BIORTHOGONAL SYSTEMS OF FUNCTIONS*

BY

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Introduction.

In boundary value problems of differential equations which are not self-adjoint, biorthogonal systems of functions play the same rôle as the orthogonal systems do in the self-adjoint case. LIOUVILLE† has considered special non-self-adjoint differential equations with real characteristic values of the parameter; BIRKHOFF‡ has proved the existence of the characteristic values (in general complex) for the differential equation of the nth order and obtained the related expansions.

If the integral equation

$$u(s)=\lambda\int_a^b L(s,t)u(t)\,dt$$

with the unsymmetric kernel $L(s,t)$ has solutions $u(s)$, and therefore the integral equation

$$v(s)=\lambda\int_a^b L(t,s)v(t)\,dt$$

solutions $v(s)$, it has been shown by PLEMELJ§ and GOURSAT‖ that the solutions or functions closely related to them form a biorthogonal system. But expansions in terms of these solutions have not been obtained, and no criteria have been given for the existence of real characteristic numbers of an unsymmetric kernel.¶

The object of this paper is the development of a theory of biorthogonal systems of functions independent of their connection with integral or differential equations. In the theory frequent use is made of the theorems by RIESZ, FISCHER, and TOEPLITZ (Lemmas 1, 2, 3, and 4, § 2).

* Presented to the Society (Chicago) April 10, 1909.

† Liouville's Journal, ser. 1, vol. 3 (1838).

‡ Transactions of the American Mathematical Society, vol. 9 (1908).

§ Monatshefte für Mathematik und Physik, vol. 15 (1904).

‖ Annales de Toulouse, 1908.

¶ Since this paper was written J. MARTY has published some results for unsymmetric kernels, Comptes Rendus, February 28, March 7, April 25, and June 6, 1910. See also the note by the author in Bulletin of the American Mathematical Society, July, 1910.

Necessary and sufficient conditions for the existence of the adjoint system $\{v_i\}$ of any system of linearly independent functions $\{u_i\}$ are deduced (theorem 2, § 3). Theorems for biorthogonal systems analogous to those of RIESZ and FISCHER for orthogonal systems are: (*a*) if

$$\sum_{i=1}^{\infty}\left(\int f v_i\right)^2$$

converges for every function f which is integrable and has an integrable square, then for every system of constants $\{c_i\}$ of finite norm there exists a function g such that $c_i = \int g u_i$ (corollary 1, theorem 6); (*b*) if the system $\{u_i\}$ is complete and if certain conditions are imposed on the function g, then (theorem 8)

$$\int fg = \sum_{i=1}^{\infty}\int f v_i \int u_i g.$$

The equivalence of two biorthogonal systems is defined and a classification into types is made (§ 7). With each type satisfying a certain condition there is connected uniquely (§ 6) and in a reversible way (theorem 20, § 8) a single-valued functional transformation* $T(f)$, which transforms every function which is integrable and has an integrable square into a function of the same kind, and is defined by the properties

$$T(a_1 f_1 + a_2 f_2) = a_1 T(f_1) + a_2 T(f_2), \tag{1}$$

$$\int [f_1 T(f_2) - f_2 T(f_1)] = 0, \tag{2}$$

$$\int f T(f) \geqq 0. \tag{3}$$

The class of all orthogonal systems of functions is a special type, for which $T(f)$ is the identical transformation. Other special cases of $T(f)$ are

$$T(f) = f(s) - \frac{p(s)\int_a^b p(s) f(s)\,ds}{\int_a^b [p(s)]^2 ds},$$

$$T(f) = \int_a^b K(s, t) f(t)\,dt.$$

By means of the theorems by RIESZ, FISCHER and TOEPLITZ, it can be shown that there is a one-to-one correspondence between this functional transformation $T(f)$ and positive definite limited quadratic forms in infinitely many variables.

* Linear functional transformations have recently been studied by F. RIESZ, *Mathematische Annalen*, vol. 69 (1910), p. 449.

§ 1. *Fundamental Notations.*

We consider an arbitrary interval I: $a \leqq s \leqq b$: of the real variable s, and the corresponding square S: $a \leqq s \leqq b$, $a \leqq t \leqq b$: of the real variables s and t, and real functions of these variables. We denote functions of a single variable by f, g, φ, ϕ, etc., omitting the argument. $\mathfrak{F}$ is the class of all functions f such that f and f^2 are integrable in the sense of Lebesgue* on the interval I. As usual, we regard two functions of $\mathfrak{F}$ as equal if they differ only on a set of points of content zero.† We call two such functions "essentially equal."

When there is no ambiguity we omit the variable of integration In all cases the limits of integration are omitted, since I is the only interval of integration considered in this paper.

Constants and functions with the subscripts i, j, k denote sequences of constants and functions.

$\mathfrak{G}$ and $\mathfrak{H}$ denote subclasses of $\mathfrak{F}$ and the elements of these subclasses are denoted by the small letters g and h.

To express that a relation involving the function g holds for every function of the class we write (g) after the relation; for example

$$\int f^2 \geqq 0 \qquad (f).$$

We use a similar notation in the case of several variables; for example

$$\left(\int f_1 f_2\right)^2 \leqq \int f_1^2 \int f_2^2 \qquad (f_1, f_2).$$

Let $\{f_i\}$ denote any sequence of linearly independent functions of the class $\mathfrak{F}$, and $\{\phi_i\}$ the sequence of normalized and orthogonal ‡ functions which are obtained linearly from the functions f_i by means of the construction given by E. Schmidt, §

$$\phi_i = \frac{f_i - \sum_{k=1}^{i-1} \phi_k \int \phi_k f_i}{\sqrt{\int (f_i - \sum_{k=1}^{i-1} \phi_k \int \phi_k f_i)^2}} \qquad (i).$$

Each ϕ_i may be expressed linearly and homogeneously with constant coefficients in terms of $f_1, f_2, \cdots, f_i$, and conversely:

(1) $$\phi_i = \sum_{k=1}^{i} a_{ik} f_k, \qquad f_i = \sum_{k=1}^{i} b_{ik} \phi_k \qquad (i).$$

* Lebesgue, *Sur l'intégration*, p. 115.

† Lebesgue, l. c., p. 106.

‡ A function f is normalized if $\int f^2 = 1$. Two functions f_1 and f_2 are orthogonal if $\int f_1 f_2 = 0$.

§ *Zur Theorie der linearen und nichtlinearen Integralgleichungen.* Mathematische Annalen, vol. 63 (1907), p. 442.

The constants a_{ik}, b_{ik} have the following properties:

$$(2)\qquad \begin{aligned} a_{ii} &\neq 0 && (i), \\ a_{ik} &= 0 && (k>i), \\ b_{ii} &\neq 0 && (i), \\ b_{ik} &= 0 && (k>i). \end{aligned}$$

The matrices formed from the coefficients a_{ik} and b_{ik} we denote by $(a_{ik})_f$ and $(b_{ik})_f$ respectively, or simply by $(a)_f$ and $(b)_f$.

If $\{\bar{\phi}_i\}$ are the normalized and orthogonal functions corresponding to $\{\bar{f}_i = \lambda_i f_i\}$ where $\{\lambda_i\}$ are constants different from zero, then $\bar{\phi}_i = \phi_i$.

§ 2. *Preliminary definitions and lemmas.*

Definition. A system of functions of $\mathfrak{F}$ is called *complete* * for the interval I if there exists no function f of $\mathfrak{F}$, essentially different from zero, which is orthogonal to all the functions of the system in the interval I.

Definition. A sequence of constants $\{c_i\}$ is of *finite norm* if the sum of the squares,

$$\sum_{i=1}^{\infty} c_i^2,$$

converges, and it has the norm M if

$$\sum_{i=1}^{\infty} c_i^2 = M.$$

For any system $\{\phi_i\}$ of normalized and orthogonal functions of $\mathfrak{F}$ we have the three following lemmas.†

Lemma 1. The sequence of constants

$$\left\{\int f\phi_i\right\}$$

is of finite norm for every function f of $\mathfrak{F}$.

Lemma 2. If the system $\{\phi_i\}$ is complete,

$$\int f_1 f_2 = \sum_{i=1}^{\infty} \int f_1\phi_i \int \phi_i f_2 \qquad (f_1, f_2).$$

* F. RIESZ, Comptes Rendus, November, 1906, p. 738.

† E. SCHMIDT, l. c., p. 439; F. RIESZ, Comptes Rendus, March, 1907, p. 615, and April, 1907, p. 734; E. FISCHER, Comptes Rendus, May, 1907, p. 1022.

Lemma 3. To every system of constants $\{c_i\}$ of finite norm there corresponds a function f of $\mathfrak{F}$ such that

$$c_i = \int f\phi_i \qquad (i),$$

and this function f is essentially unique if the system $\{\phi_i\}$ is complete.

Definition. A matrix (a_{ik}) is *limited* if

$$\sum_{i=1}^{\infty} a_{ik} x_i$$

converges for every k, and

$$\left\{ \sum_{i=1}^{\infty} a_{ik} x_i \right\}$$

is of finite norm for every system of values of $\{x_i\}$ of finite norm.

