

# ON SEMI-DISCRIMINANTS OF TERNARY FORMS\*

BY

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## § 1. *Introduction.*

It is well known that the number of independent conditional relations which must exist among the coefficients of a ternary form of order  $m$  in order that it should be factorable into linear factors, distinct term for term, is  $\frac{1}{2}m(m-1)$ . Several writers, † among them BRILL, and GORDAN, have published methods for the determination of such sets of relations. Their results are, as a rule, expressed in the form of a covariant, the identical vanishing of which gives necessary and sufficient conditions for the factorability.

These methods are somewhat indirect, and from certain standpoints are unsatisfactory for the additional reason that the set of conditions given by the identical vanishing of such a covariant is always redundant.

Our aim in this paper has been to develop a direct method of attacking this problem. Our method leads to a set of conditional relations containing the exact minimum number  $\frac{1}{2}m(m-1)$ ; that is, it leads to a set of  $\frac{1}{2}m(m-1)$  independent seminvariants of the form, whose simultaneous vanishing gives necessary and sufficient conditions for the factorability. We shall call these seminvariants *semi-discriminants* ‡ of the form. They are all of the same degree  $2m-1$ ; and are readily formed for any order  $m$  as simultaneous invariants of a certain set of *binary* quantics related to the original ternary form.

If a polynomial,  $f_{3m}$ , of order  $m$ , and homogeneous in three variables  $(x_1, x_2, x_3)$  is factorable into linear factors, its terms in  $(x_1, x_2)$  must furnish the  $(x_1, x_2)$  terms of those factors. Call these terms collectively  $a_{0x}^m$ , and the terms linear in  $x_3$  collectively  $x_3 a_{1x}^{m-1}$ . Then if the factors of the former were known, and were distinct, say

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† THAER, "Ueber die Zerlegbarkeit einer ebenen Linie dritter Ordnung in gerade Linien," *Mathematische Annalen*, vol. 14 (1879); GORDAN, "Das Zerfallen der Curven in gerade Linien," *Mathematische Annalen*, vol. 45 (1894); BRILL, "Ueber die Zerfällung der Ternärformen in lineare Factoren," *Mathematische Annalen*, vol. 50 (1898).

‡ It seems desirable to introduce a new term for the members of such a minimum set of functions of the form's coefficients. Since the name discriminant is already in common use, and since discriminants are *invariants*, it seems natural to adopt here the term *semi-discriminant*.

$$a_{0x}^m \equiv a_{00} \prod_{i=1}^m (r_2^{(i)} x_1 - r_1^{(i)} x_2) \div \prod_{i=1}^m (r_2^{(i)}),$$

the second would give by rational means the terms in  $x_3$  required to complete the several factors. For we could find rationally the numerators of the partial fractions in the decomposition of  $a_{1x}^{m-1}/a_{0x}^m$ , viz.:

$$\frac{a_{1x}^{m-1}}{a_{0x}^m} \equiv \frac{\prod_{i=1}^m r_2^{(i)}}{a_{00}} \sum_{i=1}^m \frac{\alpha_i}{r_2^{(i)} x_1 - r_1^{(i)} x_2},$$

and the factors of the complete form will be, of course,

$$r_2^{(i)} x_1 - r_1^{(i)} x_2 + \alpha_i x_3 \quad (i=1, 2, \dots, m).$$

Further, the coefficients of all other terms in  $f_{3m}$  are rational integral functions of the  $r^{(i)}$  on the one hand, and of the  $\alpha_i$  on the other — symmetrical in the sets  $(r_2^{(i)}, -r_1^{(i)}, \alpha_i)$ . We shall show in general that all these coefficients in the case of any linearly factorable form are *rationally* expressible in terms of those occurring in  $a_{0x}^m, a_{1x}^{m-1}$ . Hence will follow the important theorem (§ 5):

**Theorem 1.** *If a ternary form  $f_{3m}$  is decomposable into linear factors, all its coefficients, after certain  $2m$ , are expressible rationally in terms of those  $2m$  coefficients. That is, in the space whose coördinates are all the coefficients of ternary forms of order  $m$ , the forms composed of linear factors fill a rational spread of  $2m$  dimensions.*

We shall thus obtain the explicit form of the general ternary quantic which is factorable into linear factors (§ 5). Moreover, in case  $f_{3m}$  is not factorable a similar development will give the theorem (§ 3),

