ON A THEOREM OF FEJÉR'S AND AN ANALOGON TO GIBBS' PHENOMENON*

BY

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§ 1. *Introduction.*

Fejér † has obtained the following general result concerning the partial sums of a Fourier series:

Let $f(x)$ be a function, finite and integrable for $0 \leqq x \leqq 2\pi$; suppose that

(1) $$|f(x)| \leqq M, \qquad 0 \leqq x \leqq 2\pi,$$

and that the Fourier coefficients of $f(x)$ satisfy the inequalities

(2) $$|a_n| \leqq \frac{A}{n}, \qquad |b_n| \leqq \frac{B}{n},$$

which is the case, for instance, when $f(x)$ is a function of limited fluctuation. Let

(3) $$s_n(x) = a_0 + a_1 \cos x + b_1 \sin x + \cdots + a_n \cos nx + b_n \sin nx$$

be the $(n+1)$th partial sum of the Fourier series for $f(x)$, and

(4) $$S_n(x) = \frac{1}{n+1}[s_0(x) + s_1(x) + \cdots + s_n(x)];$$

then ‡

(5) $$|S_n(x)| \leqq M, \qquad 0 \leqq x \leqq 2\pi,$$

and from the identity

(6) $$s_n(x) = S_n(x) + \frac{1}{n+1}\sum_{\nu=1}^{n} \nu(a_\nu \cos \nu x + b_\nu \sin \nu x)$$

we conclude by the aid of (2) and (5) that

(7) $$|s_n(x)| \leqq M + \frac{1}{n+1}\left(A\sum_{\nu=1}^{n}|\cos \nu x| + B\sum_{\nu=1}^{n}|\sin \nu x|\right),$$

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* Presented to the Society (Chicago), April 5, 1912.

† L. Fejér, *Sur les sommes partielles de la série de Fourier*, Comptes Rendus, 23 mai, 1910, and *Sur les singularités de la série de Fourier des fonctions continues*, Annales de l'École Normale, ser. 3: vol. 28 (1911), pp. 63–103.

‡ L. Fejér, *Untersuchungen über trigonometrische Reihen*, Mathematische Annalen, vol. 58 (1904), pp. 51–69.

whence Fejér's inequality

$$|s_n(x)| \overline{\gtrless} M + \frac{n}{n+1}(A+B) < M + A + B.$$

In § 2 of the present paper, it is shown that the above inequality may be replaced by the following closer ones:

$$|s_n(x)| < M + \sqrt{A^2 + B^2} \tag{8}$$

for any values of A and B, and

$$|s_n(x)| < M + \frac{1}{2}A + \frac{\sqrt{3}}{2}B \tag{9}$$

for $B \overline{\gtrless} \sqrt{3}\,A$.

We then proceed to investigate the expressions occurring in (7):

$$\begin{aligned} U_n(x) &= \frac{1}{n+1}(|\cos x| + |\cos 2x| + \cdots + |\cos nx|), \\ V_n(x) &= \frac{1}{n+1}(|\sin x| + |\sin 2x| + \cdots + |\sin nx|). \end{aligned} \tag{10}$$

The first question demanding a solution is obviously that of the existence of $\lim_{n=\infty} U_n(x)$ and $\lim_{n=\infty} V_n(x)$. In § 3 it is shown that for $x = 0$

$$\lim_{n=\infty} U_n(0) = 1, \qquad \lim_{n=\infty} V_n(0) = 0;$$

that for $x = (p/q)\pi$, where $p\ (\neq 0)$ and q are integers and relative primes

$$\begin{aligned} \lim_{n=\infty} U_n\left(\frac{p}{q}\pi\right) &= \frac{\sin \dfrac{\left[\frac{q}{2}\right] + \frac{1}{2}}{q}\pi}{q \sin \dfrac{\pi}{2q}} = \frac{1}{2q}\left(\cot\frac{\pi}{4q} + (-1)^{q+1}\tan\frac{\pi}{4q}\right) \\ &= \begin{cases} \dfrac{1}{q}\cot\dfrac{\pi}{2q}, & q \text{ even}, \\ \dfrac{1}{q\sin\dfrac{\pi}{2q}}, & q \text{ odd}, \end{cases} \end{aligned} \tag{11}$$

$$\lim_{n=\infty} V_n\left(\frac{p}{q}\pi\right) = \frac{1}{q}\cot\frac{\pi}{2q};$$

and finally that for $x : \pi$ irrational

$$\lim_{n=\infty} U_n(x) = \lim_{n=\infty} V_n(x) = \frac{2}{\pi}. \tag{12}$$

Thus the expressions $\lim_{n=\infty} U_n(x)$ and $\lim_{n=\infty} V_n(x)$ are simple examples of point-wise discontinuous functions of x, and it is interesting to note that they arise quite naturally from the elementary problem in Fourier's series considered above, while most of the known examples of such functions have to be constructed in a more or less artificial way.

