# APPLICATIONS OF BIORTHOGONAL SYSTEMS OF FUNCTIONS TO THE THEORY OF INTEGRAL EQUATIONS*

BY

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## *Introduction.*

In this paper we give a sufficient condition that the characteristic numbers of an unsymmetric kernel exist and be real, and prove the expansibility of arbitrary functions in terms of the corresponding characteristic functions. This sufficient condition is stated in terms of a functional transformation† $T(f)$ defined by certain general properties (§ 1), and for the special case $T(f) = f$ we obtain the known theory of the orthogonal integral equations. The method employed is that of infinitely many variables and is based to some extent on an earlier paper.‡ However, the present paper, with the exception of a few footnotes, can be read independently of (I) if Theorems 20 and 8 which are here stated in § 1 are accepted; and if the suggestion in Remark 1 of § 1 is carried out, the only reference to (I) that is necessary is Theorem 20.

## § 1. *Preliminary notation and theorems.*

We denote by $I$ the interval $a \leqq s \leqq b$ of the real variable $s$, and by $S$ the square

$$a \leqq s \leqq b, \qquad a \leqq t \leqq b,$$

* Presented to the Society, September, 1909.

† Since the original manuscript (the present paper is essentially unchanged in content and method, but somewhat revised in form) was sent to the editors in March, 1910, J. MARTY has announced some results for unsymmetric kernels in the Comptes Rendus. His sufficient condition (April, 1910), which is also necessary (June, 1910) is stated in terms of a special $T(f)$, namely $T(f) = \int K(s, t) f(t) dt$, which satisfies the conditions of § 1 imposed on the $T(f)$ of this paper. Special examples show that the condition ($c$) of § 1 is more or less essential, see also MARTY's correction (ibid., June, 1910). The method indicated by him is a generalization of that used by E. SCHMIDT to prove the existence and is not essentially different from the method referred to by the author in the footnote at the beginning of § 3. MARTY states no expansion theorems. In May, 1910, the author sent a note to the Bulletin of the American Mathematical Society which appeared in July, 1910, and which gives necessary and sufficient conditions in terms of the general $T(f)$.

‡ A. J. PELL, these Transactions, this number. It will be referred to as (I).

of the real variables $s$ and $t$. We denote by $\mathfrak{F}$ the class of all real functions of $s$ which are integrable and whose squares are integrable on the interval $I$ in the sense of LEBESGUE, and regard two functions of $\mathfrak{F}$ as equal if they differ only on a set of points of content zero. The argument of the function is omitted whenever possible. The integral

$$\int_a^b \varphi(s)\,ds$$

is denoted by $\int \varphi$. A sequence of real numbers is of finite norm if the sum of the squares is convergent.

Let $T(f)$ be a single valued functional transformation with the following properties:

($a$) $T(f)$ transforms every function of $\mathfrak{F}$ into a function of $\mathfrak{F}$.

($b$) For any two functions $f_1$ and $f_2$ of $\mathfrak{F}$,

$$\int f_1 T(f_2) = \int f_2 T(f_1).$$

From this property follows the linearity of $T(f)$:

$$T(a_1 f_1 + a_2 f_2) = a_1 T(f_1) + a_2 T(f_2).$$

($c$) For every function $f$ of $\mathfrak{F}$,

$$\int f T(f) \geqq 0,$$

the equality sign holding only for functions $f$ such that $T(f) \equiv 0$.

Theorems 20 and 8 of (I) give the existence* of a biorthogonal system $(U_i, V_i)$ such that

($d$) $$V_i = T(U_i).$$

($e$) For every function $f$ of $\mathfrak{F}$,

$$\sum_i \left(\int f V_i\right)^2 \leqq \int f T(f).$$

($f$) For any functions† $f_1$ and $g_2 = T(f_2)$,

$$\int f_1 g_2 = \sum_i \int f_1 V_i \int U_i g_2.$$

For the purposes of this paper we suppose further that $T(f)$ has the following properties:

* The theory developed in (I) shows that every biorthogonal system $(u_i, v_i)$ such that $\sum_i \left(\int f v_i\right)^2$ converges for every $f$ of $\mathfrak{F}$ gives rise to a functional transformation $T(f)$ with the properties ($a$), ($b$), ($c$) and such that $v_i = T(u_i)$.