**Theorem 2.** *Every ternary form  $f_{3m}$ , for which the discriminant  $D$  of  $a_{0x}^m$  does not vanish, can be expressed as the sum of the product of  $m$  distinct linear forms, plus the square of an arbitrarily chosen linear form, multiplied by a "satellite" form of order  $m-2$  whose coefficients are, except for the factor  $D^{-1}$ , integral rational seminvariants of the original form  $f_{3m}$ .*

## § 2. A class of ternary seminvariants.

Let us write the general ternary quantic in homogeneous variables as follows:

$$f_{3m} = a_{0x}^m + a_{1x}^{m-1} x_3 + a_{2x}^{m-2} x_3^2 + \dots + a_{m0} x_3^m,$$

where

$$a_{ix}^{m-i} \equiv a_{i0} x_1^{m-i} + a_{i1} x_1^{m-i-1} x_2 + a_{i2} x_1^{m-i-2} x_2^2 + \dots + a_{im-i} x_2^{m-i} \quad (i=1, 2, \dots, m).$$

Then write

$$\frac{a_{1x}^{m-1}}{a_{0x}^m} = \frac{a_{1x}^{m-1}}{\prod_{k=1}^m (r_2^{(k)} x_1 - r_1^{(k)} x_2)} = \sum_{k=1}^m \frac{\alpha_k}{r_2^{(k)} x_1 - r_1^{(k)} x_2} \quad (a_{00} = r_2^{(1)} r_2^{(2)} \dots r_2^{(m)});$$

and we have in consequence, assuming that  $D \neq 0$ , and writing

$$a'_{0r(k)} = \left[ \frac{\partial a'_{0x}}{\partial x_1} \right]_{x_1=r_1^{(k)}, x_2=r_2^{(k)}}, \quad a'_{0r(k)} = \left[ \frac{\partial a'_{0x}}{\partial x_2} \right]_{x_1=r_1^{(k)}, x_2=r_2^{(k)}},$$

the results

$$(1) \quad \alpha_k = r_2^{(k)} a'_{1r(k)}{}^{m-1} / a'_{0r(k)}{}^m = -r_1^{(k)} a'_{1r(k)}{}^{m-1} / a'_{0r(k)}{}^m.$$

Hence also

$$(2) \quad a'_{0r(k)}{}^m = -\frac{r_1^{(k)}}{r_2^{(k)}} a'_{0r(k)}{}^m.$$

The discriminant of  $a'_{0x}$  can be expressed in the following form :

$$(3) \quad D = \prod_{j=1}^m a'_{0r(j)}{}^m / a_{00} (-1)^{\frac{1}{2}m(m-1)},$$

and therefore

$$(4) \quad \alpha_k = \frac{r_2^{(k)} a'_{1r(k)}{}^{m-1} a'_{0r(1)}{}^m a'_{0r(2)}{}^m \cdots a'_{0r(k-1)}{}^m a'_{0r(k+1)}{}^m \cdots a'_{0r(m)}{}^m}{a_{00} (-1)^{\frac{1}{2}m(m-1)} D},$$

and in like manner we get

$$(5) \quad \prod_{k=1}^m \alpha_k = a'_{1r(1)}{}^{m-1} a'_{1r(2)}{}^{m-1} \cdots a'_{1r(m)}{}^{m-1} / (-1)^{\frac{1}{2}m(m-1)} D.$$

The numerator of the right hand member of this last equality is evidently the resultant (say  $R_m$ ) of  $a'_{0x}$  and  $a'_{1x}$ .

Consider next the two differential operators

$$\Delta_1 = m a'_{00} \frac{\partial}{\partial a_{10}} + (m-1) a_{01} \frac{\partial}{\partial a_{11}} + \cdots + a_{0m-1} \frac{\partial}{\partial a_{1m-1}},$$

$$\Delta_2 = m a_{0m} \frac{\partial}{\partial a_{1m-1}} + (m-1) a_{0m-1} \frac{\partial}{\partial a_{1m-2}} + \cdots + a_{01} \frac{\partial}{\partial a_{10}};$$

and particularly their effect when applied to  $a'_{1x}{}^{m-1}$ . We get [see (2)]

$$(6) \quad \Delta_1 a'_{1r(k)}{}^{m-1} = a'_{0r(k)}{}^m, \quad \Delta_2 a'_{1r(k)}{}^{m-1} = a'_{0r(k)}{}^m = -\frac{r_1^{(k)}}{r_2^{(k)}} a'_{0r(k)}{}^m,$$

