtain

$$4\frac{a_0(x)a_2(x+1)}{a_1(x)a_1(x+1)}-1=-\nu^2-\frac{\mu}{x+1}-\theta(x),$$

in which

(55) $$\nu=\frac{\sqrt{a_1^2-4a_0a_2}}{a_1},$$

(56) $$\mu=4\frac{2a_0a_2b_1-a_1a_2b_0-a_0a_1b_2}{a_1^3},$$

and $\theta(x)$ vanishes to at least the second order when $x=\infty$.

Thus, for the function $z_1(x)$ we have

(57) $$z_1(x)=\prod_{x_1=a}^{x_1=x}\left(1+\nu+\frac{\mu}{2\nu x_1}\right)=g_1(1+\nu)^{x+1}\frac{\Gamma\left(x+1+\frac{\mu}{2\nu(1+\nu)}\right)}{\Gamma(x+1)}$$

where $g_1$ is a constant and $\mu$ and $\nu$ are given by (56) and (55). Similarly,

(58) $$z_2(x)=g_2(1-\nu)^{x+1}\frac{\Gamma\left(x+1-\frac{\mu}{2\nu(1-\nu)}\right)}{\Gamma(x+1)}$$

Let us next consider what conditions ($a$) and ($b$) of Theorem I become in the present instance. Since

$$\nu(\nu^2-1)=\frac{4a_0a_2}{a_1^3}\sqrt{a_1^2-4a_0a_2},$$

they will be satisfied if $a_0a_1a_2(a_1^2-4a_0a_2)\neq 0$. Moreover, the roots of the quadratic equation $a_0\lambda^2+a_1\lambda+a_2=0$ are

$$\lambda_1=(1/2a_0)(-a_1+\sqrt{a_1^2-4a_0a_2}),\qquad \lambda_2=(1/2a_0)(-a_1-\sqrt{a_1^2-4a_0a_2}),$$

so that

$$1+\nu=-\frac{2a_2}{a_1\lambda_1},\qquad 1-\nu=-\frac{2a_2}{a_1\lambda_2}.$$

Also we have

$$\frac{\mu}{2\nu(1+\nu)}=\frac{\mu'}{a_0a_1(a_1\lambda_2+2a_2)},\qquad \frac{-\mu}{2\nu(1-\nu)}=\frac{\mu'}{a_0a_1(a_1\lambda_1+2a_2)},$$

where $\mu'=-\mu a_1^3/4=a_0a_1b_2+a_1a_2b_0-2a_0a_2b_1$.

Noting that $a_1^2-4a_0a_2\neq 0$ whenever $\lambda_1\neq\lambda_2$, we see that if in equation (51) the roots $\lambda_1$, $\lambda_2$ of the quadratic $a_0\lambda^2+a_1\lambda+a_2=0$ are of unequal modulus and $a_0a_1a_2\neq 0$, there are two solutions of the same equation which, when considered for all positive integral values of $x$ sufficiently large, take the forms

$$(59)\quad y_1(x)=x^g\frac{\Gamma(x)\lambda_1^x(1+\epsilon_1(x))}{\Gamma\left(x+\dfrac{p}{q\lambda_2+r}\right)},\qquad y_2(x)=x^g\frac{\Gamma(x)\lambda_2^x(1+\epsilon_2(x))}{\Gamma\left(x+\dfrac{p}{q\lambda_1+r}\right)},$$

wherein $\lim_{x=\infty}\epsilon_1(x)=\lim_{x=\infty}\epsilon_2(x)=0$, and the constants have the values

$$g=\frac{a_1b_2-a_2b_1}{a_1a_2},\quad p=a_0a_1b_2+a_1a_2b_0-2a_0a_2b_1,\quad q=a_0a_1^2,\quad r=2a_0a_1a_2.$$

Moreover, for the cases in which $\lambda_1$, $\lambda_2$ are distinct but of equal modulus we see from (57) and (58) that the result just obtained will continue to hold true, by Theorem I, provided that for all $x\geqq\alpha$ we have

$$|x+\mu/2\nu(1+\nu)|=|x-\mu/2\nu(1-\nu)|,$$

i. e. $|x+c_1|=|x+c_2|$ where

$$c_1=\frac{p}{q\lambda_2+r},\qquad c_2=\frac{p}{q\lambda_1+r}.$$

But if this latter condition is satisfied, it is evident that $c_1$ and $c_2$ are conjugate imaginaries, and conversely. Thus the result already obtained when $|\lambda_1|\neq|\lambda_2|$ will continue to hold true when $|\lambda_1|=|\lambda_2|$, provided that $\lambda_1\neq\lambda_2$, and either $c_1$, $c_2$ are conjugate imaginaries or $c_1=c_2=0$.