Definition. A bilinear form of the infinitely many variables x_i and y_i

$$\sum_{k=1}^{\infty} \sum_{i=1}^{\infty} a_{ik} x_i y_k$$

is limited if for some positive constant M

$$\left| \sum_{k=1}^{n} \sum_{i=1}^{n} a_{ik} x_i y_k \right| \leqq M \qquad (n)$$

for all values of $\{x_i\}$ and $\{y_i\}$ of norm $\leqq 1$ (H, IV, p. 176 *).

Lemma 4. The matrix of the coefficients of a limited bilinear form is limited (H, IV, p. 179); conversely, in virtue of a theorem by TOEPLITZ † a bilinear form whose coefficients are the elements of a limited matrix is limited.

Lemma 5. If the matrix (a_{ik}) is limited the matrix (a_{ki}) is limited also.

Lemma 6. The product of two limited matrices is a limited matrix (H, IV, p. 179).

Definition. A matrix (a_{ik}) is orthogonal if

$$\sum_{i=1}^{\infty} a_{ij} a_{ik} = \begin{cases} 1 & j = k \\ 0 & i \neq k \end{cases} \qquad (j, k),$$

$$\sum_{i=1}^{\infty} a_{ji} a_{ki} = \begin{cases} 1 & i = k \\ 0 & j \neq k \end{cases} \qquad (j, k).$$

Lemma 7. An orthogonal matrix is limited (H, IV, p. 180). If some of the elements of the orthogonal matrix are replaced by zeros, the resulting matrix is limited.

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* The references to the memoirs by HILBERT in the Göttinger Nachrichten, 1904–1906, are denoted by (H, I, II, $\cdots$, V).

† E. SCHMIDT, Palermo Rendiconti, vol. 25 (1908), p. 2; HELLINGER and TOEPLITZ, Göttinger Nachrichten, 1906, p. 351, and Mathematische Annalen, vol. 69 (1910), p. 289.

Lemma 8. The value of a limited bilinear form for arguments $\{x_i\}$ and $\{y_i\}$ of norm 1 is obtained by summation either by rows or by columns (*H*, IV, p. 180).

Lemma 9. If a_{ik} are the elements of a limited matrix, and if $a_{ik}=a_{ki}$, then the system of homogeneous linear equations

$$x_i-\lambda\sum_{k=1}^{\infty}a_{ik}x_k=0 \qquad (i)$$

has a solution $\{x_i\}$ of finite norm when and only when λ is a "characteristic number," that is, a root of a certain associated equation, and the number of linearly independent solutions is equal to the multiplicity of the root (*H*, IV, p. 198).

Definition. Two systems* (u_i) and (v_i) of functions of $\mathfrak{F}$ form a *biorthogonal system* of functions (u_i, v_i) if a one-to-one correspondence can be established between them such that the integral of the product of two corresponding functions is equal to unity and the integral of the product of two non-corresponding functions is equal to zero; i. e.,

$$\text{(3)} \qquad \int u_i v_j=\begin{cases}1 & (i=j),\\ 0 & (i\neq j).\end{cases}$$

Each one of the two systems (u_i), (v_i) is called an *adjoint* system of the other.

The biorthogonal system (u_i, v_i) is *complete as to* u, v, respectively, if the system (u_i), (v_i), respectively, is complete.

For a biorthogonal system (u_i, v_i) the functions u_i and also v_i are linearly independent. Suppose there existed a linear relation

$$c_1u_1+c_2u_2+\cdots+c_nu_n=0.$$

Multiply by $v_i(i=1, 2, \cdots, n)$ and integrate; from the properties (3) we obtain

$$c_i=0 \qquad (i=1, 2, \cdots, n).$$

§ 3. *Denumerability and existence of the adjoint system.*

Theorem 1. *A system (u_i) of functions of $\mathfrak{F}$ having an adjoint system (v_i) is denumerable.*

We follow the method used by RIESZ † to prove that an orthogonal system of functions is denumerable. The proof is based on the following theorem: If for

* Until after theorem 1 the subscripts i and j are used to denote not only sequences but any system of functions.

† Comptes Rendus, November, 1906, p. 738.

every sequence $\{f_i\}$ of linearly independent functions selected from a given system

$$\int (f_{i_1} - f_{i_2})^2 \geqq 1 \qquad (i_1, i_2),$$

then the given system is denumerable.

Without loss of generality we may assume that

$$\int v_i^2 = 1 \qquad (i).$$

Let $\{u_i\}$ be any sequence of distinct functions selected from the system (u_i) and let v_i be the corresponding functions of the adjoint system. By Bessel's inequality we have

$$1 = \left[\int v_{i_1} (u_{i_1} - u_{i_2}) \right]^2 \leqq \int (u_{i_1} - u_{i_2})^2 \qquad (i_1, i_2).$$

The hypotheses of the theorem are satisfied and the system (u_i) is denumerable. When the system (u_i) is complete it cannot be finite, and is therefore denumerably infinite.

Let $\{\varphi_i\}$ and $\{\psi_i\}$ be the systems of normalized orthogonal functions constructed linearly from the functions $\{u_i\}$ and $\{v_i\}$, respectively, of the biorthogonal system (u_i, v_i). Connected with these systems of functions we have the four matrices $(a)_u$, $(b)_u$, $(a)_v$, $(b)_v$. (§ 1.)

Theorem 2. *The necessary and sufficient condition that a given system of functions* $\{u_i\}$ *of* $\mathfrak{F}$ *have an adjoint system is, that the* a_{ik} *of* $(a)_u$ *be of finite norm for every* k.

Suppose the condition is satisfied; then from lemma 3 follows the existence of a system of functions $\{v_i\}$ of $\mathfrak{F}$ such that

$$(4) \qquad a_{ik} = \int \varphi_i v_k \qquad (i, k).$$

The equations connecting $\{u_i\}$ and $\{\varphi_i\}$ are

$$(5) \qquad \varphi_i = \sum_{k=1}^{i} a_{ik} u_k \qquad (i)$$

and

$$(6) \qquad u_i = \sum_{k=1}^{i} b_{ik} \varphi_k .$$

Multiply the equation (6) by v_j and integrate:

$$\int u_i v_j = \sum_{k=1}^{i} b_{ik} \int \varphi_k v_j .$$

By substitution from the equation (4):

$$\int u_i v_j = \sum_{k=1}^{i} b_{ik} a_{kj} .$$

Since the matrices (b_{ik}) and (a_{kj}) are each the inverse of the other we obtain

$$\int u_i v_j = \begin{cases} 0 & i = j, \\ 1 & i \neq j. \end{cases}$$

Thus by definition the system $\{v_i\}$ is an adjoint system of $\{u_i\}$.

Suppose conversely that an adjoint system $\{v_i\}$ exists. Multiply the equations (5) by v_k and integrate

$$a_{ik} = \int \varphi_i v_k \qquad (i, k).$$

Applying lemma 1, we see that the condition is necessary.

Example. Let $u_i = s^{i-1}$, $a = -1$, $b = +1$. Since the Legendrian polynomials $P_i(s)$ form an orthogonal system of functions for the interval $(-1, +1)$, and since $P_i(s)$ is linear in $1, s, s^2, \cdots, s^i$, it is clear that the functions φ_i defined by the equations (5) are the Legendrian polynomials multiplied by the proper constants. The value* of $a_{2m,1}$ is given by

$$a_{2m,1} = \sqrt{\frac{4m+1}{2}} \, \frac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{2 \cdot 4 \cdots 2m}.$$

Hence

$$a_{2m,1}^2 > \frac{4m+1}{2(2m+1)} \cdot \frac{2}{\pi}.$$

And the constants $\{a_{2m,1}\}$ cannot be of finite norm. This is sufficient to show that the system of the powers of s has no adjoint system in the interval $(-1, +1)$.

§ 4. *Limited matrices.*

Lemma 1, § 2, can be extended to biorthogonal systems only when certain conditions indicated in the following theorem are satisfied.

Theorem 3. *If $\{f_i\}$ is any sequence of linearly independent functions of $\mathfrak{F}$, the necessary and sufficient condition that the sequence $\left\{\int ff_i\right\}$ be of finite norm for every function f of $\mathfrak{F}$ is that the matrix $(b)_f$ be limited.*

Multiply the second equation of (1) by any function f of $\mathfrak{F}$ and integrate:

$$\int ff_i = \sum_{k=1}^{i} b_{ik} \int \phi_k f \qquad (i, f).$$

By lemma 1, $\left\{\int \phi_k f\right\}$ is of finite norm for every f of $\mathfrak{F}$, and therefore $\left\{\int ff_i\right\}$ is of finite norm for every f of $\mathfrak{F}$ if the matrix $(b)_f$ is limited.

On the other hand, if

$$\left\{\int ff_i\right\}$$

* BYERLY, *Fourier's Series*, pp. 180 and 70.

is of finite norm for every f, then

$$\left\{ \sum_{k=1}^{i} b_{ik} x_k \right\}$$

is of finite norm for every $\{x_i\}$ of finite norm because

$$\int f f_i = \sum_{k=1}^{i} b_{ik} x_k,$$

where f is the function of $\mathfrak{F}$ which by lemma 3 corresponds to $\{x_i\}$.