and from these relations we deduce the following:

$$(7) \quad \Delta_1 \prod_{k=1}^m \alpha_k = \frac{\Delta_1 R_m}{(-1)^{\frac{1}{2}m(m-1)} [1] D} = a_{00} \Sigma \frac{a'_{1r(1)}{}^{m-1} a'_{1r(2)}{}^{m-1} \cdots a'_{1r(m-1)}{}^{m-1}}{a'_{0r(1)}{}^m a'_{0r(2)}{}^m \cdots a'_{0r(m-1)}{}^m},$$

or, from (1)

$$(8) \quad \frac{\Delta_1 R_m}{(-1)^{\frac{1}{2}m(m-1)} [1] D} = \Sigma \alpha_1 \alpha_2 \cdots \alpha_{m-1} r_2^{(m)}.$$

In (7) the symmetric function  $\Sigma$  is to be read with reference to the  $r$ 's, the super-

scripts of the  $r$ 's replacing the subscripts usual in a symmetric function. Let us now operate with  $\Delta_2$  on both members of (7). This gives

$$\frac{\Delta_1 \Delta_2 R_m}{(-1)^{\frac{1}{2}m(m-1)} |1 \ D|} = a_{00} \sum \frac{a_{1r(1)}^{m-1} a_{1r(2)}^{m-1} \cdots a_{1r(m-2)}^{m-1}}{a'_{0r(1)}{}^m a'_{0r(2)}{}^m \cdots a'_{0r(m-2)}{}^m} \left( -\frac{r_1^{(m-1)}}{r_2^{(m-1)}} - \frac{r_1^{(m)}}{r_2^{(m)}} \right).$$

Let  $\Sigma_h$  represent an elementary symmetric function of the two groups of homogeneous variables  $r_1, r_2$  which involves  $h$  distinct letters of each group, viz:  $r_i^{(m-j+1)}$  ( $j = 1, 2, \dots, h$ ). Then we have

$$(9) \quad \frac{\Delta_1 \Delta_2 R_m}{(-1)^{\frac{1}{2}m(m-1)} |1 \ 1 \ D|} = \Sigma [(-1) \alpha_1 \alpha_2 \cdots \alpha_{m-2} \Sigma_2 r_1^{(m-1)} r_2^{(m)}].$$

We are now in position to prove by induction the following fundamental formula:

$$(10) \quad \frac{\Delta_1^{m-s-t} \Delta_2^t R_m}{(-1)^{\frac{1}{2}m(m-1)} |m-s-t \ t \ D|} = \Sigma [(-1)^t \alpha_1 \alpha_2 \cdots \alpha_s \Sigma_{m-s} r_1^{(s+1)} r_1^{(s+2)} \cdots r_1^{(s+t)} r_2^{(s+t+1)} \cdots r_2^{(m)}] \\ (s=0, 1, \dots, m; t=0, 1, \dots, m-s),$$

where the outer summation covers all subscripts from 1 to  $m$ , superscripts of the  $r$ 's counting as subscripts in the symmetric function. Representing by  $J_{m-s-t, t}$  the left hand member of this equality we have from (6)

$$\Delta_2 J_{m-s-t, t} = \Sigma \left( (-1)^{t+1} \frac{a_{1r(1)}^{m-1} a_{1r(2)}^{m-1} \cdots a_{1r(s-1)}^{m-1}}{a'_{0r(1)}{}^m a'_{0r(2)}{}^m \cdots a'_{0r(s-1)}{}^m} \right. \\ \left. \times r_2^{(1)} r_2^{(2)} \cdots r_2^{(s)} \frac{r_1^{(s)}}{r_2^{(s)}} \Sigma_{m-s} r_1^{(s+1)} \cdots r_1^{(s+t)} r_2^{(s+t+1)} \cdots r_2^{(m)} \right).$$