We note also that the form of the solutions (59) may be somewhat simplified by making use of the well known asymptotic relation

$$\Gamma(x)\sim\sqrt{2\pi}e^{-x}x^{x-\frac{1}{2}}.$$

Thus for any constant $l$ we have

$$\frac{\Gamma(x+l)}{\Gamma(x)}\sim e^{-l}(x+l)^l\left(1+\frac{l}{x}\right)^{x-\frac{1}{2}}\sim e^{-l}x^l\left(1+\frac{l}{x}\right)^x\sim x^l,$$

so that relations (59) may be replaced by

$$(60)\qquad y_1(x)=x^{h_1}\lambda_1^x(1+\epsilon_1(x)),\qquad y_2(x)=x^{h_2}\lambda_2^x(1+\epsilon_2(x)),$$

wherein

$$h_1 = g - c_1 = \frac{a_1 b_2 - a_2 b_1}{a_1 a_2} - \frac{p}{q\lambda_2 + r}, \qquad h_2 = \frac{a_1 b_2 - a_2 b_1}{a_1 a_2} - \frac{p}{q\lambda_1 + r}.$$

8. Besides the cases already considered in which $a_0 a_1 a_2 \neq 0$ it is deserving of note that whenever $a_0 a_2 \neq 0$, $a_1 = b_1 = 0$, the nature of the solutions of (51) for large positive values of $x$ may still be found by the application of known results. For after making the transformation

$$y(x) = x^g\left(1 + \frac{h}{x}\right)v(x)$$

the same equation takes the form

$$A_0(x)v(x+2) + A_1(x)v(x+1) + A_2(x)v(x) = 0,$$

where

$$A_0(x) = a_0 + (b_0 + ha_0)\frac{1}{x} + \cdots,$$

$$A_1(x) = \frac{c_1}{x^2} + \cdots,$$

$$A_2(x) = a_2 + (b_2 - 2ga_2 + ha_2)\frac{1}{x} + \cdots.$$

Then by choosing $h$ and $g$ so that the coefficients of $1/x$ here appearing vanish, i. e.,

$$g = \frac{a_0 b_2 - a_2 b_0}{2a_0 a_2}, \qquad h = -\frac{b_0}{a_0},$$

we may at once apply to equation (51) the Theorem III of the aforesaid memoir. As the roots of the equation $a_0\lambda^2 + a_2 = 0$ are unequal but of equal modulus, we conclude that there are two solutions under the present hypotheses having respectively the forms

$$(61) \quad y_1(x) = x^g\lambda_1^x(1+\epsilon_1(x)), \quad y_2(x) = x^g\lambda_2^x(1+\epsilon_2(x)) \quad (\lim_{x=\infty}\epsilon_1(x) = \lim_{x=\infty}\epsilon_2(x) = 0),$$

where

$$g = (a_0 b_2 - a_2 b_0)/2a_0 a_2.$$

9. Returning now to the original equation (48) and recalling that

$$u(x)[\Gamma(x)]^k y(x) = \left(\frac{x}{e}\right)^{kx} x^{-\frac{1}{2}k}(2\pi)^{\frac{1}{2}k} y(x)\left(1 + \epsilon(x)\right),$$

where $\lim_{x=\infty}\epsilon(x) = 0$, and making use of equations (50) we reach in summary the following theorem:

THEOREM II. *Given the equation*

$$P_0(x)u(x+2) + x^k P_1(x)u(x+1) + x^{2k}P_2(x)u(x) = 0,$$

*where $k$ is any integer, positive, negative or zero, while* $P_0(x)$, $P_1(x)$, $P_2(x)$

*are convergent series for sufficiently large values of* $x$ , *or are any functions of* $x$ *developable asymptotically in the form*

$$P_\lambda(x) = a_\lambda + \frac{b_\lambda}{x} + \frac{c_\lambda}{x^2} + \cdots + \frac{p_\lambda + \omega_\lambda(x)}{x^n} \qquad (\lim_{x=\infty} \omega_\lambda(x) = 0).$$