From (11) and (12) it is seen that for any fixed value of x

$$(13)\qquad \begin{aligned} \frac{1}{2} \overline{\gtrless} \lim_{n=\infty} U_n(x) \overline{\gtrless} 1, \\ 0 \overline{\gtrless} \lim_{n=\infty} V_n(x) \overline{\gtrless} \frac{2}{\pi}. \end{aligned}$$

Let x_n' be the value of x which makes $U_n(x)$ an absolute minimum (the absolute maximum is obviously $n/(n+1)$, for $x = 0$), and x_n'' the value of x making $V_n(x)$ an absolute maximum (the absolute minimum $= 0$ being attained for $x = 0$); it will be shown in § 4 that

$$(14)\qquad \lim_{n=\infty} U_n(x_n') = \frac{1}{2},$$

and in § 5 that

$$(15)\qquad V_n(x_n'') < \lim_{n=\infty} V_n(x_n'') = \sin z_0,$$

where z_0 is the smallest positive root of the equation

$$\tan\frac{z_0}{2} = z_0.$$

On account of $\sin z_0 > 2/\pi$, it is shown by (13), (14) and (15) that $V_n(x)$ presents an analogon to Gibbs' phenomenon, while there is none in the case of $U_n(x)$. Furthermore, $V_n(x_n'')$ increases monotonously with n, and, for $A = 0$, the inequality (9) may be replaced by

$$|s_n(x)| < M + B\sin z_0 < M + \frac{3}{4}B.$$

§ 2. *Demonstration of the relations* (8) *and* (9).

The identity

$$n(c_1^2 + c_2^2 + \cdots + c_n^2) = (c_1 + c_2 + \cdots + c_n)^2 + \frac{1}{2}\sum_{\mu=1}^{n}\sum_{\nu=1}^{n}(c_\mu - c_\nu)^2$$

gives, when $c_1, \cdots, c_n$ are any real quantities,

$$(16)\qquad \frac{c_1 + c_2 + \cdots + c_n}{n} \overline{\gtrless} \sqrt{\frac{c_1^2 + c_2^2 + \cdots + c_n^2}{n}},$$

and making $c_\nu = |\cos \nu x|$ and $c_\nu = |\sin \nu x|$ respectively, we obtain from (7)

$$|s_n(x)| \overline{\gtrless} M + \frac{n}{n+1}\left(A\sqrt{\frac{\cos^2 x + \cos^2 2x + \cdots + \cos^2 nx}{n}} \right.$$
$$\left. + B\sqrt{\frac{\sin^2 x + \sin^2 2x + \cdots + \sin^2 nx}{n}}\right),$$

or, writing

(17) $$y = \sqrt{\frac{\sin^2 x + \sin^2 2x + \cdots + \sin^2 nx}{n}},$$

(18) $$| s_n(x) | \overline{\gtrless} M + \frac{n}{n+1}(A\sqrt{1-y^2} + By).$$

The expression $A\sqrt{1-y^2} + By$ becomes a maximum for

$$-\frac{Ay}{\sqrt{1-y^2}} + B = 0,$$

whence

$$y^2 = \frac{B^2}{A^2+B^2}, \qquad 1 - y^2 = \frac{A^2}{A^2+B^2},$$

$$A\sqrt{1-y^2} + By = \sqrt{A^2+B^2},$$

and by comparison with (18)

(19) $$| s_n(x) | \overline{\gtrless} M + \frac{n}{n+1}\sqrt{A^2+B^2},$$

from which (8) immediately follows. For $A = 0$ or $B = 0$, (8) coincides with Fejér's inequality.

We shall now proceed to prove (9) which, for $B \overline{\gtrless} \sqrt{3}A$, gives a closer limitation than either of the preceding ones. From

$$\sum_{\nu=1}^{n} \sin^2 \nu x = \frac{1}{2}\sum_{\nu=1}^{n}(1 - \cos 2\nu x) = \frac{n}{2} + \frac{1}{4} - \frac{\sin(2n+1)x}{4\sin x},$$

(20) $$y^2 = \frac{1}{2} + \frac{1}{4n} - \frac{\sin(2n+1)x}{4n \sin x}.$$

As we obviously have

$$y^2(\pi + x) = y^2(\pi - x) = y^2(x),$$

we need consider the interval $0 \overline{\gtrless} x \overline{\gtrless} \pi/2$ only. First, making

$$x = \frac{2m\pi + z}{2n+1}, \qquad 0 \overline{\gtrless} m \overline{\gtrless} \left[\frac{n}{2}\right], \qquad 0 \overline{\gtrless} z \overline{\gtrless} \pi,$$

we find

$$y^2 = \frac{1}{2} + \frac{1}{4n} - \frac{\sin z}{4n \sin \dfrac{2m\pi + z}{2n+1}},$$

whence

(21) $$y^2 \overline{\gtrless} \frac{1}{2} + \frac{1}{4n} \text{ for } \frac{2m\pi}{2n+1} \overline{\gtrless} x \overline{\gtrless} \frac{(2m+1)\pi}{2n+1}, \qquad 0 \overline{\gtrless} m \overline{\gtrless} \left[\frac{n}{2}\right].$$