This equals

$$\Sigma (-1)^{t+1} \alpha_1 \alpha_2 \cdots \alpha_{s-1} S,$$

where  $S$  is a symmetric function each term of which involves  $t+1$  letters  $r_1$  and  $m-s-t$  letters  $r_2$ . The number of terms in an elementary symmetric function of any number of groups of homogeneous variables equals the number of permutations of the letters occurring in any one term when the subscripts (here superscripts) are removed. Hence the number of terms in  $\Sigma_{m-s}$  is

$$\frac{|m-s|}{|m-s-t \ t|},$$

and the number of terms in  $S$  is

$$(m-s+1) |m-s|/|t \ m-s-t|.$$



Then  $I_{10}$  is annihilated by  $\Omega_{x_2x_1}$  but not by  $\Omega_{x_1x_2}$ ,  $I_{01}$  is annihilated by  $\Omega_{x_1x_2}$  but not by  $\Omega_{x_2x_1}$ , and  $I_{00}$  is annihilated by  $\Omega_{x_1x_2}$  but not by  $\Omega_{x_2x_1}$ . In general  $I_{m-s-t}$  fails of annihilation when operated upon by a general operator  $\Omega_{x_ix_j}$  which contains a partial derivative with respect to  $a_{it}$ . We have now proved theorem 2.

#### § 4. The semi-discriminants.

A necessary and sufficient condition that  $f_{3m}$  should degenerate into the product of  $m$  distinct linear factors is that  $\mu_{m-2}$  should vanish identically. Hence, since the number of coefficients in  $\mu_{m-2}$  is  $\frac{1}{2}m(m-1)$ , these equated to zero give a minimum set of conditions in order that  $f_{3m}$  should be factorable in the manner stated. As previously indicated we refer to these seminvariants as a set of semi-discriminants of the form  $f_{3m}$ . They are

$$(13) \quad I_{m-s-t} = Da_{st} - \frac{\Delta_1^{m-s-t} \Delta_2^t R_m}{(-1)^{\frac{1}{2}m(m-1)} [t] [m-s-t]} \quad \left( \begin{matrix} s=2, 3, \dots, m; \\ t=0, 1, \dots, m-s \end{matrix} \right).$$

They are obviously independent since each one contains a coefficient ( $a_{st}$ ) not contained in any other. They are free from adventitious factors, and each one is of degree  $2m-1$ .

In the case where  $m=2$  we have

$$I_{00} = -a_{20} \begin{vmatrix} 2a_{00} & a_{01} \\ a_{01} & 2a_{02} \end{vmatrix} + \begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & 0 \\ 0 & a_{10} & a_{11} \end{vmatrix}.$$

This is also the *ordinary* discriminant of the ternary quadratic.

The three semi-discriminants of the ternary cubic have been computed by the author by another method.\* Corresponding results for the case  $m=4$  have not been published. They are the following:

$$I_{00} = \frac{1}{27}a_{40}(4i_1^3 - J_1^2) - R_4,$$

where

$$i_1 = a_{02}^2 - 3a_{01}a_{03} + 12a_{00}a_{04},$$

$$J_1 = 27a_{01}^2a_{04} + 27a_{00}a_{03}^2 + 2a_{02}^3 - 72a_{00}a_{02}a_{04} - 9a_{01}a_{02}a_{03},$$

$$R_4 = \begin{vmatrix} a_{10} & a_{11} & a_{12} & a_{13} & 0 & 0 \\ 0 & a_{10} & a_{11} & a_{12} & a_{13} & 0 \\ 0 & 0 & a_{10} & a_{11} & a_{12} & a_{13} \\ a_{01}a_{10} - a_{00}a_{11} & a_{02}a_{10} - a_{00}a_{12} & a_{03}a_{10} - a_{00}a_{13} & a_{04}a_{10} & 0 & 0 \\ a_{00} & a_{01} & a_{02} & a_{03} & a_{04} & 0 \\ 0 & a_{00} & a_{01} & a_{02} & a_{03} & a_{04} \end{vmatrix},$$

\*American Journal of Mathematics, vol. 32 (1910), p. 89.

the other members of the set being obtained by operating upon  $R_4$  with powers of  $\Delta_1, \Delta_2$ :

$$\begin{aligned}\Delta_1 &= 4a_{00} \frac{\partial}{\partial a_{10}} + 3a_{01} \frac{\partial}{\partial a_{11}} + 2a_{02} \frac{\partial}{\partial a_{12}} + a_{03} \frac{\partial}{\partial a_{13}}, \\ \Delta_2 &= 4a_{04} \frac{\partial}{\partial a_{13}} + 3a_{03} \frac{\partial}{\partial a_{12}} + 2a_{02} \frac{\partial}{\partial a_{11}} + a_{01} \frac{\partial}{\partial a_{10}},\end{aligned}$$

according to the formula

$$I_{4-s-t, t} = a_{tt} D - \frac{\Delta_1^{4-s-t} \Delta_2^t R_4}{[4-s-t][t]} \quad (s=2, 3, 4; t=0, 1, \dots, 4-s).$$