CASE I. *If* $a_0 a_1 a_2 \neq 0$ *and if the roots* $\lambda_1$, $\lambda_2$ *of the quadratic equation* $a_0\lambda^2 + a_1\lambda + a_2 = 0$ *are of unequal modulus, there are two solutions of the given equation which, when considered for all positive integral values of* $x$ *sufficiently large, take the respective forms*

$$u_1(x) = \left(\frac{x}{e}\right)^{kx} x^{\rho+\sigma_2}\lambda_1^x\left(1+\epsilon_1(x)\right), \quad u_2(x) = \left(\frac{x}{e}\right)^{kx} x^{\rho+\sigma_1}\lambda_2^x\left(1+\epsilon_2(x)\right)$$

$$(\lim_{x=\infty} \epsilon_1(x) = 0, \lim_{x=\infty} \epsilon_2(x = 0)).$$

*wherein* $\rho$, $\sigma_1$, $\sigma_2$ *are constants defined as follows:*

$$\rho = \frac{a_1 b_2 - a_2 b_1}{a_1 a_2} - \frac{k}{2},$$

$$\sigma_i = \frac{2a_0 a_2 b_1 - a_0 a_1 b_2 - a_1 a_2 b_0 - k a_0 a_1 a_2}{a_0 a_1 (a_1 \lambda_i + 2a_2)} \qquad (i = 1, 2).$$

*Moreover, the same result holds true whenever the roots* $\lambda_1$, $\lambda_2$ *are unequal but of equal modulus, provided that* $\sigma_1$ *and* $\sigma_2$ *are conjugate imaginaries, including the case in which* $\sigma_1 = \sigma_2 = 0$.

CASE II. *If* $a_0 a_2 \neq 0$, $a_1 = b_1 = 0$ *and if we represent by* $\lambda_1$, $\lambda_2$ *the roots of the quadratic equation* $a_0\lambda^2 + a_2 = 0$, *there are two solutions of the given equation which when considered for values of* $x$ *sufficiently large take the forms*

$$u_1(x) = \left(\frac{x}{e}\right)^{kx} x^{\rho}\lambda_1^x\left(1+\epsilon_1(x)\right), \qquad u_2(x) = \left(\frac{x}{e}\right)^{kx} x^{\rho}\lambda_2^x\left(1+\epsilon_2(x)\right)$$

$$(\lim_{x=\infty} \epsilon_1(x) = \lim_{x=\infty} \epsilon_2(x) = 0),$$

*where* $\rho$ *is defined by the relation*

$$\rho = \frac{a_0 b_2 - a_2 b_0}{2a_0 a_2} - \frac{k}{2}.$$

**10. The results obtained under Case I of the above theorem are in accord with those obtained by HORN.* To show this we evidently need to show merely that our values of $\rho + \sigma_1$, $\rho + \sigma_2$ are equal respectively to the quantities $\rho_1$, $\rho_2$ employed by HORN, and defined† by the relation**

$$\rho_1 = -\frac{k(a_1 + 4a_0\lambda_1)}{2(a_1 + 2a_0\lambda_1)} - \frac{b_2 + b_1\lambda_1 + b_0\lambda_1^2}{a_1\lambda_1 + 2a_0\lambda_1^2},$$

**with a similar formula for $\rho_2$ obtained by replacing $\lambda_1$ by $\lambda_2$.**

* Loc. cit., p. 192.

† Loc. cit., p. 191, footnote.

Now in the sum $\rho + \sigma_1$ the coefficient of $\frac{1}{2}k$ is $-1 - 2a_2/(a_1\lambda_2 + 2a_2)$. But the coefficient of $-\frac{1}{2}k$ in $\rho_1$ may be written in the form

$$-1 - \frac{2a_0\lambda_1}{a_1 + 2a_0\lambda_1} = -1 - \frac{2a_0\lambda_1\lambda_2}{a_1\lambda_1 + 2a_0\lambda_1\lambda_2},$$

which becomes the same as the coefficient of $-\frac{1}{2}k$ in $\rho + \sigma_1$ since $\lambda_1\lambda_2 = a_2/a_0$. It remains only to show that

$$\frac{a_1b_2 - a_2b_1}{a_1a_2} + \frac{2a_0a_2b_1 - a_0a_1b_2 - a_1a_2b_0}{a_0a_1(a_1\lambda_2 + 2a_2)} = -\frac{b_2 + b_1\lambda_1 + b_0\lambda_1^2}{a_1\lambda_1 + 2a_0\lambda_1^2}.$$