In the second place, making

$$x = \frac{(2m+1)\pi + z}{2n+1}, \qquad 0 \overline{\gtrless} m \overline{\gtrless} \left[\frac{n-1}{2}\right], \qquad 0 \overline{\gtrless} z \overline{\gtrless} \pi,$$

we find

$$y^2 = \frac{1}{2} + \frac{1}{4n} + \frac{\sin z}{4n \sin \frac{(2m+1)+z}{2n+1}}, \tag{22}$$

and to make this expression a maximum, z must be a root of the equation

$$\tan \frac{(2m+1)\pi + z}{2n+1} - \frac{1}{2n+1} \tan z = 0,$$

which obviously has a root in the interval $0 < z \overline{\gtrless} \pi/2$, but none in the interval $\pi/2 < z \overline{\gtrless} \pi$. Making $z = \pi/2 - \zeta$, the equation becomes

$$\tan \zeta - \frac{1}{2n+1} \cot \frac{(2m+1)\pi + \pi/2 - \zeta}{2n+1} = 0,$$

and as $0 < \zeta \overline{\gtrless} \pi/2$,

$$\zeta < \tan \zeta = \frac{1}{2n+1} \cot \frac{(2m+1)\pi + \frac{\pi}{2} - \zeta}{2n+1}$$

$$< \frac{1}{(2m+1)\pi + \pi/2 - \zeta} < \frac{1}{(2m+1)\pi},$$

whence

$$\frac{\pi}{2} - \frac{1}{(2m+1)\pi} < z \overline{\gtrless} \frac{\pi}{2}. \tag{23}$$

Now we first suppose $2m + 1 = n$; the equation determining ζ becomes

$$\tan \zeta - \frac{1}{2n+1} \tan \frac{\zeta}{2n+1} = 0$$

or

$$\zeta \tan \zeta - \frac{\zeta}{2n+1} \tan \frac{\zeta}{2n+1} = 0,$$

and as $x \tan x$ is a monotonously increasing function for $0 \overline{\gtrless} x \overline{\gtrless} \pi/2$, the above equation is satisfied only by $\zeta = 0$, whence $z = \pi/2$,

$$\frac{(2m+1)\pi + z}{2n+1} = \frac{\pi}{2}$$

and from (22)

$$y^2 = \frac{1}{2} + \frac{1}{4n} + \frac{1}{4n}$$

$$= \frac{1}{2} + \frac{1}{4n} + \frac{2n+1}{4n} \cdot \frac{1}{4m+3}. \tag{24}$$

In the remaining case, we have $1 \overline{\gtrless} 2m + 1 \overline{\gtrless} n - 1$, and for all integral values of m satisfying this inequality, the expression $(2m+1)(n - 2m - 1)\pi$

attains its minimum value $= (n-1)\pi$ for $2m+1 = 1$ and $2m+1 = n-1$; as we have

$$(n-1)\pi > \frac{2n}{\pi}$$

for $n \geqq 2$, we find

$$(2m+1)(n-2m-1)\pi > \frac{2n}{\pi},$$

or

$$\frac{\pi}{2} - \frac{1}{(2m+1)\pi} > \frac{2m+1}{2n}\pi.$$

By the aid of (23), this gives

$$z > \frac{2m+1}{2n}\pi$$

or

$$z > \frac{(2m+1)\pi + z}{2n+1},$$

whence, the expression $\sin z / z$ decreasing monotonously for z increasing from 0 to π,

$$\frac{\sin z}{z} < \frac{\sin\dfrac{(2m+1)\pi + z}{2n+1}}{\dfrac{(2m+1)\pi + z}{2n+1}}.$$

This inequality compared with (22) gives

$$y^2 < \frac{1}{2} + \frac{1}{4n} + \frac{2n+1}{4n} \cdot \frac{z}{(2m+1)\pi + z},$$

and as, for z limited by (23), the last fraction attains its maximum for $z = \pi/2$,

$$y^2 < \frac{1}{2} + \frac{1}{4n} + \frac{2n+1}{4n} \cdot \frac{1}{4m+3} \quad \text{for} \quad \frac{(2m+1)\pi}{2n+1} \leqq x \leqq \frac{(2m+2)\pi}{2n+1} \quad (n \geqq 2).$$

The comparison of this result with (21) and (24) finally gives

$$y^2 \leqq \frac{2n+1}{4n}\left(1 + \frac{1}{4m+3}\right) \tag{25}$$

for

$$\frac{2m\pi}{2n+1} \leqq x \leqq \frac{(2m+2)\pi}{2n+1}, \qquad 0 \leqq m \leqq \left[\frac{n}{2}\right] \qquad (n \geqq 2),$$

whence

$$y^2 \leqq \frac{2n+1}{4n}\left(1 + \frac{1}{3}\right) \leqq \frac{3}{4} \quad \text{for} \quad n \geqq 4.$$

Now, for $B \geqq \sqrt{3}A$, the expression $A\sqrt{1-y^2} + By$ attains its maximum in the interval $0 \overline{\overline{<}} y^2 \overline{\overline{<}} \frac{3}{4}$ for $y^2 = \frac{3}{4}$, and (18) gives

$$| s_n(x) | \overline{\overline{<}} M + \frac{n}{n+1}\left(\frac{1}{2}A + \frac{\sqrt{3}}{2}B\right) \qquad (n \overline{\overline{>}} 4),$$

whence (9) is proved for $n \overline{\overline{>}} 4$; for $n = 1, 2, 3$ it is easily verified by numerical calculation.

§ 3. *Determination of* $\lim_{n=\infty} U_n(x)$ *and* $\lim_{n=\infty} V_n(x)$.

In order to obtain these limits, we have to consider three cases: 1°, $x = 0$; 2°, $x : \pi$ rational and $\neq 0$; 3°, $x : \pi$ irrational.

