

ON THE ORDER OF LINEAR HOMOGENEOUS GROUPS*

(FOURTH PAPER)

BY

H. F. BLICHFELDT

1. In the writer's theorems † on finite groups of linear homogeneous substitutions of determinant unity, a group of special nature called a *self-conjugate* (or *invariant*) *subgroup* H plays an important rôle. There is a lack of completeness to these theorems due to the fact it has not been proved that H is actually less than the transitive (irreducible) group G in which, under certain conditions, it is contained; i. e., the groups G and H may be identical so far as the theorems are concerned. Account had to be taken of this fact in constructing the collineation-groups for the plane and space.‡

The relation between G and H is as follows. Let the number of variables be n , and let V be any substitution of G . The sum of the multipliers of V (weight, characteristic) we shall indicate by (V) , which, therefore, represents the sum of n roots of unity:

$$(V) = \sum_{i=1}^n \alpha_i \beta_i.$$

Each of these roots we write as the product of one α_i , whose index is prime to a given prime number p , and one β_i , whose index is a power of p . By $(V)_p$, we indicate the quantity obtained by replacing in (V) every root β_i by unity:

$$(V)_p = \sum_{i=1}^n \alpha_i.$$

Then, under certain conditions, as mentioned above, there is in G a self-conjugate subgroup H whose substitutions T have the property

$$(VT)_p \equiv (V)_p \pmod{p};$$

from which, in particular,

$$(T)_p \equiv n \pmod{p}.$$

* Presented to the Society (San Francisco), September 24, 1910.

† The theorems chiefly concerned are as follows: Theorems 10 and 11 of the article *On the Order of Linear Homogeneous Groups (Second Paper)*, these *Transactions*, vol. 5 (1904), pp. 315, 317; Theorem 14 of the article *On the Order of Linear Homogeneous Groups (Supplement)*, *ibid.*, vol. 7 (1906), p. 523. These two articles will be referred to by *LGII* and *LGS* respectively.

‡ *Mathematische Annalen*, vol. 60 (1905), p. 214, § 16; vol. 63 (1907), p. 563, § 15.

2. The writer has recently been able to prove that H (or more strictly, a modified form of it) is actually less than G when this group is primitive, so that in all such cases G cannot be a simple group. Furthermore, it follows that

$$n \equiv 0 \pmod{p}.$$

Theorem 14 of *LGS* may be modified to read:

THEOREM 14'. *If for $n > 1$ a primitive group G in n variables has an abelian subgroup K of order $p^a \geq p^n$, then G will have an invariant subgroup H which contains at least p^{a-n+1} substitutions of K , but which does not contain K entirely. If T be any substitution of H , and V any substitution of G whose weight (V) is zero,* then*

$$(VT)_p \equiv 0 \pmod{p}.$$

Moreover,

$$(T)_p \equiv 0, \quad \text{and} \quad n \equiv 0 \pmod{p}.$$

Let the substitutions of K be represented by $S_0 = \text{identity}$, S_1, \dots, S_{p^a-1} .

In proving Theorem 14, *LGS*, it was first proved that for at least p^{a-n+1} substitutions of K ,† say

$$(1) \quad S_0, S_1, \dots, S_i, \dots,$$

we have equations of the form

$$(VS_i) - (V) + (1 - \theta)X_i = 0,$$

V being any substitution of G , θ a root of the equation

$$\theta^{p^a} - 1 = 0,$$

and X_i a sum of the weights

$$(2) \quad (VS_j) \quad (j=0, 1, \dots, p^a-1),$$

multiplied by integral functions of θ , the numerical coefficients entering being integers or fractions whose denominators are prime to p .

Now let V be any substitution of G such that (V) is zero. Then

$$(3) \quad (VS_i) + (1 - \theta)X_i = 0$$

for all substitutions S_i of the series (1).

Consider all the weights (2).

* BURNSIDE has proved the following theorem: "In any irreducible group of linear substitutions of finite order, other than a cyclical group in a single variable, at least one of its characteristics is zero," *Proceedings of the London Mathematical Society*, ser. 2, vol. 1 (1903), p. 115.

† Attention is here called to an omission in *LGS*, page 524. First line below the matrix reads, "Now, to this matrix may be added p^{a-m+1} rows . . ." This should read: "Now, to this matrix may be added $p^{a-m+1} - 1$ rows . . ."