This reduces to an identity by virtue of the relations

$$a_1\lambda_2 + 2a_2 = \frac{a_2}{a_0\lambda_1}(a_1 + 2a_0\lambda), \qquad a_0\lambda_1^2 + a_1\lambda_1 + a_2 = 0.$$

Similarly, by using $\lambda_2$ instead of $\lambda_1$ we obtain $\rho + \sigma_2 = \rho_2$.

It is to be observed that HORN's work concerns only case I.

11. As an illustration of the application of the preceding theorem, let us consider the equation

$$(62)\quad \left(1 + \frac{2}{x}\right)u(x+2) - \left(2 + \frac{3}{x}\right)zu(x+1) + \left(1 + \frac{1}{x}\right)u(x) = 0,$$

one of whose solutions is Legendre's function of the first kind $P_x(z)$. For simplicity, we shall confine ourselves to real values of $z$.

Here we have $k = 0$, $a_0 = 1$, $a_1 = -2z$, $a_2 = 1$, $b_0 = 2$, $b_1 = -3z$, $b_2 = 1$. Whence the roots $\lambda_1, \lambda_2$ are those of the quadratic $\lambda^2 - 2z\lambda + 1 = 0$; i. e., $\lambda_1 = z + \sqrt{z^2 - 1}$, $\lambda_2 = z - \sqrt{z^2 - 1}$. Thus, we shall have $|\lambda_1| \neq |\lambda_2|$ if $|z| > 1$, while we shall have $|\lambda_1| = |\lambda_2|$ but $\lambda_1 \neq \lambda_2$ if $|z| < 1$. Applying Case I of Theorem II, observing that $a_0a_1a_2 \neq 0$ when $z \neq 0$ and that in the present instance $\rho = -\frac{1}{2}$, $\sigma_1 = \sigma_2 = 0$, we find that for all real values of $z$ except $z = 0$ the general solution of the above equation, when considered for all positive integral values of $x$ sufficiently large takes the form

$$u(x) = \frac{1}{\sqrt{x}}\left[k_1(z + \sqrt{z^2 - 1})^x\big(1 + \epsilon_1(x)\big) + k_2(z - \sqrt{z^2 - 1})^x\big(1 + \epsilon_2(x)\big)\right]$$

$$\left(\lim_{x=\infty}\epsilon_1(x) = \lim_{x=\infty}\epsilon_2(x) = 0\right).$$

$k_1, k_2$ being arbitrary constants.

Moreover, precisely the same result holds when $z = 0$, as appears directly by applying Case II of the same theorem.

If, in particular, $-1 < z < 1$, we may place $z = \cos\xi$ and write $(z + \sqrt{z^2 - 1})^x = \cos x\xi + i\sin x\xi$. The solution $u(x)$ then takes the form

$$u(x) = \frac{1}{\sqrt{x}}\left[k_1\big(1 + \epsilon_1(x)\big)\cos x\xi + k_2\big(1 + \epsilon_2(x)\big)\sin x\xi\right],$$

where $k_1$, $k_2$, $\epsilon_1$, $\epsilon_2$ have the properties already mentioned. Moreover, upon determining two constants $\lambda$, $\mu$ from the equations $\lambda \sin \mu = k_1$, $\lambda \cos \mu = k_2$, and inserting for the constants $k_1$, $k_2$ these expressions, we obtain for $u(x)$ the form

$$u(x) = \frac{\lambda}{\sqrt{x}} [\sin(x\xi + \mu) + \epsilon(x)] \qquad (\lim_{x=\infty} \epsilon(x) = 0)$$

where $\lambda$ and $\mu$ are arbitrary constants.

This last result for the special case in which $u(x) = P_x(z)$ agrees with other well known results respecting the behavior of Legendre's function of the first kind for large values of $x$. Previous investigations upon the subject, however, appear to have been from the standpoint of the *differential* equation satisfied by $P_x(z)$ rather than from that of the *difference* equation (62).*

* Cf. DINI, *Studi sulle equazioni differenziali*, Annali di Matematica, ser. 3, vol. 3 (1899), p. 178.