† Theorem 8 of (I) is more general than this in some cases.

$(a_1)$ $T(f)$ transforms every continuous function into a continuous function; and for every continuous function $f$

$$\int |T(f)| \leqq k \int |f|,$$

where $k$ is a constant independent of $f$.

$(c_1)$ There is at most* one function $p$, not essentially zero, such that $T(p) \equiv 0$ and this function $p$ is continuous.

In virtue of $(a_1)$ and $(c_1)$ we can assume that the biorthogonal system $(U_i, V_i)$ is such that

$(h)$ The functions $U_i$ and $V_i$ are continuous and the system $U_i$ is complete.

*Remark* 1. If the conditions on $T(f)$ were stated for continuous functions only in the form of $(a_1)$, $(b)$ and $(c)$, we could exhibit a biorthogonal system $(u_i, v_i)$ such that the properties $(d)$, $(e)$ and $(h)$ hold for continuous functions by referring to Theorem 20 of (I), and the property $(f)$ could be shown to hold for all continuous functions $f_1$ and $f_2$ by a simple generalization of HILBERT's proof for the analogous theorem for orthogonal functions ($H$, V, pp. 443–445).† The generalization consists of the systematic substitution of $\int fT(f)$ for $\int f^2$.

*Remark* 2. If $K(s, t)$ is continuous on $S$, then the transformed function with respect to either variable is also continuous on $S$. This follows from the conditions‡ $(a_1)$ and $(b)$ and a theorem § by Professor E. H. MOORE, namely; the necessary and sufficient condition that the sequence of continuous functions $f_n(s)$ converge uniformly is, that for every $e$ there exist an $n_e$ depending on $e$ such that for $n_1 \geqq n_e$ and $n_2 \geqq n_e$

$$\left| \int (f_{n_1} - f_{n_2}) f \right| \leqq k_1 \int |f|$$

for every continuous function $f$.

*Remark* 3. For every function $K(s, t)$ continuous on $S_1$ we have

$$T_s \int K(s, t) dt - \int T_s K(s, t) dt = 0,$$

* Throughout the paper $p$ denotes a function such that $T(p) \equiv 0$. The condition $(c_1)$ is added for simplicity; the results obtained would hold with slight modifications if there were more than one such function $p$. See Remark (7), § 3.

† HILBERT's fourth and fifth memoirs in Göttinger Nachrichten are referred to as $H$, IV, and $H$, V.

‡ These conditions are used as follows:

$$\left| \int f(t)[T_t K(s', t) - T_t K(s, t)] dt \right| = \left| \int T[f(t)][K(s', t) - K(s, t)] dt \right|$$
$$\leqq k_1 \int |T(f)| \leqq kk_1 \int |f|.$$

§ *Introduction to a form of General Analysis*, in The New Haven Mathematical Colloquium (1910), p. 5; Atti del IV congresso internazionale dei matematici, vol. 2 (1909), p. 103.

where $T_s$ denotes that $T$ operates on the argument $s$. To prove this, multiply the left-hand side of the equation by an arbitrary continuous function, integrate and apply property $(b)$.

In a similar manner it can be shown that

$$T_s T_t K(s, t) = T_t T_s K(s, t).$$

A particular case of this is that $T_s T_t K(s, t)$ is symmetric if $K(s, t)$ is symmetric.

§ 2. *Integral equation with general kernel.*

Let $K(s, t)$ be a function continuous in $S$, $f$ a function continuous on $I$, $T(f)$ the functional transformation defined in § 1and $u$ an unknown function. Consider the integral equations

$$u(s) + \int K(s, t) T[u(t)]\, dt = f(s), \tag{1}$$

$$u(s) + \int K(s, t) T[u(t)]\, dt = 0, \tag{2}$$

and their adjoints

$$u(s) + \int K(t, s) T[u(t)]\, dt = f(s), \tag{3}$$

$$u(s) + \int K(t, s) T[u(t)]\, dt = 0. \tag{4}$$

By means of the biorthogonal system $(U_i, V_i)$ defined in § 1 we establish a one-to-one correspondence between the equations (1), (2), (3), (4) and a system of linear equations in infinitely many variables $\alpha_i$,