In the first case, we obviously have

$$U_n(0) = \frac{n}{n+1}, \qquad V_n(0) = 0,$$

whence

$$\lim_{n=\infty} U_n(0) = 1, \qquad \lim_{n=\infty} V_n(0) = 0.$$

In the second case, we have $x = \pi p/q$, where p and q are relative primes, and we may obviously suppose $q > 0$. As $|\cos \nu x|$ has the period π, we obtain

$$\sum_{\nu=1}^{n} \left|\cos \frac{\nu p}{q}\pi\right| = \sum_{\nu=1}^{q} + \sum_{\nu=q+1}^{2q} + \cdots + \sum_{\nu=[n/q]q-q+1}^{[n/q]q} + \sum_{\nu=[n/q]q+1}^{n}$$

$$= \left[\frac{n}{q}\right]\sum_{\nu=1}^{q}\left|\cos\frac{\nu p}{q}\pi\right| + \sum_{\nu=[n/q]q+1}^{n}\left|\cos\frac{\nu p}{q}\pi\right|.$$

We further have $[n/q] = (n/q) - \theta$, $0 \overline{\overline{<}} \theta < 1$, and

$$\sum_{\nu=[n/q]q+1}^{n}\left|\cos\frac{\nu p}{q}\pi\right| \overline{\overline{<}} \sum_{\nu=[n/q]q+1}^{n} 1 = n - \left[\frac{n}{q}\right]q = \theta q,$$

whence

$$\lim_{n=\infty} U_n\left(\frac{p}{q}\pi\right) = \frac{1}{q}\sum_{\nu=1}^{q}\left|\cos\frac{\nu p}{q}\pi\right|.$$

Now the numbers $p, 2p, \cdots, qp$ have the same residues modulo q (in a different order) as the numbers $1, 2, \cdots, q$, so that

$$\lim_{n=\infty} U_n\left(\frac{p}{q}\pi\right) = \frac{1}{q}\sum_{\nu=1}^{q}\left|\cos\frac{\nu p}{q}\pi\right| = \frac{1}{q}\sum_{\nu=1}^{q}\left|\cos\frac{\nu\pi}{q}\right|$$

$$= \frac{1}{q}\sum_{\nu=1}^{[q/2]}\cos\frac{\nu\pi}{q} - \frac{1}{q}\sum_{\nu=[q/2]+1}^{q}\cos\frac{\nu\pi}{q}$$

$$= \frac{1}{2q\sin\frac{\pi}{2q}}\left(\sin\frac{\left[\frac{q}{2}\right]+\frac{1}{2}}{q}\pi - \sin\frac{\pi}{2q}\right)$$

$$- \frac{1}{2q\sin\frac{\pi}{2q}}\left(\sin\frac{q+\frac{1}{2}}{q}\pi - \sin\frac{\left[\frac{q}{2}\right]+\frac{1}{2}}{q}\pi\right)$$

$$= \frac{\sin\frac{\left[\frac{q}{2}\right]+\frac{1}{2}}{q}\pi}{q\sin\frac{\pi}{2q}}$$

$$= \begin{cases} \dfrac{1}{q}\cot\dfrac{\pi}{2q}, & q \text{ even}, \\ \dfrac{1}{q\sin\frac{\pi}{2q}}, & q \text{ odd}, \end{cases}$$

and the last two expressions are obviously contained in

$$\frac{\cos^2\frac{\pi}{4q} + (-1)^{q+1}\sin^2\frac{\pi}{4q}}{q\sin\frac{\pi}{2q}} = \frac{1}{2q}\left(\cot\frac{\pi}{4q} + (-1)^{q+1}\tan\frac{\pi}{4q}\right).$$

In exactly the same way we find

$$\sum_{\nu=1}^{n}\left|\sin\frac{\nu p}{q}\pi\right| = \sum_{\nu=1}^{q} + \sum_{\nu=q+1}^{2q} + \cdots + \sum_{\nu=[n/q]q-q+1}^{[n/q]q} + \sum_{\nu=[n/q]q+1}^{n}$$

$$= \left[\frac{n}{q}\right]\sum_{\nu=1}^{q}\left|\sin\frac{\nu p}{q}\pi\right| + \sum_{\nu=[n/q]q+1}^{n}\left|\sin\frac{\nu p}{q}\pi\right|,$$

$$\lim_{n=\infty} V_n\left(\frac{p}{q}\pi\right) = \frac{1}{q}\sum_{\nu=1}^{q}\left|\sin\frac{\nu p}{q}\pi\right| = \frac{1}{q}\sum_{\nu=1}^{q}\left|\sin\frac{\nu\pi}{q}\right| = \frac{1}{q}\sum_{\nu=1}^{q}\sin\frac{\nu\pi}{q} = \frac{1}{q}\cot\frac{\pi}{2q}.$$

We finally consider the third case, where $x:\pi$ is irrational. In order to evaluate our expressions, we begin by developing the even function $|\cos x|$, which obviously satisfies Dirichlet's conditions for $0 \overline{\overline{<}} x \overline{\overline{<}} 2\pi$, in a Fourier series:

$$|\cos x| = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} a_\nu \cos\nu x,$$

where

$$a_\nu = \frac{1}{\pi}\int_0^{2\pi} |\cos x| \cos \nu x dx = \frac{1+(-1)^\nu}{\pi}\int_{-\pi/2}^{\pi/2} \cos x \cos \nu x dx,$$

whence, performing the integration,

$$(26) \qquad |\cos x| = \frac{2}{\pi} + \frac{2}{\pi}\sum_{\nu=1}^{\infty}(-1)^\nu\left(\frac{1}{2\nu+1} - \frac{1}{2\nu-1}\right)\cos 2\nu x,$$

and on account of the periodicity of both members, this formula is valid for all real values of x. Consequently, we have

$$(27) \quad |\cos x| = \frac{2}{\pi} + \frac{2}{\pi}\sum_{\nu=1}^{N}(-1)^\nu\left(\frac{1}{2\nu+1} - \frac{1}{2\nu-1}\right)\cos 2\nu x + R_N,$$

where

$$(28) \qquad |R_N| \overline{\gtrless} \frac{\pi}{2}\sum_{\nu=N+1}^{\infty}\left(\frac{1}{2\nu-1} - \frac{1}{2\nu+1}\right) = \frac{2}{\pi}\cdot\frac{1}{2N+1}.$$