### § 5. Proof of Theorem 1.

The factors of  $a_{0x}^m$  being assumed distinct we can always solve  $I_{m-s-t, t} = 0$  for  $a_{tt}$ , the result being obviously rational in the coefficients occurring in  $a_{0x}^m, a_{1x}^{m-1}$ . This proves theorem 1, as far as the case  $D \neq 0$  is concerned. Moreover by carrying the resulting values of  $a_{tt}$  ( $s=2, 3, \dots, m; t=0, 1, \dots, m-s$ ) back into  $f_{3m}$  we get the general form of a ternary quantic which is factorable into linear forms. In the result  $a_{0x}^m, a_{1x}^{m-1}$  are perfectly general (the former, however, subject to the negative condition  $D \neq 0$ ), whereas

$$(-1)^{\frac{1}{2}m(m-1)} D a_{jx}^{m-j} \equiv \frac{\Delta_1^{m-j} R_m}{[m-j]} x_1^{m-j} + \frac{\Delta_1^{m-j-1} \Delta_2 R_m}{[m-j-1][1]} x_1^{m-j-1} x_2 + \dots + \frac{\Delta_2^{m-j} R_m}{[m-j]} x_2^{m-j} \quad (j=2, 3, \dots, m).$$

Assume next that  $D = 0$ . Then there are two cases to consider. First,  $a_{0x}^m$  has multiple factors but  $f_{3m}$  as a whole has no multiple linear ternary factors, and a mere interchange of subscripts of the variables ( $x_1, x_2, x_3$ ) transforms  $f_{3m}$  into a new quantic whose binary  $a_{0x}^m$  has no multiple factors. For this new quantic, then,  $D \neq 0$ . Secondly,  $f_{3m}$  as a whole may have repeated ternary linear factors. Let there be one such  $D$  factor, of multiplicity two. Then  $J_{m-s-t, t}$ , which we now write in the briefer form

$$J_{m-s-t, t} = \frac{\Delta_t R_m(a_{00}, a_{01}, \dots)}{D(a_{00}, a_{01}, \dots)},$$

is indeterminate. In fact, in this case, two of the  $\alpha_i$  in the right hand member of (10), say  $\alpha_1, \alpha_2$ , are to be replaced by one and the same quantity

$$\alpha_{12} \equiv \frac{r_2^{(1)} \frac{\partial}{\partial r_1^{(1)}} a_{1r}^{m-1}}{\frac{\partial}{\partial r_1^{(1)}} a_{0r}^{m-1}}.*$$

\*See Bulletin of the American Mathematical Society, vol. 17 (1911), p. 451, theorem in § 2.

Then it is not difficult to show that the corresponding true value\* of  $J_{m-s-t, t}$ , and hence of  $a_{it}$ , is

$$a_{it} = \frac{\Delta_{it} \frac{\partial}{\partial a_{0i}} R_m(a_{00}, \dots)}{\frac{\partial}{\partial a_{0i}} D(a_{00}, \dots)} \quad (i = \text{any number of the set } 0, 1, \dots, m).$$

Likewise, when  $f_{3m}$  contains a linear factor of multiplicity three, three of the  $\alpha_i$  in (10), say  $\alpha_1, \alpha_2, \alpha_3$  are to be replaced by the same quantity, viz.:

$$\alpha_{123} \equiv \frac{r_2^{(1)} \frac{\partial^2}{\partial r_1^{(1)2}} a_{1r^{(1)}}^{m-1}}{\frac{\partial^2}{\partial r_1^{(1)2}} a_{0r^{(1)}}^m}.$$

and then we get

$$a_{it} = \frac{\Delta_{it} \frac{\partial^2}{\partial a_{0i}^2} R_m(a_{00}, \dots)}{\frac{\partial^2}{\partial a_{0i}^2} D(a_{00}, \dots)} \quad (i = \text{any number of the set } 0, 1, \dots, m).$$

Similarly in the case of a factor of  $f_{3m}$  of multiplicity higher than three. Hence in these cases also  $a_{it}$  is rational in the coefficients of  $a_{0x}^m, a_{1x}^{m-1}$ .

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\* Cf. SALMON, *Modern Higher Algebra* (fourth edition), p. 90.