$$\alpha_i + \sum_{j=1}^{\infty} k_{ji} \alpha_j = a_i, \tag{5}$$

$$\alpha_i + \sum_{j=1}^{\infty} k_{ji} \alpha_j = 0, \tag{6}$$

and their adjoints. In these equations the sequence $\{a_i\}$ is of finite norm and

$$k_{ij} = \int\int K(s, t) V_i(t) V_j(s)\, ds\, dt.$$

We proceed by forming the following functions and constants

$$k_i(s) = \int K(s, t) V_i(t)\, dt, \qquad k_{ij} = \int k_i V_j, \qquad a_i = \int f V_i,$$

$$L(s, t) = T_t K(s, t), \qquad L_1(s, t) = T_s K(s, t),$$

$$M(s, t) = T_s L(s, t) = T_t L_1(s, t),$$

$$l_i(s) = T[k_i(s)] = \int L(s, t) V_i(t)\, dt = \int M(s, t) U_i(t)\, dt.$$

The functions $k_i$ and $l_i$ are continuous on $I$ and $L(s,t)$, $L_1(s,t)$ and $M(s,t)$ are continuous on $S$. From the property $(f)$ of § 1 we obtain

$$\sum_{j=1}^{\infty}[k_j(s)]^2=\int K(s,t)L(s,t)dt,$$

$$\sum_{j=1}^{\infty}[l_j(s)]^2=\int M(s,t)L_1(s,t)dt,$$

$$\sum_{j=1}^{\infty}k_{ij}^2=\int k_il_i,$$

and since

$$\sum_{i=1}^{n}\sum_{j=1}^{\infty}k_{ij}^2=\sum_{i=1}^{n}\int k_il_i\leqq\sum_{i=1}^{n}\int(k_i^2+l_i^2)$$

we obtain finally

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}k_{ij}^2\leqq\int\int K(s,t)L(s,t)ds\,dt+\int\int M(s,t)L_1(s,t)ds\,dt.$$

This last inequality shows that the bilinear form

$$\sum k_{ij}x_iy_j$$

is continuous ($H$, IV, p. 203). Therefore Hilbert's theorems ($H$, IV, p. 219, and $H$, V, p. 449) can be applied to the equations (5) and (6).

*Theorem* 1. *The non-homogeneous equation* (1) *has a unique continuous solution if the homogeneous equation* (2) *has no solution. If the homogeneous equation* (2) *has a continuous solution* $u$, *it is of finite multiplicity* $n$, *and the adjoint equation* (4) *has the same number* $n$ *of linearly independent solutions* $\bar{u}$; *and the necessary and sufficient conditions that the non-homogeneous equations* (1) *and* (3) *have solutions are respectively*

$$\int fT\bar{u}_k=0 \qquad (k=1,2,\cdots,n),$$

*and*

$$\int fT(u_k)=0 \qquad (k=1,2,\cdots,n).$$

After the correspondence has been established the proof follows from an application of Hilbert's theorems mentioned above. If the equation (5) has a solution $\{a_i\}$ of finite norm the function

$$\alpha=\sum_{j=1}^{\infty}a_jk_j \tag{7}$$

is continuous, since from property $(e)$ the series converges uniformly ($H$, V, p. 442). Multiply (7) by $V_i$ and integrate,

$$\int\alpha V_i=\sum_{i=1}^{\infty}a_jk_{ji}=a_i-\alpha_i.$$

Let

$$u = f - \alpha,$$

then

$$\int u V_i = \alpha_i.$$

The function $u$ is continuous and is a solution of the equation (1), for making the substitution

$$f_1(t) = K(s, t), \qquad g_2 = T(u)$$

in the equation $(f)$, § 1, we obtain

$$\int K(s, t)\, T[u(t)]\, dt = \sum_{j=1}^{\infty} \alpha_j k_j(s) = f(s) - u(s).$$

Conversely if $u$ is a continuous solution of (1), the sequence of constants

$$\alpha_i = \int u V_i$$

is of finite norm and satisfies the equation (5), for the series

$$\sum_{i=1}^{\infty} \alpha_i k_i(s) = \int K(s, t) T[u(t)]\, dt$$

is uniformly convergent, and after multiplication by $V_i$ and integration, we obtain the equation (5).