Remark. We can always multiply the functions $\{f_i\}$ by constants $\lambda_i \neq 0$ so that

$$\left\{ \lambda_i \int f f_i \right\} \qquad (f)$$

is of finite norm. This is evident since it follows from Bessel's inequality that

$$\sum_{i=1}^{\infty} \left(\lambda_i \int f f_i \right)^2 \leqq \int f^2 \sum_{i=1}^{\infty} \lambda_i^2 \int f_i^2 \qquad (f).$$

Corollary 1. If $\{f_i\}$ is any sequence of linearly independent functions of $\mathfrak{F}$, the necessary and sufficient condition that the sequence $\left\{ \int f f_i \right\}$ be of finite norm for every function f of $\mathfrak{F}$ is that

$$\sum \left(\int f f_i \right)^2 \leqq M \int f^2,$$

where M is a constant depending only on the sequence $\{f_i\}$.

This is a consequence of the property of a limited matrix that

$$\sum_i \left(\sum_k a_{ik} x_k \right)^2 \leqq M \sum x_i^2,$$

where M does not depend on $\sum x_i^2$ (lemma 4).

Corollary 2. If $\{u_i\}$ and $\{v_i\}$ form a biorthogonal system of functions, the necessary and sufficient condition that $\left\{ \int f u_i \right\}$, $\left\{ \int f v_i \right\}$, respectively, be of finite norm $\leqq M \int f^2$ for every function f of $\mathfrak{F}$ is that $(b)_u$, $(b)_v$, respectively, be limited.

Theorem 4. *If the biorthogonal system* (u_i, v_i) *is complete as to* u, v, *respectively, the necessary and sufficient condition that* $\left\{ \int f v_i \right\}$, $\left\{ \int f u_i \right\}$, *respectively, be of finite norm for every function* f *of* $\mathfrak{F}$ *is that the matrices* $(a)_u$, $(a)_v$, *respectively, be limited.*

We prove the theorem for the case that $\{u_i\}$ is complete. From the equations (5) it is clear that the system $\{\varphi_i\}$ is also complete. Applying lemma 2 and the equations (4) we obtain the following equations:

$$(7) \qquad \int f v_i = \sum_{k=1}^{\infty} a_{ki} \int \varphi_k f \qquad (i, f).$$

The remainder of the proof is precisely the same as for theorem 3.

Corollary. When the biorthogonal system (u_i, v_i) is complete as to u, v, respectively, a limited matrix $(a)_u$, $(a)_v$, respectively, implies a limited matrix $(b)_v$, $(b)_u$, respectively, and conversely.

Theorem 5. *If* $\{f_i\}$ *is any sequence of linearly independent functions of* $\mathfrak{F}$, *the necessary and sufficient condition, that for every sequence of constants* $\{c_i\}$ *of finite norm there may exist a function* f *of* $\mathfrak{F}$ *such that*

$$c_i = \int ff_i \qquad (i),$$

is that the matrix $(a)_f$ *be limited.*

Multiply the first set of equations (1) by an arbitrary function f of $\mathfrak{F}$ and integrate:

$$\int f\phi_i = \sum_{k=1}^{i} a_{ik} \int f_k f \qquad (i). \tag{8}$$

Suppose first that the matrix $(a)_f$ is limited; then the sequence of constants

$$\left\{ \sum_{k=1}^{i} a_{ik} c_k \right\}$$

is of finite norm and there exists [lemma 3] a function f of $\mathfrak{F}$ such that

$$\int f\phi_i = \sum_{k=1}^{i} a_{ik} c_k \qquad (i).$$

Taking into consideration the properties (2), § 1, it is clear from the equations (8) that

$$c_i = \int f_i f \qquad (i).$$

We now make the assumption that to every sequence of constants $\{c_i\}$ of finite norm there corresponds a function f of $\mathfrak{F}$ such that

$$c_i = \int f_i f \qquad (i).$$

By means of the equations (8) and lemma 1 we find that

$$\sum_{k=1}^{i} a_{ik} c_k$$

is of finite norm for every sequence $\{c_i\}$ of finite norm, and thus by definition the matrix $(a)_f$ is limited.

If the system $\{f_i\}$ is complete, then

$$\int f^2 \leqq M \sum_i c_i^2 .$$

Corollary. If $\{u_i\}$ and $\{v_i\}$ form a biorthogonal system, the necessary and sufficient condition, that for every sequence of constants $\{c_i\}$ of finite norm there exists a function f of $\mathfrak{F}$ such that for every i, $c_i = \int u_i f$, $c_i = \int v_i f_i$, respectively, is that the matrix $(a)_u$, $(a)_v$, respectively, be limited.

Theorem 6. *If* $\{u_i\}$ *and* $\{v_i\}$ *form a biorthogonal system of functions, a limited matrix* $(b)_v$, $(b)_u$, *respectively, implies a limited matrix* $(a)_u$, $(a)_v$, *respectively.*

We give the proof for a limited matrix $(b)_v$. The matrix $(b)_v$ is formed from the coefficients in the equations

$$v_i = \sum_{j=1}^{i} b_{ij}\psi_j \qquad (i).$$

Multiply these equations by φ_k and integrate:

$$a_{ki} = \sum_{j=1}^{i} b_{ij}\int \psi_j\varphi_k \qquad (i, k), \tag{9}$$

where by equations (4) a_{ki} are the elements of the matrix $(a_{ki})_u$. The matrix $(a_{ki})_u$ is the product of the two matrices $(b_{ij})_v$ and $\left(\int \psi_j\varphi_k\right)$ which are both limited (lemma 7), and is therefore (lemma 6) itself limited.

For a biorthogonal system (u_i, v_i), the functions u_i can be multiplied by constants $\lambda_i \neq 0$ so that the resulting matrix $(a)_{\lambda u}$ is limited.

Corollary 1. If $\{u_i\}$ and $\{v_i\}$ form a biorthogonal system such that the sequence $\left\{\int fv_i\right\}$ is of finite norm for every f of $\mathfrak{F}$, then for every sequence of constants $\{c_i\}$ of finite norm there exists a function g of $\mathfrak{F}$ such that

$$c_i = \int gu_i.$$

Corollary 2. When the biorthogonal system (u_i, v_i) is complete as to u, the elements of the matrices $(a)_u$ and $(b)_v$ are connected by the relations

$$\sum_k a_{ki}^2 = \sum_k b_{ik}^2 \qquad (i). \tag{10}$$

A similar relation exists between the elements of the matrices $(a)_v$ and $(b)_u$ when the biorthogonal system is complete as to v.

We give below four examples to illustrate the different cases which may arise.

Example 1. Let $\{\phi_i\}$ be a system of normalized orthogonal functions for the interval I, and $\{a_i\}$ a system of constants such that $|a_i| \leqq M$ where M is some finite positive quantity. Then $\{u_i\}$ and $\{v_i\}$, defined by

$$u_{2j-1} = \phi_{2j-1}, \qquad u_{2j} = \phi_{2j} + a_j\phi_{2j-1}, \qquad v_{2j-1} = \phi_{2j-1} - a_j\phi_{2j}, \qquad v_{2j} = \phi_{2j},$$

form a biorthogonal system (u_i, v_i) whose matrices $(b)_u$, $(b)_v$ and therefore $(a)_v$, $(a)_u$ are limited.

Example 2. The same biorthogonal system as in example 1, except that the constants $\{a_i\}$ are such that $\{1/a_i\}$ is of finite norm. In this case neither of the matrices $(b)_u$ and $(b)_v$ is limited.

Example 3. Let $\{\phi_i\}$ be a system of normalized orthogonal functions for the interval I: $\{a_i\}$ a sequence of constants such that $\{1/a_i\}$ is of finite norm. For the biorthogonal system (u_i, v_i) defined by the equations

$$u_i = a_i \phi_i \qquad (i),$$

$$v_i = \frac{\phi_i}{a_i} \qquad (i),$$

the matrix $(b)_v$ is limited, but the matrix $(b)_u$ is not.

Example 4. The systems $\{\phi_i\}$ and $\{a_i\}$ are conditioned as in example (3). For the biorthogonal system (u_i, v_i) defined by the equations

$$u_i = \phi_{2i-1} \qquad (i),$$

$$v_i = \phi_{2i-1} + a_i \phi_{2i} \qquad (i),$$

the matrix $(a)_u$ is limited but the matrix $(b)_v$ is not. In this example the matrices $(a)_v$ and $(b)_u$ are also limited but the system $\{u_i\}$ is not complete: compare with theorem 4.

§ 5. *Convergence and evaluation of*

$$\sum_{i=1}^{\infty} \int f_1 u_i \int v_i f_2 .$$

Definition. For the biorthogonal system (u_i, v_i), $\mathfrak{G}$ is the class of all functions g for which there exists a system of constants $\{\mu_i \neq 0\}$ such that $\left\{\mu_i \int g u_i\right\}$ is of finite norm, and $\left\{\int f v_i/\mu_i\right\}$ is of finite norm for *every function* f of $\mathfrak{F}$.

Example. For the biorthogonal system given in example 2, § 4, the class $\mathfrak{G}$ is the class of all functions g such that $\left\{c_i \int g u_i\right\}$ is of finite norm, where $c_{2i-1} = a_i$, $c_{2i} = 1$. It is evident that the matrix $(b)_{v/\mu}$ is limited. Any other system of constants $\{\mu_i\}$ which would make $(b)_{v/\mu}$ limited is such that the ratio c_i/μ_i is less than a fixed constant for every i, and therefore for any function g such that $\left\{\mu_i \int g u_i\right\}$ is of finite norm, the sequence $\left\{c_j \int g u_i\right\}$ is also of finite norm.