By (27), we obtain

$$(29) \quad U_n(x) = \frac{2}{\pi}\cdot\frac{n}{n+1} + \frac{2}{\pi}\sum_{\nu=1}^{N}(-1)^\nu\left(\frac{1}{2\nu+1} - \frac{1}{2\nu-1}\right)\cdot\frac{1}{n+1}\sum_{\lambda=1}^{n}\cos 2\lambda\nu x + \epsilon_N,$$

where, on account of (28),

$$(30) \quad \begin{aligned} |\epsilon_N| &\overline{\gtrless} \frac{1}{n+1}\Big(|R_N(x)| + |R_N(2x)| + \cdots + |R_N(nx)|\Big) \\ &\overline{\gtrless} \frac{n}{n+1}\cdot\frac{2}{\pi}\cdot\frac{1}{2N+1} < \frac{2}{\pi}\cdot\frac{1}{2N+1}. \end{aligned}$$

Now, $x:\pi$ being irrational, none of the expressions $\sin \nu x$ ($\nu = 1, 2, \cdots, N$) is equal to zero, so that

$$\lim_{n=\infty}\frac{1}{n+1}\sum_{\lambda=1}^{n}\cos 2\lambda\nu x = \lim_{n=\infty}\frac{1}{n+1}\,\frac{\sin n\nu x \cos(n+1)\nu x}{\sin \nu x} = 0,$$

and (29) and (30) consequently give

$$\limsup_{n=\infty} U_n(x) \overline{\gtrless} \frac{2}{\pi} + \frac{2}{\pi}\cdot\frac{1}{2N+1},$$

$$\liminf_{n=\infty} U_n(x) \overline{\gtrless} \frac{2}{\pi} - \frac{2}{\pi}\cdot\frac{1}{2N+1}.$$

As the left hand members are independent of N, we finally obtain, by increasing N indefinitely,

$$\lim_{n=\infty} U_n(x) = \frac{2}{\pi}.$$

In equation (27), we change x into $x + \pi/2$, whence

$$|\sin x| = \frac{2}{\pi} + \frac{2}{\pi}\sum_{\nu=1}^{N}\left(\frac{1}{2\nu+1} - \frac{1}{2\nu-1}\right)\cos 2\nu x + R'_N,$$

$$|R'_N| < \frac{2}{\pi}\cdot\frac{1}{2N+1},$$

and from these formulæ we obtain, in the same way as before,

$$\lim_{n=\infty} V_n(x) = \frac{2}{\pi}.$$

§ 4. *On the range of variation of* $U_n(x)$.

On account of the obvious relations

$$U_n(x+\pi) = U_n(\pi - x) = U_n(x),$$

it is sufficient, in studying the range of variation of $U_n(x)$, to consider the interval

$$0 \leqq x \leqq \frac{2}{\pi}.$$

Let x'_n be the value of x in this interval which makes $U_n(x)$ an absolute minimum; we shall now proceed to show that

$$\lim_{n=\infty} U_n(x'_n) = \frac{1}{2}.$$

To this purpose, we begin by determining, for each positive integral n, integers $m(n)$, $l_0(n)$, $l_1(n)$, $\cdots$, $l_\lambda(n)$ $\cdots$ such that

(31)
$$\frac{2m\pi}{2n+1} \leqq x'_n < \frac{2(m+1)\pi}{2n+1},$$

and

$$l_0 = 0,$$

$$l_1 x'_n \leqq \frac{\pi}{2} < (l_1 + 1)x'_n,$$

(32,
$$l_2 x'_n \leqq \frac{3\pi}{2} < (l_2 + 1)x'_n,$$

. . . .

$$l_\lambda x'_n \leqq (2\lambda - 1)\frac{\pi}{2} < (l_\lambda + 1)x'_n,$$

. . . .

and also an index $\mu(n)$ such that

$$l_\mu < n \overline{\gtrless} l_{\mu+1}. \tag{33}$$

We then have

$$|\cos \nu x_n'| = (-1)^\lambda \cos \nu x_n' \quad \text{for} \quad l_\lambda + 1 \overline{\gtrless} \nu \overline{\gtrless} l_{\lambda+1},$$

and consequently

$$U_n(x_n') = \frac{1}{n+1}\sum_{\nu=1}^{n}|\cos \nu x_n'| = \frac{1}{n+1}\sum_{\lambda=0}^{\mu-1}(-1)^\lambda \sum_{\nu=l_\lambda+1}^{l_\lambda+1} \cos \nu x_n' + \frac{(-1)^\mu}{n+1}\sum_{\nu=l_\mu+1}^{n} \cos \nu x_n'$$

$$= \frac{\sum_{\lambda=0}^{\mu-1}(-1)^\lambda\left(\sin\frac{2l_{\lambda+1}+1}{2}x_n' - \sin\frac{2l_\lambda+1}{2}x_n'\right) + (-1)^\mu\left(\sin\frac{2n+1}{2}x_n' - \sin\frac{2l_\mu+1}{2}x_n'\right)}{2(n+1)\sin\frac{x_n'}{2}} \tag{34}$$

$$= \frac{-\sin\frac{x_n'}{2} + 2\sum_{\lambda=0}^{\mu-1}(-1)^\lambda \sin\frac{2l_{\lambda+1}+1}{2}x_n' + (-1)^\mu \sin\frac{2n+1}{2}x_n'}{2(n+1)\sin\frac{x_n'}{2}}.$$