In exactly the same manner the equivalence * of the other sets of equations can be shown.

Applying property $(b)$ of $T(f)$ on the equations (1) and (2), and operating with $T$ on the equations (3) and (4), theorem 1 can be argued from the Fredholm theory.

## § 3. *Integral equation with symmetric kernel.*†

From the hypothesis

$$K(s, t) = K(t, s)$$

follows immediately

$$k_{ij} = k_{ji},$$

* In virtue of theorem 16, (I), any other biorthogonal system of functions satisfying $(d)$, $(e)$ and $(f)$ leads to the same solutions of the integral equations as that obtained by using the system $(U_i, V_i)$ $(H$, V, p. 451).

† The results obtained in §§ 3, 4 could be obtained by a proper generalization of the method developed by E. SCHMIDT in his dissertation and in Mathematische Annalen, vol. 63. In place of SCHWARZ's inequality we have

$$\left(\int f_1 T(f_2)\right)^2 \leqq \int f_1 T(f_1) \int f_2 T(f_2)$$

(see § 8, I), and for the iterated kernels

$$K^{(n)}(s, t) = \int K(s, r)\, T_r K^{n-1}(r, t)\, dr.$$

where the constants $k_{ij}$ are defined in § 2. By means of an orthogonal transformation of the variables* $\{x_i\}$ the continuous (§ 2) quadratic form,

$$K(x) = \sum k_{ij} x_i x_j,$$

if one assumes that not all of the coefficients $k_{ij}$ are zero, is transformed into the normal form ($H$, IV, p. 201):

$$K(x) = \sum_{i=1}^{\infty} c_i x_i'^2 .$$

All of the coefficients $c_i$ are real. But some may have the value zero, let the corresponding linear forms be

$$x'_{h_i} = \sum_{k=1}^{\infty} m_{ik} x_k = M_i(x).$$

Denote the other values of $c_i$ by $1/\lambda_i$, and the corresponding linear transformations by

$$x'_{g_i} = \sum_{k=1}^{\infty} l_{ik} x_k = L_i(x).$$

The linear forms $L_i(x)$ and $M_i(x)$ are connected by the following relations ($H$, IV, p. 202):

$$\sum_{j=1}^{\infty} l_{ij} l_{kj} = \begin{cases} 1 & (i=k), \\ 0 & (i \neq k), \end{cases} \tag{8}$$

$$\sum_{j=1}^{\infty} l_{ij} m_{kj} = 0, \tag{9}$$

$$\sum_{i=1}^{\infty} x_i y_i = \sum_{i=1}^{\infty} L_i(x) L_i(y) + \sum_{i=1}^{\infty} M_i(x) M_i(y). \tag{10}$$

The relations connecting the linear forms $L_i(x)$ and $M_i(x)$ and the quadratic form $K(x)$ are as follows:

$$\sum_{j=1}^{\infty} l_{jk} k_{ji} = \frac{l_{ki}}{\lambda_k}, \tag{11}$$

$$\sum_{j=1}^{\infty} m_{jk} k_{ji} = 0. \tag{12}$$

*Remark* 4. We need to consider the exceptional cases $k_{ij} = 0$ for all $i$ and $j$. Since

$$k_{ij} = \int\int K(s, t) V_i(s) V_i(t) ds\, dt = \int\int T_s T_t K(s, t) U_i(s) U_i(t) ds\, dt,$$

it follows from $(h)$ that

$$T_s T_t K(s, t) \equiv 0,$$

* By $\{x_i\}$ and $\{y_i\}$ we denote sequences of finite norm.

and therefore, if there exists a $p$, $K(s, t)$ has the form

$$K(s, t) = \alpha(s)p(t) + \alpha(t)p(s) + kp(s)p(t),$$

where $\alpha$ is any continuous function and $k$ is any constant.