When $(b)_v$ is limited, $\mathfrak{G}$ contains the class of all functions g of $\mathfrak{F}$ such that $\left\{\int g u_i\right\}$ is of finite norm.

Definition. For the biorthogonal system (u_i, v_i), $\mathfrak{H}$ is the class of all functions h for which there exists a system of constants $\{\nu_i \neq 0\}$ such that $\left\{\nu_i \int h v_i\right\}$ is of finite norm, and $\left\{\int f u_i / \nu_i\right\}$ is of finite norm for *every function* f of $\mathfrak{F}$.

When $(b)_u$ is limited, $\mathfrak{H}$ contains the class of all functions h of $\mathfrak{F}$ such that $\left\{\int h v_i\right\}$ is of finite norm.

In the definition the function g is assumed to belong to the class $\mathfrak{F}$, but any function g satisfying those conditions must belong to $\mathfrak{F}$ since the matrix $(b)_{v/\mu}$ and therefore the matrix $(a)_{\mu v}$ corresponding to g is limited, and by the equations (5), $\left\{\int g \varphi_i\right\}$ is of finite norm and therefore g belongs to $\mathfrak{F}$. Similarly the functions h must necessarily belong to $\mathfrak{F}$.

Theorem 7. *If* (u_i, v_i) *is a biorthogonal system, then for every function* g *of* $\mathfrak{G}$ *and for every function* f *of* $\mathfrak{F}$ *the expression*

$$\sum_{i=1}^{\infty} \int g u_i \int v_i f$$

is convergent; and for every function h *of* $\mathfrak{H}$ *and every function* f *of* $\mathfrak{F}$ *the expression*

$$\sum_{i=1}^{\infty} \int f u_i \int v_i h$$

is convergent.

The convergence follows directly from the two inequalities:

$$\left(\sum_{i=1}^{\infty} \int g u_i \int v_i f\right)^2 \leqq \sum_{i=1}^{\infty}\left(\mu_i \int g u_i\right)^2 \sum_{i=1}^{\infty}\left(\frac{\int f v_i}{\mu_i}\right)^2 \qquad (f, g),$$

$$\left(\sum_{i=1}^{\infty} \int f u_i \int v_i h\right)^2 \leqq \sum_{i=1}^{\infty}\left(\frac{\int f u_i}{\nu_i}\right)^2 \sum_{i=1}^{\infty}\left(\nu_i \int h v_i\right)^2 \qquad (f, h).$$

Theorem 8. *When the biorthogonal system* (u_i, v_i) *is complete as to* u, *or when the functions* v_i *and* g *are orthogonal to the functions which together with the* u_i *form a complete system, the integral* $\int g f$ *may be represented as follows:*

$$\int g f = \sum_{i=1}^{\infty} \int g u_i \int v_i f \qquad (f, g); \tag{11}$$

when it is complete as to v, *or when the functions* u_i *and* h *are orthogonal to the functions which together with the* v_i *form a complete system, the integral* $\int h f$ *may similarly be represented:*

$$\int h f = \sum_{i=1}^{\infty} \int f u_i \int v_i h \qquad (f, h). \tag{12}$$

We give the proof for the first part of the theorem.

From the definition of $\mathfrak{G}$ the matrix $(b)_{v/\mu}$ is limited, and therefore by theorem 6 the matrix $(a)_{\mu u}$ is limited.

The elements of the limited matrix $(a)_{\mu u}$ are a_{ik}/μ_k. Consider the bilinear form

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\frac{a_{ik}}{\mu_k}x_k y_i.$$

Make the substitution

$$x_k=\mu_k\int gu_k \qquad (k),$$

$$y_i=\int\varphi_i f \qquad (i).$$

Since

$$\frac{\int fv_k}{\mu_k}=\sum_{k=1}^{\infty}\frac{a_{ik}}{\mu_k}\int\varphi_i f \qquad (k)$$

we obtain from summation by rows

$$\sum_{k=1}^{\infty}\int gu_k\int v_k f.$$

Since

$$\int f\varphi_i=\sum_{k=1}^{i}\frac{a_{ik}}{\mu_k}\mu_k\int u_k f$$

we obtain from summation by columns

$$\sum_{i=1}^{\infty}\int g\varphi_i\int\varphi_i f.$$

By lemma 7 these two sums are equal, and applying lemma 2 we have the desired result:

$$\int gf=\sum_{i=1}^{\infty}\int gu_i\int v_i f \qquad (f, g).$$

Corollary 1. When the biorthogonal system (u_i, v_i) is complete as to u, then

(13) $$\int g^2=\sum_{i=1}^{\infty}\int gu_i\int v_i g \qquad (g)$$

when it is complete as to v, then

(14) $$\int h^2=\sum_{i=1}^{\infty}\int hu_i\int v_i h \qquad (h).$$

These relations do not necessarily hold for every function f of $\mathfrak{F}$ unless $\mathfrak{G}=\mathfrak{F}$ and $\mathfrak{H}=\mathfrak{F}$, respectively. This is illustrated by the example at the end of this section.

Corollary 2. If the function p is orthogonal to all the functions v_i, u_i respectively, then it is orthogonal to all the functions of the classes $\mathfrak{G}$, $\mathfrak{H}$ respectively.

Corollary 3. If the biorthogonal system (v_i, u_i) is complete as to u, v, respectively, and the function p is orthogonal to all the functions v_i, u_i, respectively, then the matrices $(b)_u$ and $(b)_v$ cannot both be limited.

Theorem 9. *When the biorthogonal system (u_i, v_i) is complete as to u, v respectively there exists no function in $\mathfrak{G}$, $\mathfrak{H}$, respectively, essentially different from zero which is orthogonal to the system $\{v_i\}$, $\{u_i\}$, respectively.*

This is an immediate consequence of the equations (13) and (14).

Corollary. When $\mathfrak{G} = \mathfrak{F}$, $\mathfrak{H} = \mathfrak{F}$, respectively, a complete system $\{u_i\}$, $\{v_i\}$ respectively, implies a complete system $\{v_i\}$, $\{u_i\}$ respectively.

Theorem 10. *If the biorthogonal system (u_i, v_i) is complete as to u, v, respectively and the functions u_i, v_i, respectively, belong to the class $\mathfrak{G}$, $\mathfrak{H}$, respectively, then the system $\{v_i\}$, $\{u_i\}$ respectively is also complete.*

A function f of $\mathfrak{F}$, orthogonal to the functions v_i, would be orthogonal to the functions $\{u_i\}$ also, as is seen from the equations (11).

The following example is that of a biorthogonal system (u_i, v_i) complete as to u but not complete as to v.

Example. Let $\{\phi_i\}$ form a complete system of normalized orthogonal functions, and let $p \neq 0$ be a function belonging to $\mathfrak{F}$ but linearly dependent on no finite number of functions ϕ_i and therefore

$$\int p^2 - \sum_{k=i}^{n} \left(\int p \phi_k \right)^2 > 0 \qquad (n).$$

Construct the functions

$$(15) \qquad u_i = \frac{\left[\int p^2 - \sum_{k=1}^{i-1} \left(\int p\phi_k \right)^2 \right] \phi_i + \int p\phi_i \sum_{k=1}^{i-1} \phi_k \int p\phi_k}{\sqrt{\int p^2 - \sum_{k=1}^{i} \left(\int p\phi_k \right)^2} \sqrt{\int p^2 - \sum_{k=1}^{i-1} \left(\int p\phi_k \right)^2}} \qquad (i)$$

$$v_i = u_i - \frac{p \int pu_i}{\int p^2} \qquad (i).$$

The functions u_i and v_i form a biorthogonal system (u_i, v_i) complete as to u but the function p is orthogonal to all the functions v_i. The functions v_i form a system of orthogonal and normalized functions and therefore the matrix $(b)_v$ is limited. The matrix $(b)_u$ is not limited; otherwise $\int p^2$ could be expressed by

$$\int p^2 = \sum_{i=1}^{\infty} \int pu_i \int v_i p,$$

which is impossible.

This construction could be generalized, so that the system $\{u_i\}$ would be complete and all the functions v_i orthogonal to a finite number of functions $p_1, p_2, \cdots, p_n$.

§ 6. *Associated functions.*

Definition. The function f_1 is a *vu-associated* function of the function f with respect to the biorthogonal system $(u_i,\ v_i)$ if it satisfies the following relation:

$$\int fv_i = \int u_i f_1 \qquad (i). \tag{16}$$

Remark 1. If the system $\{u_i\}$ is complete, then f_1 is the only *vu*-associated function of f, and for a function p orthogonal to all the functions v_i the *vu*-associated function is zero, and the function $f + \mu p$, where μ is an arbitrary constant, has the same *vu*-associated function as f.

Remark 2. The functions v_i are *vu*-associated functions of u_i. For an orthogonal system a *vu*-associated function is the function itself.

Remark 3. The sum of two functions, which are *vu*-associated functions of two given functions, is a *vu*-associated function of the sum of the two given functions.