To find the inferior limit of the expression (34) for $n = \infty$, it will be convenient to distinguish two cases:

Case 1: From the set of integers $1, 2, 3, \cdots$ we may extract a set

$$n_1^{(1)},\ n_2^{(1)},\ n_3^{(1)},\ \cdots$$

where

$$n_{\nu+1}^{(1)} > n_\nu^{(1)}, \qquad \lim_{\nu=\infty} n_\nu^{(1)} = \infty,$$

and when n assumes these values, x_n' is such that in (31) we have

$$m < n^{\frac{1}{3}}.$$

Case 2: From the set of integers $1, 2, 3, \cdots$ we may extract a set

$$n_1^{(2)},\ n_2^{(2)},\ n_3^{(2)},\ \cdots,$$

where

$$n_{\nu+1}^{(2)} > n_\nu^{(2)}, \qquad \lim_{\nu=\infty} n_\nu^{(2)} = \infty,$$

and when n assumes these values, x_n' is such that in (31) we have

$$m \overline{\gtrless} n^{\frac{1}{3}}.$$

Case 1: n assumes the value in the set $n_1^{(1)}, n_2^{(1)}, n_3^{(1)}, \cdots$, or briefly, the set $\{1\}$, and for all these values of n we have

$$m < n^{\frac{1}{3}}. \tag{35}$$

We shall first develop some consequences of (31), (32), (33) and (35). By the last inequality (32), we have

$$\frac{2\lambda - 1}{l_\lambda + 1}\frac{\pi}{2} < x_n' \leqq \frac{2\lambda - 1}{l_\lambda}\frac{\pi}{2}, \tag{36}$$

which may also be written

$$-\frac{2\lambda - 1}{l_\lambda + 1}\frac{\pi}{4} < \frac{2l_\lambda + 1}{2}x_n' - (2\lambda - 1)\frac{\pi}{2} \leqq \frac{2\lambda - 1}{l_\lambda}\frac{\pi}{4}. \tag{37}$$

From (31) and (36) we find

$$\frac{2\lambda - 1}{l_\lambda + 1} < \frac{4(m + 1)}{2n + 1}; \qquad \frac{2\lambda - 1}{l_\lambda} \geqq \frac{4m}{2n + 1}, \tag{38}$$

and from (37) and (38)

$$\left|\frac{2l_\lambda + 1}{2}x_n' - (2\lambda - 1)\frac{\pi}{2}\right| \leqq \frac{2\lambda - 1}{l_\lambda}\frac{\pi}{4} < \frac{\pi(m + 1)}{2n + 1}\frac{l_\lambda + 1}{l_\lambda} \leqq \frac{2\pi(m + 1)}{2n + 1}. \tag{39}$$

From the first inequality (38) we obtain, making $\lambda = \mu$ and using (33),

$$2\mu - 1 < \frac{4(l_\mu + 1)(m + 1)}{2n + 1} \leqq \frac{4n}{2n + 1}(m + 1) < 2(m + 1),$$

$$2\mu - 1 \leqq 2m + 1,$$

$$\mu \leqq m + 1,$$

and from the second inequality (38), making $\lambda = \mu + 1$ and using (33),

$$2\mu + 1 \geqq \frac{4l_{\mu+1}m}{2n + 1} \geqq \frac{4nm}{2n + 1} = 2m\left(1 - \frac{1}{2n + 1}\right)$$

or on account of (35), for n sufficiently large

$$2\mu + 1 \geqq 2m,$$

whence

$$2\mu + 1 \geqq 2m + 1,$$

$$\mu \geqq m,$$

or finally, combining the two inequalities obtained,

$$m \leqq \mu \leqq m+1. \tag{40}$$

Furthermore, (32) and (33) give

$$(2\mu-1)\frac{\pi}{2} < (l_\mu + 1)\, x_n' \leqq n x_n' \leqq \frac{n}{l_1}\frac{\pi}{2},$$

and

$$(2\mu+1)\frac{\pi}{2} \geqq l_{\mu+1}\, x_n' \geqq n x_n' > \frac{n}{l_1+1}\frac{\pi}{2} \geqq \frac{n}{2l_1}\frac{\pi}{2},$$

whence

$$\frac{1}{4}\left(\frac{n}{l_1} - 2\right) < \mu < \frac{1}{2}\left(\frac{n}{l_1} + 1\right). \tag{41}$$