*Theorem* 2. *For a symmetric continuous kernel $K(s, t)$, which is not identically zero and which is not of the form $\alpha(s)p(t) + \alpha(t)p(s) + kp(s)p(t)$ if there is a function $p$, there exists at least one value of $\lambda$ which is necessarily real and for which the integral equation*

$$u(s) = \lambda \int K(s, t)\, T[u(t)]\, dt \tag{13}$$

*has a continuous solution $u(s)$ which is not identically zero. If there is a $p$ and if $K(s, t)$ has the form*

$$\alpha(s)p(t) + \alpha(t)p(s) + kp(s)p(t),$$

*there are no solutions $u$.*

The theorem follows directly from the existence of the constants $\lambda_i$ and the solutions $l_{ik}$ of (11), and an application of the process employed in § 2. We put

$$u_i(s) = \lambda_i \sum_{j=1}^{\infty} l_{ij} k_j(s) \tag{14}$$

and show that

$$u_i(s) = \lambda_i \int K(s, t) T[u_i(t)]\, dt.$$

*Remark* 5. If there is a function $p$, then the existence theorem for the equation (13) is equivalent to an existence theorem for the equation

$$w(s) + \mu p(s) = \lambda \int K(s, t) T[w(t)]\, dt, \qquad \int wp = c, \tag{15}$$

where $\mu$ is a parameter and $c$ is any given constant.

*Corollary* 1. There exists at least one characteristic number $\lambda$, which is necessarily real, for an unsymmetric kernel $L(s, t)$ which is of the form $T_t K(s, t)$, where $K(s, t)$ is symmetric and is not of the form

$$\alpha(t)p(s) + \alpha(s)p(t) + kp(s)p(t)$$

if a function $p$ exists. Corresponding to the characteristic numbers $\lambda_i$ the kernel $L(s, t)$, and the transposed kernel $L(t, s)$, have continuous characteristic functions $u_i$, and $v_i$, respectively, which form a biorthogonal system $(u_i, v_i)$ of the type* $T(f)$.

Applying property $(b)$ to the right-hand side of the equation (13) we obtain

$$u(s) = \lambda \int L(s, t) u(t)\, dt. \tag{16}$$

* That is, $v_i = T(u_i)$.

Operating with $T(f)$ on the equation (15) we obtain after substituting $v = T(u) = T(w + \mu p)$

$$(17) \qquad v(s) = \lambda \int L(t, s) v(t)\, dt.$$

In both cases the processes are reversible. That the functions $u_i$ and $v_i$ form a biorthogonal system follows from the equations (8).

*Remark* 6. The existence theorem for the equation (15) is equivalent to an existence theorem for the equations

$$(18) \qquad T(w) = \lambda \int M(s, t) w(t)\, dt, \qquad \int wp = c,$$

where $c$ is any given constant, and $M(s, t) = T_s T_t K(s, t)$ and is therefore symmetric (Remark 3).

*Remark* 7. If there were more than one function $p$, the exceptional cases in theorem 2 would be given * by $T_s T_t K(s, t) \equiv 0$.

If the functions $f(s)$ and $K(s, t)$ were integrable and with integrable squares, the theorems 1 and 2 would hold for any functional transformation $T(f)$ with the properties $(a)$, $(b)$ and $(c)$; the solutions being integrable with integrable squares.

## § 4. *Development of arbitrary functions.*

*Theorem* 3. *Any function $f$ expressible in the form* †

$$f(s) = \int K(s, t) g_1(t)\, dt = \int L(s, t) f_1(t)\, dt,$$

*where $f_1$ is any continuous function and $g_1 = T(f_1)$, can be developed into the uniformly convergent series*

$$(19) \qquad f(s) = \sum_i u_i(s) \int f v_i + \nu_f p(s).$$

In the linear form $M(x)$ make the substitution

$$x_i = k_i(s);$$

the resulting series is uniformly convergent. Multiply by $V_j$ and integrate; from the relation (9) we obtain

$$\int M_i[k(s)]\, V_j(s)\, ds = 0$$

* The functions $U_i$ and $p_i$ would form a complete system.

† The theorem could be stated for any function $g_1$ such that $\Sigma\left(\int g_1 u_i\right)^2$ converges, or even more broadly in some cases (Theorem 8, (I)).

and therefore, if there is a $p$,

$$M_i[k(s)] = \sum_{j=1}^{\infty} m_{ij} k_j(s) = \nu_i p(s),$$

otherwise, $M_i[k(s)] = 0$.