Definition. The function f_1 is a *uv-associated* function of the function f with respect to the biorthogonal system $(u_i,\ v_i)$ if it satisfies the following relation:

$$\int f_1 v_i = \int u_i f \qquad (i). \tag{17}$$

Remark 4. A statement of the immediate consequences of this definition is obtained by interchanging u and v in the remarks (1), (2), and (3).

Theorem 11. *If the biorthogonal system (u_i, v_i) has a limited matrix $(b)_v$, $(b)_u$, respectively, then for every function f of $\mathfrak{F}$ there exists a vu-associated g which belongs to $\mathfrak{G}$, a uv-associated function h which belongs to $\mathfrak{H}$, respectively.*

This theorem is a consequence of the theorems 3, 5, and 6. Take any function f of $\mathfrak{F}$ and let $c_i = \int fv_i$; by theorem 3, the sequence $\{c_i\}$ is of finite norm, and by theorems 6 and 5 there exists a function g of $\mathfrak{G}$ such that

$$c_i = \int fv_i = \int u_i g \qquad (i).$$

Corollary. If the biorthogonal system $(u_i,\ v_i)$ has a limited matrix $(b)_v$, then for every function f of $\mathfrak{F}$ there exists a *uv*-associated function when and only when the matrix $(b)_u$ is also limited.

Theorem 12. *If for a biorthogonal system, complete as to u, the vu-associated function g exists for every function f of $\mathfrak{F}$, then the matrix $(b)_v$ is limited.*

This theorem is a consequence of lemma 3 and the equations (5) and (7). Multiply the equations (5) by g and integrate and then substitute from the equations (7) for $\int gu_i$. By lemma 3 the matrix $(\sum u_{ik}u_{jk})$ is limited and by reason of the properties (2), § 2, it follows that $(a)_u$ is also limited. Corollary to theorem 4 shows that $(b)_v$ is limited.

Theorem 13. *If the biorthogonal system* (u_i, v_i) *is complete as to u and has a limited matrix* $(b)_v$, *and if g is the vu-associated function of f, then the integral* $\int fg$ *can be expressed as the sum of squares:*

$$\int fg = \sum_{i=1}^{\infty}\left(\int fv_i\right)^2 \qquad (f). \tag{18}$$

This follows from a direct application of the theorem 8 and equations (11).

Remark. If the biorthogonal system (u_i, v_i) is complete as to u and has a limited matrix $(b)_v$ and if g is the vu-associated function of some function f, and the function p is orthogonal to all the functions v_i, then p is also orthogonal to g. This is a special case of the corollary to theorem 8.

Theorem 14. *If the biorthogonal system* (u_i, v_i) *is complete as to u and has a limited matrix* $(b)_v$, *and if* g_1 *is the vu-associated function of* f_1, *and* g_2 *that of* f_2, *then*

$$\int f_1 g_2 = \int f_2 g_1 \qquad (f_1, f_2). \tag{19}$$

This is a consequence of theorem 8.

If the biorthogonal system (u_i, v_i) has a limited matrix $(b)_v$, then by theorem 11 the vu-associated functions define a functional transformation T,

$$g = T(f) \qquad (f), \tag{20}$$

which is applicable to the class $\mathfrak{F}$, and is single-valued if the system $\{u_i\}$ is complete.

A particular case of equation (20) is

$$v_i = T(u_i) \qquad (i). \tag{21}$$

When the system $\{u_i\}$ is complete, T is linear:

$$T(a_1 f_1 + a_2 f_2) = a_1 T(f_1) + a_2 T(f_2) \qquad (f_1, f_2). \tag{22}$$

In terms of T the theorems 14 and 13 may be expressed as follows:

$$\int [f_1 T(f_2) - f_2 T(f_1)] = 0 \qquad (f_1, f_2) \tag{23}$$

and

$$\int f T(f) \geqq 0 \qquad (f), \tag{24}$$

the equality sign in (24) holding only for $f = 0$ or $T(f) = 0$.

We give below some examples of this functional transformation T.

Example 1. For a complete orthogonal system $T(f)=f$.

Example 2. For the complete biorthogonal system

$$v_i = \alpha u_i \qquad (i),$$

where α is a positive function of $\mathfrak{F}$, the transformation T is

$$T(f) = \alpha f.$$

Example 3. Let (u_i, v_i) be the biorthogonal system defined in the example in § 5. The transformation T is given by

$$T(f) = f - \frac{p\int pf}{\int p^2}.$$

Example 4. Let $K(s, t)$ be a kernel which is continuous, symmetric, has positive characteristic numbers $\{\mu_i^2\}$, and for which there exists no function f in $\mathfrak{F}$ such that $\int K(s, t)f(t)dt = 0$. Let the characteristic functions be $\{\phi_i\}$:

$$\phi_i(s) = \mu_i^2 \int K(s, t)\phi_i(t)dt \qquad (i).$$

The biorthogonal system

$$u_i = \mu_i \phi_i, \qquad v_i = \frac{\phi_i}{\mu_i} \qquad (i)$$

is complete as to u, and has a limited matrix $(b)_v$ because the sequence of numbers $\{1/\mu_i\}$ is of finite norm. The functional transformation T is

$$g(s) = \int K(s, t)f(t)dt.$$

§ 7. *Equivalent biorthogonal systems.*

Definition. The biorthogonal systems (u_i, v_i) and $(\bar{u}_i, \bar{v}_i)$ complete as to u and $\bar{u}$, respectively, and satisfying the relation

$$\int u_i \bar{v}_k = \int v_i \bar{u}_k \qquad (i, k),$$

are called equivalent.

As a trivial case, all orthogonal systems are equivalent. The definition may be applied to some biorthogonal systems which are not complete.

Theorem 15. *If the biorthogonal system (u_i, v_i) is complete as to u and has a limited matrix $(b)_v$, then two biorthogonal systems $(\bar{u}_i, \bar{v}_i)$ and $(\bar{\bar{u}}_i, \bar{\bar{v}}_i)$ each equivalent to (u_i, v_i) are equivalent to each other.*

This follows directly from the equations (19).

Theorem 16. *If the biorthogonal systems* (u_i, v_i) *and* $(\bar{u}_i, \bar{v}_i)$ *are complete as to* u *and* $\bar{u}$ *respectively, and if the matrices* $(b)_v$ *and* $(b)_{\bar{v}}$ *are limited, then the necessary and sufficient condition that the matrix* $\int \bar{u}_i v_k$ *be orthogonal, is that the two given systems be equivalent.*

If the two systems are equivalent, the conditions for the orthogonality (§ 2) of the matrix $\int \bar{u}_i v_k$ are satisfied; for

$$\sum_{k=1}^{\infty} \int \bar{u}_i v_k \int v_k \bar{u}_j = \sum_{k=1}^{\infty} \int \bar{u}_i v_k \int u_k \bar{v}_j = \int \bar{u}_i \bar{v}_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \quad (i, j),$$

$$\sum_{k=1}^{\infty} \int v_i \bar{u}_k \int \bar{u}_k v_j = \sum_{k=1}^{\infty} \int v_i \bar{u}_k \int \bar{v}_k u_j = \int u_i v_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \quad (i, j).$$

To prove the converse, let $\bar{\bar{v}}_i$ be the vu-associated function of $\bar{u}_i$. Then

$$\int \bar{u}_i \bar{\bar{v}}_j = \sum_{k=1}^{\infty} \int \bar{u}_i v_k \int u_k \bar{\bar{v}}_j = \sum_{k=1}^{\infty} \int \bar{u}_i v_k \int v_k \bar{u}_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$$

and therefore

$$\int \bar{u}_i \bar{\bar{v}}_j = \int \bar{v}_j \bar{u}_i \quad (i, j).$$

Since the system $\{\bar{u}_i\}$ is complete we obtain

$$\bar{\bar{v}}_i = \bar{v}_i \quad (i)$$

and the two systems are equivalent.

Theorem 17. *If the biorthogonal system* (u_i, v_i) *is complete as to* u *and the matrix* $(b)_v$ *is limited, then for any biorthogonal system* $(\bar{u}_i, \bar{v}_i)$ *complete as to* u *and equivalent to the system* (u_i, v_i) *the matrix* $(b)_{\bar{v}}$ *is limited.*

Let f be any function of $\mathfrak{F}$ and write the equations

$$\int f\bar{v}_i = \sum_{k=1}^{\infty} \int f v_k \int u_k \bar{v}_i \quad (i).$$

From this system of equations we see that $\left\{\int f\bar{v}_i\right\}$ is of finite norm for every f and therefore the matrix $(b)_{\bar{v}}$ is limited.

Theorem 18. *If the equivalent biorthogonal systems* (u_i, v_i) *and* $(\bar{u}_i, \bar{v}_i)$ *are complete as to* u *and* $\bar{u}$ *respectively, and the matrices* $(b)_v$ *and* $(b)_{\bar{v}}$ *are limited, then the vu-associated function* g *of any given function* f *is equal to the* $\bar{v}\bar{u}$*-associated function* $\bar{g}$ *of* f.

We have

$$\int \bar{g}\bar{u}_i = \int f\bar{v}_i = \sum_{k=1}^{\infty} \int f v_k \int u_k \bar{v}_i = \sum_{k=1}^{\infty} \int g u_k \int v_k \bar{u}_i = \int g \bar{u}_i .$$

Since the system $\bar{u}_i$ is complete, $\bar{g} = g$.