We now have

$$\sum_{\lambda=0}^{\mu-1} (-1)^\lambda \sin \frac{2l_{\lambda+1}+1}{2} x_n' - \sum_{\lambda=0}^{\mu-1} (-1)^\lambda \sin (2\lambda+1)\frac{\pi}{2}$$

$$= \sum_{\lambda=0}^{\mu-1} (-1)^\lambda \left(\frac{2l_{\lambda+1}+1}{2} x_n' - (2\lambda+1)\frac{\pi}{2} \right) \cos \left[(2\lambda+1)\frac{\pi}{2} \right.$$

$$\left. + \theta_\lambda \left(\frac{2l_{\lambda+1}+1}{2} x_n' - (2\lambda+1)\frac{\pi}{2} \right) \right], \qquad (0 < \theta_\lambda < 1),$$

whence, by (39), (40) and (35),

$$\left| \sum_{\lambda=0}^{\mu-1} (-1)^\lambda \sin \frac{2l_{\lambda+1}+1}{2} x_n' - \mu \right|$$

$$\begin{aligned} &= \left| \sum_{\lambda=0}^{\mu-1} (-1)^\lambda \sin \frac{2l_{\lambda+1}+1}{2} x_n' - \sum_{\lambda=0}^{\mu-1} (-1)^\lambda \sin (2\lambda+1)\frac{\pi}{2} \right| \\ &\leqq \sum_{\lambda=0}^{\mu-1} \left| \frac{2l_{\lambda+1}+1}{2} x_n' - (2\lambda+1)\frac{\pi}{2} \right| \leqq \sum_{\lambda=0}^{\mu-1} \frac{2\pi(m+1)}{2n+1} = \frac{2\pi\mu(m+1)}{2n+1} \\ &\leqq \frac{2\pi(m+1)^2}{2n+1} < \frac{8\pi m^2}{2n} < \frac{4\pi}{n^{\frac{1}{3}}}. \end{aligned} \tag{42}$$

Now first suppose that from the set $\{1\}$ we may detach a set $\{3\}$:

$$n_1^{(3)},\ n_2^{(3)},\ n_3^{(3)},\ \cdots;\ n_{\nu+1}^{(3)} > n_\nu^{(3)},\ \lim_{\nu=\infty} n_\nu^{(3)} = \infty,$$

such that, when n assumes the values of the set $\{3\}$,

$$\lim_{\{3\}} \frac{n}{l_1} = \infty \tag{43}$$

We have, by (34) and (42),

$$U_n(x_n') > \frac{2\left(\mu - \frac{4\pi}{n^{\frac{1}{3}}}\right) + (-1)^{\mu} \sin \frac{2n+1}{2} x_n'}{2(n+1)\sin\frac{x_n'}{2}} - \frac{1}{2(n+1)}$$

$$\overline{\gtrless} \frac{2\left(\mu - \frac{4\pi}{n^{\frac{1}{3}}}\right) - 1}{2(n+1)\frac{x_n'}{2}} - \frac{1}{2(n+1)},$$

and by (40) and (31),

$$U_n(x_n') > \frac{2\left(m - \frac{4\pi}{n^{\frac{1}{3}}}\right) - 1}{\frac{2(n+1)(m+1)\pi}{2n+1}} - \frac{1}{2(n+1)}.$$

On account of (40) and (41), it follows from (43) that

$$\lim_{\{3\}} m = \infty,$$

and consequently the preceding inequality gives

$$\liminf_{\{3\}} U_n(x_n') \overline{\gtrless} \frac{2}{\pi} > \frac{1}{2}. \tag{44}$$

In the second place, suppose that from the set $\{1\}$ we may detach a set $\{4\}$

$$n_1^{(4)},\ n_2^{(4)},\ n_3^{(4)},\ \cdots;\ n_{\nu+1}^{(4)} > n_\nu^{(4)},\ \lim_{\nu=\infty} n_\nu^{(4)} = \infty,$$

such that, when n assumes the values of the set $\{4\}$,

$$\lim_{\{4\}} \frac{n}{l_1} = \alpha, \tag{45}$$

where α is finite. The inequalities (41) then show that for the set $\{4\}$, μ is enclosed between finite limits, and consequently we may detach from $\{4\}$ a set $\{5\}$

$$n_1^{(5)},\ n_2^{(5)},\ n_3^{(5)},\ \cdots;\ n_{\nu+1}^{(5)} > n_\nu^{(5)},\ \lim_{\nu-\infty} n_\nu^{(5)} = \infty,$$

such that

$$\lim_{\{5\}} \frac{n}{l_1} = \lim_{\{4\}} \frac{n}{l_1} = \alpha, \qquad \lim_{\{5\}} \mu = \mu_0, \tag{46}$$

where μ_0 is finite . The inequalities from which we derived (41) were

$$(2\mu - 1)\frac{\pi}{2} \overline{\lessgtr} \frac{n}{l_1}\frac{\pi}{2}, \qquad (2\mu + 1)\frac{\pi}{2} > \frac{n}{l_1+1}\frac{\pi}{2};$$

as (41) gives

$$l_1 > \frac{n}{4\mu + 2},$$

we have for n sufficiently large in the set $\{5\}$

$$l_1 > \frac{n}{8\mu_0 + 4}, \tag{47}$$

and consequently the preceding inequalities give, when combined with (46),

$$2\mu_0 - 1 \overline{\gtrless} \alpha \overline{\gtrless} 2\mu_0 + 1. \tag{48}$$