In the equation (10) make the substitution

$$x_i = \int g_1 U_i, \qquad y_i = \int K(s, t) V_i(t) dt.$$

The result is

$$f(s) = \int K(s, t) g_1(t) dt = \int L(s, t) f_1(t) dt$$

$$= \sum_i \left( \sum_{j=1}^{\infty} l_{ij} \int g_1 u_j \right) \left( \sum_{j=1}^{\infty} l_{ij} k_j(s) \right) + \nu_f p(s)$$

$$= \sum_i \frac{u_i(s) \int g_1 u_i}{\lambda_i} + \nu_f p(s)$$

$$= \sum_i u_i(s) \int v_i f + \nu_f p(s).$$

*Theorem* 4. *Any function $f$ expressible in the form*

$$f(s) = \int M(s, t) f_1(t) dt = \int L(t, s) g_1(t) dt,$$

*where $f_1$ is any continuous function and* $g_1 \equiv T(f_1)$, can be developed into the uniformly convergent series*

$$f(s) = \sum_i v_i(s) \int u_i f. \tag{20}$$

This development may be obtained either by substituting in the equation (10)

$$x_i = \int f_1 V_i, \qquad y_i = \int M(s, t) U_i(t) dt,$$

or by operating on the development in theorem 3 with the transformation $T(f)$.

*Remark* 8. The kernel $L(s, t)$ has no characteristic number $\lambda$ not equal to $\lambda_i$; suppose it had, and let $u$ be the corresponding characteristic function, then $u$ could be developed into a uniformly convergent series, but the coefficients $\int uv_i$ would be zero, and therefore $u = 0$ or $u = p$.

* This restriction on the function $g_1$ is not necessary. See footnote to theorem 3.

Similarly we can show that there are no solutions corresponding to a characteristic number $\lambda_i$, which are not linear in the characteristic functions belonging to $\lambda_i$.

*Remark* 9. If there are only a finite number $n$ of characteristic numbers $\lambda_i$ it follows from the developments (19) and (20) that the kernels have the form

$$K(s, t) = \sum_{i=1}^{n} \frac{u_i(s)u_i(t)}{\lambda_i} + \alpha(s)p(t) + \alpha(t)p(s) + kp(s)p(t),$$

$$L(s, t) = \sum_{i=1}^{n} \frac{u_i(s)v_i(t)}{\lambda_i} + p(s)\beta(t),$$

$$M(s, t) = \sum_{i=1}^{n} \frac{v_i(s)v_i(t)}{\lambda_i}.$$

*Theorem* 5. *The maximum value which the expression*

$$\left|\int\int K(s, t)\, T[f(s)]\, T[f(t)]\, dsdt\right|$$

*can assume for all continuous functions $f$ such that*

$$\int fT(f) = 1$$

*is equal to the reciprocal of that $\lambda_i$ which is smallest in absolute value, and this maximum is attained for the corresponding characteristic function.*

## § 5. *Examples.*

As examples of the different forms of integral equations for which § 3 gives existence theorems we give the following:

*Example* 1.

$$u(s) = \lambda \int K(s, t)\alpha(t)u(t)dt,$$

where $\alpha$ is a continuous positive function.

*Example* 2.

$$u(s) = \lambda \int\int K(s, t)K^*(s, t)u(t_1)dt\,dt,$$

where $K^*(s, t)$ is a positive-definite continuous kernel.

*Example* 3.

$$u(s) + \mu q(s) = \lambda \int K(s, t)T[u(t)]dt, \qquad \int uq = 0,$$

where $q$ is an an invariant function of the functional transformation $T(f)$, i. e., $q = kT(q)$. The equations are derived from the equations (15) where

$c = 0$ and the functional transformation * is

$$\bar{T}(f) = T(f) - \frac{q\int qT(f)}{\int q^2},$$

which satisfies all the conditions of § 1.