Corollary. The functional transformations T and $\bar{T}$, corresponding to two biorthogonal systems (u_i, v_i) and $(\bar{u}_i, \bar{v}_i)$ which satisfy the conditions of theorem 17, are equal.

Remark. If two biorthogonal systems have the same functional transformation T, they are equivalent.

§ 8. *Functional transformations T.*

In this section we consider functional transformations T, which transform every function of $\mathfrak{F}$ into a function of $\mathfrak{F}$ and which are single-valued and possess the following three properties: *

$$T(a_1 f_1 + a_2 f_2) = a_1 T(f_1) + a_2 T(f_2) \qquad (a_1, a_2, f_1, f_2), \tag{22}$$

$$\int [f_1 T(f_2) - f_2 T(f_1)] = 0 \qquad (f_1, f_2), \tag{23}$$

$$\int f T(f) \geqq 0 \qquad (f), \tag{24}$$

the equality sign in (24) holding only for $f = 0$ or $T(f) = 0$.

Theorem 19. *Any biorthogonal system* (u_i, v_i) *such that*

$$v_i = T(u_i) \qquad (i),$$

where T has the three properties (22), (23), (24), *has a limited matrix* $(b)_v$.

Let f be any function of $\mathfrak{F}$; then by the property (24)

$$\int \left(f - \sum_{k=1}^{i} u_k \int u_k T(f)\right)\left(T(f) - \sum_{k=1}^{i} v_k \int v_k f\right) \geqq 0 \qquad (i, f).$$

Transforming this inequality, we obtain

$$\sum_{k=1}^{i} \left(\int v_k f\right)^2 \leqq \int f T(f) \qquad (i, f), \tag{25}$$

and therefore the matrix $(b)_v$ is limited.

Corollary. For any two functions f_1 and f_2 of $\mathfrak{F}$,

$$\left(\int f_1 T(f_2)\right)^2 \leqq \int f_1 T(f_1) \int f_2 T(f_2) \qquad (f_1, f_2). \tag{26}$$

In the inequality (25) make the substitution $i = 1$, $f = f_2$, and

$$v_1 = \frac{T(f_1)}{\int f_1 T(f_1)},$$

where f_1 and f_2 are any two functions of $\mathfrak{F}$, such that $T(f_1) \neq 0$ and $T(f_2) \neq 0$.

* Since T is single-valued, the linearity (22) is a consequence of (23).

Theorem 20. *For any functional transformation T which has the three properties* (22), (23), (24), *and which does not transform every function into zero, there exist biorthogonal systems* (u_i, v_i) *for which*

$$v_i = T(u_i) \qquad (i).$$

Let $\{f_i\}$ be any sequence of linearly independent functions of $\mathfrak{F}$, for which $T(f_i) \neq 0$ and also are linearly independent. Construct the following system:

$$(27) \qquad \begin{aligned} u_i &= \frac{f_i - \sum_{k=1}^{i-1} u_k \int u_k T(f_i)}{\sqrt{\int\left(f_i - \sum_{k=1}^{i-1} u_k \int u_k T(f_i)\right)\left(T(f_i) - \sum_{k=1}^{i-1} v_k \int v_k f_i\right)}} \qquad (i), \\ v_i &= \frac{T(f_i) - \sum_{k=1}^{i-1} v_k \int v_k f_i}{\sqrt{\int\left(f_i - \sum_{k=1}^{i-1} u_k \int u_k T(f_i)\right)\left(T(f_i) - \sum_{k=1}^{i-1} v_k \int v_k f_i\right)}} \qquad (i). \end{aligned}$$

For $i = 1$ it is clear that the denominators do not vanish and that $v_i = T(u_i)$. Assume that the statement is true for $i - 1$; then

$$T(f_i) - \sum_{k=1}^{i-1} v_k \int v_k f_i = T\left(f_i - \sum_{k=1}^{i-1} u_k \int u_k T(f_i)\right),$$

and therefore by (24) the denominators of u and v cannot vanish unless

$$f_i - \sum_{k=1}^{i-1} u_k \int u_k T(f_i) = 0$$

or

$$T(f_i) - \sum_{k=1}^{i-1} v_k \int v_k f_i = 0.$$

Either one of these equations is contrary to the hypothesis on $\{f_i\}$. Therefore

$$v_i = T(u_i) \qquad (i).$$

It can now be argued that the systems $\{u_i\}$ and $\{v_i\}$ form a biorthogonal system. It is easily seen that the relations

$$\int u_i v_j = \begin{cases} 0 & (i \neq j), \\ 1 & (i = j), \end{cases}$$

hold for $i, j = 1, 2$. Assume that they hold for $i, j = 1, 2, \cdots, n-1$ and prove that they then hold for $i, j = 1, 2, \cdots, n$.

Since in a biorthogonal system (u_i, v_i) the functions u_i and also v_i are necessarily linearly independent, the vanishing of the denominator of u_i and v_i gives a sufficient condition that $\{f_i\}$ and $\{T(f_i)\}$ be linearly dependent.

The biorthogonal system (u_i, v_i) defined by the equations (27) has by theorem 18 a limited matrix $(b)_v$; a vu-associated function g of any function f exists and

$$g = T(f).$$

Biorthogonal systems constructed from different sequences $\{f_i\}$ are equivalent or belong to the same type.

There exist functional transformations T with the properties (22), (23), (24) for which there does not exist a complete system of linearly independent functions $\{f_i\}$ such that the functions $T(f_i)$ are linearly independent. For example, the functional transformation

$$T(f) = \int K(s, t) f(t)\, dt,$$

where

$$K(s, t) = \sum_{i=1}^{n} \frac{\phi_i(s)\phi_i(t)}{\mu_i^2}$$

transforms every function f into a linear combination of the n functions ϕ_i $(i = 1, \cdots, n)$. If, however, there exists a complete system of linearly independent functions $\{f_i\}$ such that the functions $T(f_i)$ are linearly independent, then a biorthogonal system can be constructed by (27) which is complete as to u. In all cases however,

$$\int gf = \sum_{i=1}^{\infty} \int gu_i \int v_i f.$$

Theorem 21. *A single-valued functional transformation T which has the three properties* (22), (23), *and* (24), *and for which there exists a complete system of linearly independent functions $\{f_i\}$ such that $T(f_i)$ are also linearly independent, is equivalent to a type of biorthogonal systems (u_i, v_i) complete as to u and with limited matrices $(b)_v$.*

The formulæ for u_i and v_i may be expressed in the following form:

$$u_i = \frac{\begin{vmatrix} \int f_1 T(f_1) & \int f_1 T(f_2) & \cdots & \int f_1 T(f_{i-1}) & f_1 \\ \int f_2 T(f_1) & \int f_2 T(f_2) & \cdots & \int f_2 T(f_{i-1}) & f_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \int f_i T(f_1) & \int f_i T(f_2) & \cdots & \int f_i T(f_{i-1}) & f_i \end{vmatrix}}{\sqrt{\begin{vmatrix} \int f_1 T(f_1) \int f_1 T(f_2) \cdots \int f_1 T(f_{i-1}) \\ \int f_2 T(f_1) \int f_2 T(f_2) \cdots \int f_2 T(f_{i-1}) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \int f_{i-1} T(f_1) \int f_{i-1} T(f_2) \cdots \int f_{i-1} T(f_{i-1}) \end{vmatrix} \cdot \begin{vmatrix} \int f_1 T(f_1) \int f_1 T(f_2) \cdots \int f_1 T(f_i) \\ \int f_2 T(f_1) \int f_2 T(f_2) \cdots \int f_2 T(f_i) \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ \int f_i T(f_1) \int f_i T(f_2) \cdots \int f_i T(f_i) \end{vmatrix}}} \quad (i),$$

$$
v_i = \frac{\begin{vmatrix} \int f_1 T(f_1) & \int f_2 T(f_1) & \cdots & \int f_{i-1} T(f_1) & T(f_i) \\ \int f_1 T(f_2) & \int f_2 T(f_2) & \cdots & \int f_{i-1} T(f_2) & T(f_i) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \int f_1 T(f_i) & \int f_2 T(f_i) & \cdots & \int f_{i-1} T(f_i) & T(f_i) \end{vmatrix}}{\sqrt{\begin{vmatrix} \int f_1 T(f_1) & \int f_2 T(f_1) & \cdots & \int f_{i-1} T(f_1) \\ \int f_1 T(f_2) & \int f_2 T(f_2) & \cdots & \int f_{i-1} T(f_2) \\ \cdot & \cdot & \cdot & \cdot \\ \int f_1 T(f_{i-1}) & \int f_2 T(f_{i-1}) & \cdots & \int f_{i-1} T(f_{i-1}) \end{vmatrix} \cdot \begin{vmatrix} \int f_1 T(f_1) & \int f_2 T(f_1) & \cdots & \int f_i T(f_1) \\ \int f_1 T(f_2) & \int f_2 T(f_2) & \cdots & \int f_i T(f_2) \\ \cdot & \cdot & \cdot & \cdot \\ \int f_1 T(f_i) & \int f_2 T(f_i) & \cdots & \int f_i T(f_i) \end{vmatrix}}} \quad (i).
$$