First, they may all vanish, whatever be the substitution V chosen, so long as (V) vanishes. By the arguments of § 7, *LGII*, all the substitutions of K and all further substitutions T of G for which (VT) vanishes, form a group H , self-conjugate in G . If H had any weight (W) which vanished, then every (WT) would vanish, and therefore every (T) . But this is impossible, since $(S_0) = n$. Accordingly, every weight of H is non-vanishing, and therefore H is intransitive, by Burnside's Theorem. It follows by Theorem 8, *LGII*, and by Burnside's Theorem that H is composed of similarity-substitutions.

Second, the weights (2) do not all vanish. By (3), some of them are divisible by $1 - \theta$, the quotient being expressible as a linear function of a finite number of roots of unity, no numerical coefficient entering having a denominator which is divisible by p . Assume that all the weights (2) are divisible by $(1 - \theta)^k$, whenever V represents a substitution of G whose weight is divisible by $(1 - \theta)^k$, k ranging through the values $0, 1, 2, \dots, m$; but that the weights (2) are not all divisible by $(1 - \theta)^{m+1}$ whenever (V) is divisible by $(1 - \theta)^{m+1}$. In general, we should not expect m to be greater than zero. Now, all the substitutions T of G for which

$$(VT) = (1 - \theta)^{m+1}X,$$

whenever

$$(V) = (1 - \theta)^{m+1}Y,$$

form an invariant subgroup H . To H belongs the series (1), but not the entire group K .

Since G is assumed to be primitive, H , if not composed of similarity-substitutions, must contain a substitution W whose weight vanishes (Theorem 8, *LGII*, and Burnside's Theorem). Then

$$(WT) = (1 - \theta)^{m+1}Z$$

for every substitution T of H ; i. e., every

$$(WT)_p \equiv 0 \pmod{p},$$

and therefore every

$$(T)_p \equiv 0 \pmod{p}.$$

The Theorems 10 and 11, *LGII*, may be modified in like manner to read that the self-conjugate subgroup H is less than G , the latter being primitive. In addition,

$$(T)_p \equiv 0 \pmod{p}$$

for every substitution of H .

3. One of the most important problems in the theory of linear homogeneous groups is the determination of the maximum order; i. e., the fixing of a superior limit to λ in Jordan's Theorem, the number of variables being given. The

limit known* can now be greatly reduced in special cases. Let G be a primitive group, n the number of variables, and p a prime. Then, using Theorem 14' in conjunction with Theorem 9, *LGII*, we can prove the

COROLLARY. *If p and n are prime to each other, the highest power of p which divides the order of G must divide $n! p^{n-1}$.*

If $n = p$, an invariant subgroup H (assumed to contain and to be greater than the group of similarity-substitutions of order p) must be of order p^k and cannot be abelian, G being assumed primitive. Writing H in monomial form we readily find that, if $k > 3$, it possesses one, and only one, invariant of degree p which can be factored into p linear factors. In such a case G cannot be primitive. Hence, H is of order p^3 (or p^2 when considered as a collineation-group), being generated by the substitutions

$$\begin{aligned} A: x'_1 &= x_1, x'_2 = \theta x_2, \dots, x'_p = \theta^{p-1} x_p; \frac{\theta^p - 1}{\theta - 1} = 0; \\ B: x'_1 &= x_2, x'_2 = x_3, \dots, x'_p = x_1. \end{aligned}$$

The order of G is a factor of $(p^2 - 1)p \cdot p^2$ when considered as a collineation-group. The corollary above is therefore true also when $n = p > 2$.

THEOREM 17. *Let G be a primitive collineation-group in n variables, n being a prime > 1 . Then the order of G is a factor of*

$$(4) \quad n! (2 \cdot 3 \cdot \dots \cdot p \cdot \dots)^{n-1},$$

2, 3, ..., p, ... being the different primes not greater than the greater of the numbers $4n - 3, (n - 2)(2n + 1)$.† The only exception is the octahedral group of order 24 in two variables.

It will be noticed that a transitive group in a prime number of variables n is either primitive or of monomial type. In either case, the corresponding Jordan factor λ must divide the number (4) if $n > 2$.

* See *LGII*, pp. 310, 320-321, and *On Imprimitve Linear Homogeneous Groups, these Transactions*, vol. 6 (1905), p. 232, for an upper limit for all transitive groups. SCHUR has given an upper limit when the weights belong to a given field: *Sitzungsberichte der Kgl. Preussischen Akademie der Wissenschaften*, 1905, p. 77 ff.

† *LGS*, p. 528.