As a special case of this, for $T(f) = f$, we have the integral equation treated by CAIRNS in his dissertation.†

*Example* 4. Problems in the calculus of variations in which the condition

$$\int fT(f) = 1$$

replaces the usual condition

$$\int f^2 = 1,$$

give rise to differential-integral equations of the form

$$L(u) + \lambda T(u) = 0,$$

where

$$L(u) = \frac{d\left(p_1 \dfrac{du}{ds}\right)}{ds} + p_2 u(s),$$

and $p_1$ is a positive function which together with its first derivative is continuous, and $p_2$ is a continuous function. The equation above reduces to the integral equation

$$u(s) = \lambda \int G(s, t)\, T[u(t)]\, dt,$$

where $G(s, t)$ is a Green's function for $L(u)$ satisfying certain boundary conditions. This equation has infinitely many solutions, satisfying the same boundary conditions as $G(s, t)$, corresponding to real values of $\lambda$, if there are only a finite number of functions $p_i$ such that $T(p_i) \equiv 0$.

A special case is the non-self-adjoint differential equation

$$L[\bar{L}(v)] - \lambda v = 0,$$

where $\bar{L}(v)$ is a self-adjoint differential expression. The adjoint differential equation is

$$\bar{L}[L(u)] + \lambda u = 0.$$

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* See Theorem 16, (I).

† W. D. CAIRNS, *Die Anwendung der Integralgleichungen auf die zweite Variation bei isoperimetrischen Problemen*, Göttingen, 1907.

*Example* 5. In example (2) let the function $K^*(s, t)$ be equal to the Green's function $G(s, t)$ of example (4). Then there exists at least one value of $\lambda$ for which the equation

$$L(v) + \lambda \int K(s, t) v(t) dt = 0$$

has a solution $v$ which is continuous together with its first and second derivatives and satisfies the boundary conditions of $G(s, t)$.

The two systems $\psi_i$ and $-\psi_i''$, considered by E. SCHMIDT,* form a biorthogonal system of the type

$$T(f) = \int G(s, t) f(t) dt,$$

where $G(s, t)$ is the Green's function, for $L(v) = -v''$, which vanishes at the ends of the interval.

*Example* 6. Let $\bar{T}(f)$ be a functional transformation which satisfies the conditions $(a)$, $(b)$, and $(a_1)$ of § 1, but not necessarily $(c)$ and $(c_1)$. Let $K(s, t)$ be a symmetric, continuous kernel which is positive-semi-definite:

$$\int \int_t K(s, t) f(s) f(t) ds\, dt \geqq 0,$$

the equality sign holding at most for one function $p$. Then the functional transformation

$$T(f) = \int K(s, t) f(t) dt$$

satisfies all the conditions of § 1.

If the integral equation †

$$(A) \qquad u(s) = \lambda \int \bar{T}_s K(s, t) u(t) dt$$

has a solution $u$ for some value of $\lambda$, then the adjoint integral equation

$$v(s) = \lambda \int [\bar{T}_t K(s, t)] v(t) dt$$

has a solution $v$ such that

$$v = T(u).$$

Two sets of solutions $(u_i, v_i)$ and $(u_j, v_j)$ corresponding to two different values of $\lambda$ satisfy the condition

$$\int u_i v_j = 0;$$

* Mathematische Annalen, vol. 63 (1907), p. 473.

† A special case, namely $\bar{T}(f) = a \cdot f$, has been considered by MARTY, Comptes Rendus, February 28, 1910, p. 515.

that is, the solutions form a biorthogonal system of the type $T$, and therefore the values of $\lambda$, for which there exist solutions, are real. The existence of real characteristic numbers $\lambda$ is shown by means of the integral equation

$$(B) \qquad u(s) = \nu \int \bar{T}_s \bar{T}_t K(s, t) T[u(t)] dt,$$

which is of the type of the equation (14), for the kernel $\bar{T}_s \bar{T}_t K(s, t)$ is symmetric (Remark 3). It is evident that any solution of $(A)$ for the value $\lambda$ is a solution of $(B)$ for $\nu = \lambda^2$. And unless $\bar{T}_s \bar{T}_t K(s, t)$ is of the form

$$a(s)p(t) + a(t)p(s) + kp(s)p(t),$$

there exists at least one value of $\nu$ which is real and positive and for which the equation $(B)$ has a continuous solution $u$.