The necessary and sufficient condition that both the systems $\{f_i\}$ and $\{T(f_i)\}$ be linearly dependent is the vanishing of the determinant

$$
(28) \qquad \begin{vmatrix} \int f_1 T(f_1) & \int f_1 T(f_2) & \cdots & \int f_1 T(f_i) \\ \int f_2 T(f_1) & \int f_2 T(f_2) & \cdots & \int f_2 T(f_i) \\ \cdot & \cdot & \cdot & \cdot \\ \int f_i T(f_1) & \int f_i T(f_2) & \cdots & \int f_i T(f_i) \end{vmatrix}
$$

Theorem 22. *If, for a functional transformation T with the properties* (22), (23), *and* (24), *there exists a function p such that $p = \lambda T(p)$, then there exists a biorthogonal system $(\bar{u}_i, \bar{v}_i)$ for which the $\bar{v}\bar{u}$-associated function g of a function f orthogonal to p is given by*

$$
g = T(f).
$$

Taking into consideration the inequality (26) and the condition imposed on p, we see that the functional transformation

$$
\bar{T}(f) = T(f) - \frac{p \int p T(f)}{\int p^2}
$$

satisfies the three conditions (22), (23), and (24). By theorem 20 we can construct a biorthogonal system $(\bar{u}_i, \bar{v}_i)$ such that

$$
\bar{v}_i = \bar{T}(\bar{u}_i).
$$

The $\bar{v}\bar{u}$-associated function g of f is given by

$$
g = T(f) - \frac{T(p) \int pf}{\int p^2} = T(f) - \frac{p \int p T(f)}{\int p^2}
$$

and therefore if f_1 is orthogonal to p,

$$g_1 = T(f_1).$$

§ 9. *Solutions of $f = \lambda T(f)$.*

Let T be a single-valued functional transformation which transforms every function of $\mathfrak{F}$ into a function of $\mathfrak{F}$ and which has the properties (22), (23), and (24); and let (u_i, v_i) be a biorthogonal system (theorem 20) for which

$$v_i = T(u_i) \qquad (i).$$

Then the matrix $(b)_v$ is limited (theorem 19) and therefore also the matrix $(a)_u$.

The matrix (α_{ik}), where

$$\alpha_{ik} = \sum_{j=1}^{\infty} a_{ij} a_{kj},$$

is limited, by lemma 6, and is symmetric.

In this section we consider the functional equation

$$(31) \qquad f = \lambda T(f),$$

where λ is a parameter, and f a function to be determined.

Theorem 23. *If there exists a solution f of the equation*

$$f = \lambda T(f),$$

then

$$z_i = \int f\varphi_i \qquad (i)$$

is a solution of finite norm of the infinite system of homogeneous linear equations

$$(32) \qquad z_i - \lambda \sum_{k=1}^{\infty} \alpha_{ik} z_k = 0 \qquad (i),$$

and conversely a solution of finite norm of the equations (32) *leads to a solution of the equation* (31).

The functional equation (31) is equivalent to the set of equations

$$(33) \qquad \int f u_i = \lambda \int v_i f \qquad (i).$$

By means of the equations (5) and (7) we obtain the result that the equations (33) are equivalent to (32).

Corollary. The functional equation (31) has a solution f when and only when the quadratic form $\sum \alpha_{ik} x_i x_k$ has a characteristic number; the number of the solutions f depends on the multiplicity of the characteristic number.

This corollary follows from lemma 8.

Theorem 24. *Two solutions of*

$$f = \lambda T(f),$$

corresponding to two different values of λ, *are orthogonal.*

Let the two values of λ be λ_1 and λ_2, and the corresponding solutions f_1 and f_2:

$$f_1 = \lambda_1 T(f_1), \qquad f_2 = \lambda_2 T(f_2).$$

Multiply the first equation by f_2 and integrate. By means of the second equation and (23) we obtain

$$\int f_1 f_2 = \frac{\lambda_1}{\lambda_2} \int f_1 f_2,$$

and therefore

$$\int f_1 f_2 = 0,$$

when λ_1 and λ_2 are distinct.

If the kernel $K(s, t)$ is symmetric, positive semi-definite, and satisfies certain continuity conditions, the integral equation

$$f(s) = \lambda \int K(s, t) f(t) dt$$

is a special case of the functional equation (31). This integral equation has at least one solution, the multiplicity of a characteristic number is always finite, and if there are an infinite number of characteristic numbers the limit point is infinite.

We give examples in which the spectrum * of the quadratic form $\sum a_{ik} x_i x_k$ consists of a point spectrum with finite or infinite multiplicity, or with a finite or infinite limit point, or of a continuous spectrum.

Example 1. Same as the example in § 5.

The spectrum consists of the one point unity with an infinite multiplicity. Any function orthogonal to p is a solution of the equation (31).

Example 2. Let $\{\phi_i\}$ be a complete system of normalized and orthogonal functions; and let (u_i, v_i) be the biorthogonal system defined by

$$u_i = \sum_{k=1}^{i} \phi_k \qquad (i),$$

$$v_i = \phi_i - \phi_{i+1} \qquad (i).$$

The quadratic form $\sum a_{ik} x_i x_k$ becomes

$$x_1^2 + 2 \sum_{i=2}^{\infty} x_i^2 - 2 \sum_{i=1}^{\infty} x_i x_{i+1}$$

* *H.* IV, p. 172.

and its spectrum* is a continuous spectrum consisting of the values λ between $+1$ and $+\infty$. The equation (31) has no solution f belonging to $\mathfrak{F}$.

Example 3. Same as example 1, § 4, in which the constants a_i approach zero as a limit, and the functions ϕ_i are complete and continuous.

The spectrum consists of a point spectrum with limit point at 1:

$$\lambda_{2j-1} = \frac{2 + a_j^2 + a_j\sqrt{a_j^2 + 4}}{2}, \qquad \lambda_{2j} = \frac{2 + a_j^2 - a_j\sqrt{a_j^2 + 4}}{2} \qquad (j).$$

The corresponding solutions of (31) are

$$\alpha_{2j-1} = \frac{\phi_{2j-1}(1-\lambda_{2j-1}) - a_j\phi_{2j}}{(1-\lambda_{2j-1})^2 + a_j^2}, \qquad \alpha_{2j} = \frac{\phi_{2j-1}(1-\lambda_{2j}) - a_j\phi_{2j}}{(1-\lambda_{2j})^2 + a_j^2} \qquad (j).$$

The functions $\{\alpha_i\}$ form a complete system of normalized orthogonal functions.

It is possible to take a_i so small that

$$K(s, t) = \sum_{i=1}^{\infty} \frac{\alpha_i(s)\alpha_i(t)}{\frac{1}{1-\lambda_i}}$$

converges uniformly; $1/1 - \lambda_i$ are then the characteristic numbers of $K(s, t)$. With this restriction imposed on a_i, we obtain for any function f and its *vu*-associated function g

$$f(s) = g(s) - \int K(s, t) g(t) dt.$$

Since 1 is not a characteristic number of $K(s, t)$ we can solve this equation for g and obtain for T

$$g(s) = f(s) - \int k(s, t) f(t) dt,$$

where $k(s, t)$ is the resolvent of $K(s, t)$. The first equation gives the relation between any function g and its *uv*-associated function f.

Example 4. Same as example 1, § 4, in which however the constants a_i are all equal to the constant a, the system $\{\phi_i\}$ is complete, and the functions ϕ_i are continuous.

In this case the spectrum consists of the two points:

$$l = \frac{2 + a^2 + a\sqrt{a^2 + 4}}{2}, \qquad \frac{1}{l} = \frac{2 + a^2 - a\sqrt{a^2 + 4}}{2},$$

* H. WEYL, *Singuläre Integralgleichungen*, Dissertation, Göttingen, 1908, p. 69.

each of infinite multiplicity. Let the solutions of (31), corresponding to l and $1/l$, be $\{\alpha_i\}$ and $\{\alpha_i\}$ respectively. Each of these systems we may consider as orthogonal and normalized; and each system is orthogonal to the other. Further, the two systems together form a complete orthogonal system of continuous functions; and it is possible to choose systems of constants $\{l_i\}$ and $\{l_i^*\}$ such that they are the characteristic numbers of two symmetric kernels $K(s, t)$ and $K^*(s, t)$, whose corresponding characteristic functions are the $\{\alpha_i\}$ and $\{\alpha_i^*\}$ respectively. Then the functions u_i and v_i satisfy the equation

$$\int [K(s, t) + lK^*(s, t)]\, u_i(t)\, dt = \int [lK(s, t) + K^*(s, t)]\, v_i(t)\, dt \quad (i).$$

This is an integral equation of the first kind in both u_i and v_i. The functional transformation is given by*

$$v_i(s) = 2\int_0^\infty \left(\sum_k (-1)^k \frac{t^{2k+1}}{k!} \beta_{2k+1}^{(i)}(s) \right) dt \qquad (i),$$

where

$$\beta_k^{(i)}(s) = l \int [l^{2k} K^{(2k+2)}(s, t) + K^{*(2k+2)}(s, t)]\, u_i(t)\, dt$$

and $K^{(2k+2)}(s, t)$ and $K^{*(2k+2)}(s, t)$ are iterated kernels of K and K^* respectively.