Construct the two functions

$$2u_1(s) = u(s) + \sqrt{\nu} \int T_s K(s, t) u(t) dt,$$

$$2u_2(s) = u(s) - \sqrt{\nu} \int T_s K(s, t) u(t) dt.$$

The continuous functions $u_1$ and $u_2$ are solutions of $(A)$ for $\lambda = \sqrt{\nu}$ and $\lambda = -\sqrt{\nu}$ respectively.

*Example* 7. In certain cases the integral equation*

$$(\alpha) \qquad u(s) = \lambda \int K(s, t) u(t) dt + \mu \int H(s, t) u(t) dt,$$

where $K(s, t)$ and $H(s, t)$ are symmetric kernels continuous on $S$, $\lambda$ and $\mu$ are parameters, and $u$ the function to be determined, may be reduced to the class of integral equations considered in § 3.

Let $\lambda_1$ be that characteristic number of $K(s, t)$ which is smallest in absolute value. Then for any value of $\lambda_0$ such that

$$|\lambda_0| < |\lambda_1|,$$

there exists at least one value of $\mu$ for which the integral equation has a continuous solution $u$. For the maximum of

$$\left| \int\int K(s, t) f(s) f(t) ds\, dt \right|,$$

* MAX MASON, *Randwertaufgaben bei gewöhnlichen Differentialgleichungen*, *Dissertation*, Göttingen, 1903, p. 5.

for all continuous functions $f$ such that

$$\int f^2 = 1,$$

is equal to the reciprocal of $|\lambda_1|$, and therefore the functional transformation

$$f(s) - \lambda_0 \int K(s, t) f(t)\, dt$$

satisfies all the conditions of § 1. Since $\lambda_0$ is not a characteristic number, it can easily be shown that the inverse transformation,

$$T(f) = f(s) - \lambda_0 \int k(s, t) f(t)\, dt,$$

where $k(s, t)$ is the reciprocal of $K(s, t)$, also satisfies the conditions of § 1. Operating with $T(f)$ on the integral equation $(\alpha)$, where $\lambda = \lambda_0$, we obtain

$$(\beta) \qquad u(s) = \mu \int^{\prime} T_s H(s, t) u(t)\, dt,$$

which is of the type of the equation (17), § 3. From the equation $(\beta)$ we can pass back to the equation $(\alpha)$, and therefore the statement is proved.

Let $H(s, t)$ be a positive semi-definite kernel, and let $\lambda$ be any number not equal to a characteristic number of $K(s, t)$; then the functional transformation

$$f(s) - \lambda \int k(s, t) f(t)\, dt$$

possesses all the properties imposed on the functional transformation $\bar{T}(f)$ of Example 6. Operating with this on the equation $(\alpha)$, we obtain an equation of the type discussed in Example 6.

Let $\lambda$ take on a value equal to a characteristic number $\bar{\lambda}$ of $K(s, t)$ and let $H(s, t)$ be a positive semi-definite kernel which is orthogonal to all the characteristic functions of $K(s, t)$ corresponding to $\bar{\lambda}$. The transformation

$$f(s) - \bar{\lambda} \int K(s, t) f(t)\, dt$$

defines an inverse transformation $\bar{T}(f)$ which has the property $(b)$, § 1, and which is applicable to every function that is orthogonal to all the characteristic functions of $K(s, t)$ corresponding to $\bar{\lambda}$. Operating with this $\bar{T}(f)$ on the equation $(\alpha)$, we obtain an equation of the same form as the one considered in Example 6, and it is not difficult to see that we again have the existence of values of $\mu$ besides the obvious one, $\mu = 0$.

The following is an example of the integral equation ($\alpha$) in which the kernel $H(s, t)$ is not positive-definite, and to the value $\lambda = -1$ there corresponds no *real* value of $\mu$.

$$u(s) = \lambda \int \psi(s)\psi(t)u(t)dt + \mu \int [h_1(s)h_1(t) - 2h_2(s)h_2(t)]u(t)dt,$$

where

$$\int \psi^2 = \int h_1^2 = \int h_2^2 = 1,$$

$$\int h_1 h_2 = 0, \qquad \int h_1 \psi = 0, \qquad \int h_2 \psi = 0.$$

*March*, 1910.