§ 10. *Complete† elementary theory of six properties.*

A biorthogonal system (u_i, v_i) may have one or more of the following six properties: 1) $\{u_i\}$ complete; 2) $\{v_i\}$ complete; 3) $(a)_u$ limited; 4) $(b)_u$ limited; 5) $(a)_v$ limited; 6) $(b)_v$ limited. If we consider all the possibilities of a biorthogonal system possessing or not possessing each one of these six properties. there are $2^6 = 64$ cases to be considered. To establish a complete elementary theory we must, for each of the 64 cases, either show by theorems the impossibility of the case or exhibit an example. In this way we show that there are no general theorems relating to these six properties other than those already obtained.

Theorem 25. *There are no other theorems which are analogous to the corollary to theorem 4, theorem 6, and corollary 3 to theorem 8, and which express interdependence among the six properties mentioned above.*

To denote that a biorthogonal system possesses or does not possess the ith property $(i = 1, 2, \cdots, 6)$ we write $+$ or $-$ in the ith place.

Since by theorem 6 a limited matrix $(b)_v$, $(b)_u$ respectively implies a limited matrix $(a)_u$, $(a)_v$ respectively, the following cases are excluded

$$(\pm, \pm, -, \pm, \pm, +),$$
$$(\pm, \pm, \pm, +, -, \pm),$$

of which 28 are distinct.

* H. Bateman, *Inversion of a definite integral,* Mathematische Annalen, vol. 63 (1907).

† E. H. Moore, *Introduction to a form of General Analysis,* The New Haven Mathematical Colloquium, p. 82.

The corollary to theorem 4 excludes the cases

$$(\pm, \pm, +, \pm, \pm, -),$$
$$(\pm, \pm, \pm, -, +, \pm).$$

Of these there are 11 cases which are new and distinct.

Corollary 3 to theorem 8 excludes the two cases

$$(+, -, +, +, +, +),$$
$$(-, +, +, +, +, +).$$

There remain 23 cases, for which we exhibit examples. Two cases which differ only by an interchange of u and v may be considered as one case.

1) $$(+, +, +, +, +, +).$$

Example 1 in § 4, in which the system $\{\phi_i\}$ is complete.

This example shows the impossibility of a theorem to the effect that a biorthogonal system complete both as to u and v cannot have all the matrices limited.

2) $$(+, +, +, -, -, +),$$
3) $$(+, +, -, +, +, -).$$

Example 3 in § 4, in which the system $\{\phi_i\}$ is complete.

4). $$(+, +, -, -, -, -).$$

Example 3 in § 4, in which the system $\{\phi_i\}$ is complete.

5) $$(-, -, +, +, +, +).$$

Example 1 in § 4, in which the system $\{\phi_i\}$ is incomplete.

6) $$(-, -, -, +, +, -),$$
7) $$(-, -, +, -, -, +).$$

Example 3 in § 4, in which the system $\{\phi_i\}$ is incomplete.

8) $$(-, -, -, -, -, -).$$

Example 2 in § 4, in which the system $\{\phi_i\}$ is incomplete.

9) $$(-, -, +, +, +, -),$$
10) $$(-, -, +, -, +, +).$$

Example 4 in § 4, in which the system $\{\phi_i\}$ is incomplete.

11) $$(-, -, +, -, -, -),$$
12) $$(-, -, -, -, +, -).$$

Examples for the cases 20) and 21) below, in which the functions u_1 and v_1 are omitted.

13) $$(-, -, +, -, +, -).$$

$\{\phi_i\}$ is an incomplete system of normalized orthogonal functions. $\{a_i\}$ is a sequence of constants such that $\{1/a_i\}$ is of finite norm. $\{\lambda_i\}$ is a sequence of constants such that $\{1/\lambda_i\}$ is of finite norm and also $\{\lambda_i/a_i\}$ is of finite norm. The biorthogonal system

$$u_i = \lambda_i \phi_{2i-1}, \qquad v_i = (\phi_{2i-1} + a_i \phi_{2i}) \frac{1}{\lambda_i} \qquad (i)$$

clearly has a limited matrix $(a)_u$ and unlimited matrices $(b)_u$ and $(b)_v$. That the matrix $(a)_v$ is limited follows from the fact that, for the biorthogonal system $(\bar{u}_i, \bar{v}_i)$:

$$\bar{u}_i = \frac{\lambda_i \phi_{2i}}{a_i}, \qquad \bar{v}_i = v_i \qquad (i),$$

the matrix $(b)_{\bar{u}}$ is limited.

14) $$(+, -, +, -, +, +),$$

15) $$(-, +, +, +, +, -).$$

Example in § 5,

16) $$(+, -, +, -, -, +),$$

17) $$(-, +, -, +, +, -).$$

The biorthogonal system is given by

$$u_i = \lambda_i \bar{u}_i, \qquad v_i = \frac{\bar{v}_i}{\lambda_i} \qquad (i),$$

where $(\bar{u}_i, v_i)$ is the biorthogonal system in the example in § 5, and the sequence $\{1/\lambda_i\}$ is of finite norm.

18) $$(+, -, -, +, +, -),$$

19) $$(-, +, +, -, -, +).$$

The biorthogonal system is given by

$$u_i = \frac{\bar{u}_i}{\lambda_i}, \qquad v_i = \lambda_i \bar{v}_i \qquad (i),$$

where $(\bar{u}_i, \bar{v}_i)$ is the biorthogonal system in the example in § 5, and λ_i are chosen so that $(b)_v$ is not limited and $(b)_u$ is limited.

20) $$(+, -, -, -, +, -),$$

21) $$(-, +, +, -, -, -).$$

The biorthogonal system is given by

$$u_i = \frac{\bar{u}_i - p}{\lambda_i}, \qquad v_i = \lambda_i \bar{v}_i \qquad (i),$$

where $(\bar{u}_i, \bar{v}_i)$ is the biorthogonal system in the example in § 5 and the λ_i are constants such that $\{1/\lambda_i\}$ is not, but $\{1/\lambda_i^2\}$ is, of finite norm.

22) $$(+, -, -, -, -, -),$$

23) $$(-, +, -, -, -, -).$$

The biorthogonal system is given by

$$u_i = \frac{\bar{u}_i\left(\int p^2 - \sum_{k=1}^{i-1}\int p\bar{u}_k\int \bar{v}_k p\right) + \int p\bar{u}_i \sum_{k=1}^{i-1}\bar{u}_k\int p\bar{v}_k}{\sqrt{\left(\int p^2 - \sum_{k=1}^{i}\int p\bar{u}_k\int \bar{v}_k p\right)\left(\int p^2 - \sum_{k=1}^{i-1}\int p\bar{u}_k\int \bar{v}_k p\right)}} \qquad (i),$$

$$v_i = \frac{\left(\bar{v}_i - \frac{p\int p\bar{v}_i}{\int p^2}\right)\left(\int p^2 - \sum_{k=1}^{i-1}\int p\bar{u}_k\int \bar{v}_k p\right) + \int p\bar{v}_i \sum_{k=1}^{i-1}\left(\bar{v}_k - \frac{p\int p\bar{v}_k}{\int p^2}\right)\int p\bar{u}_k}{\sqrt{\left(\int p^2 - \sum_{k=1}^{i}\int p\bar{u}_k\int \bar{v}_k p\right)\left(\int p^2 - \sum_{k=1}^{i-}\int p\bar{u}_k\int \bar{v}_k p\right)}} \qquad (i),$$

where

$$\bar{u}_{2j} = a_j\phi_{2j}, \qquad \bar{u}_{2j+1} = \phi_{2j+1} + a_j^2\phi_{2j},$$

$$\bar{v}_{2j} = \frac{\phi_{2j} - a_j^2\phi_{2j+1}}{a_j}, \qquad \bar{v}_{2j+1} = \phi_{2j+1} \qquad (j),$$

and $\{\phi_i\}$ is a complete system of normalized orthogonal functions, $\{a_i\}$ a sequence of constants such that $\{1/a_i\}$ is of finite norm, and p a function such that

$$\int p\phi_{2j} = \frac{1}{a_j^3}, \qquad \int p\phi_{2j+1} = \frac{1}{a_j^5}.$$

The two matrices $(b)_{\bar{u}}$ and $(b)_{\bar{v}}$ are unlimited. The function p is orthogonal to all the functions v_i; the system $\{u_i\}$ is complete. The matrices $(b)_v$ and $(a)_u$ are unlimited for $\int p\bar{v}_{2j} = 0$ and therefore $v_{2j} = \bar{v}_{2j}$. Functions f orthogonal to p can be found for which $\left\{\int fu_i\right\}$ is not of finite norm and therefore $(b)_u$ is unlimited. By considering the biorthogonal system $(\bar{u}_i, \bar{v}_i)$:

$$\bar{u}_1 = \frac{p}{\sqrt{\int p^2}}, \qquad \bar{u}_{i+1} = u_i - \frac{p\int pu_i}{\int p^2},$$

$$\bar{v}_1 = \frac{p}{\sqrt{\int p^2}}, \qquad \bar{v}_{i+1} = v_i \qquad (i),$$

which is complete both as to $\bar{u}$ and as to $\bar{v}$, we find that $(a)_v$ is unlimited.

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