From (32) we obtain

$$(2n+1)\left[\frac{\pi}{4(l_1+1)} - \frac{\pi^3}{48.8l^3}\right] < (2n+1)\left[\frac{x_n'}{2} - \frac{x_n'^3}{48}\right] < (2n+1)\sin\frac{x_n'}{2}$$
$$< (2n+1)\frac{x_n'}{2} < (2n+1)\frac{\pi}{4l_1},$$

whence by (47)

$$\lim_{\{5\}} (2n+1)\sin\frac{x_n'}{2} = \frac{\alpha\pi}{2},$$

and in the same way we find

$$\lim_{\{5\}} \sin (2n+1)\frac{x_n'}{2} = \sin\frac{\alpha\pi}{2},$$

so that, by (34), (42) and (46)

$$\lim_{\{5\}} U_n(x_n') = \frac{2\mu_0 + (-1)^{\mu_0}\sin\alpha\pi/2}{\alpha\pi/2},$$

or, making

$$\frac{\alpha\pi}{2} = \mu_0\pi + z,$$

where, according to (48), $-\pi/2 \overline{\gtrless} z \overline{\gtrless} \pi/2$,

$$\lim_{\{5\}} U_n(x_n') = \frac{2\mu_0 + \sin z}{\mu_0\pi + z}.$$

For $z = \pm\pi/2$, the right hand member is equal to $2/\pi$; it therefore becomes a minimum for $z = z_0$, where $-(\pi/2) < z_0 < (\pi/2)$, z_0, being a root of the equation

$$\varphi(z) = (\mu_0\pi + z)\cos z - (2\mu_0 + \sin z) = 0,$$

and the minimum is equal to

$$\frac{2\mu_0 + \sin z_0}{\mu_0\pi + z_0} = \cos z_0.$$

For $\mu_0 = 0$, we have

$$\cos z_0 = \frac{\sin z_0}{z_0} \overline{>} \frac{\sin(\pi/2)}{(\pi/2)} = \frac{2}{\pi},$$

and for $\mu_0 \overline{>} 1$, it is seen at once that

$$\varphi(-\pi/2) < 0, \qquad \varphi(0) > 0, \qquad \varphi(\pi/2) < 0$$

and

$$\frac{d\varphi}{dz} = -(\mu_0\pi + z)\sin z \begin{matrix} > 0 \text{ for } -\pi/2 < z < \\ < 0 \text{ for } 0 < z < \pi/2, \end{matrix}$$

so that $\varphi(z) = 0$ has one and only one root in each of the intervals $-\pi/2 < z < 0$ and $0 < z < \pi/2$. We further have, for $\mu_0 \overline{\gtrless} 1$,

$$\varphi\left(-\frac{\pi}{3}\right) = \left(\mu_0\,\pi - \frac{\pi}{3}\right)\cdot\frac{1}{2} - 2\mu_0 + \frac{\sqrt{3}}{2} < 0,$$

$$\varphi\left(\frac{\pi}{3}\right) = \left(\mu_0\,\pi + \frac{\pi}{3}\right)\cdot\frac{1}{2} - 2\mu_0 - \frac{\sqrt{3}}{2} < 0,$$

so that $|z_0| < \pi/3$ and $\cos z_0 > \frac{1}{2}$. Consequently, we have, whatever may be the values of α and μ_0,

$$\lim_{\{5\}} U_n(x_n') > \frac{1}{2},$$

and combining this result with (44), we obtain

$$\liminf_{\{1\}} U_n(x_n') > \frac{1}{2}. \tag{49}$$

Case 2: n assumes the values in the set $\{2\}$, and for all these values of n we have

$$m \overline{\gtrless} n^{\frac{1}{3}}. \tag{50}$$

Then, as $|\cos \nu x| \overline{\gtrless} \cos^2 \nu x$, we have

$$U_n(x_n') \overline{\gtrless} \frac{\cos^2 x + \cos^2 2x + \cdots + \cos^2 nx}{n+1} = \frac{n}{n+1}(1 - y^2),$$

where y is given by (17), and from (25) we obtain

$$U_n(x_n') \overline{\gtrless} \frac{n}{n+1}\left(\frac{1}{2} - \frac{1}{4n} - \frac{2n+1}{4n}\cdot\frac{1}{4m+3}\right),$$

whence, by (50),

$$\liminf_{\{2\}} U_n(x_n') \overline{\gtrless} \frac{1}{2}. \tag{51}$$

Now (11) gives

$$\lim_{n=\infty} U_n\left(\frac{\pi}{2}\right) = \frac{1}{2},$$

so that, by (49) and (51), we have

$$\lim_{n=\infty} U_n(x_n') = \frac{1}{2}.$$

On the other hand, $U_n(x)$ obviously becomes a maximum $= n/(n+1)$ for $x = 0$; thus the limiting values, for $n = \infty$, of the maxima and minima of $U_n(x)$ are included in the same range (from $\frac{1}{2}$ to 1) as are the limit values of $U_n(x)$ for x independent of n, or in other words, $U_n(x)$ presents no analogon to Gibbs' phenomenon.

§ 5. *On the range of variation of $V_n(x)$ and an analogon to Gibbs' phenomenon.*

As we obviously have

$$V_n(x+\pi) = V_n(\pi - x) = V_n(x),$$

it is sufficient to consider the interval

$$0 \overline{\overline{<}} x \overline{\overline{<}} \frac{\pi}{2}.$$

We shall first prove that in the interval $0 \overline{\overline{<}} x < \pi/n$, $V_n(x)$ has a certain maximum, which increases monotonously with n towards the limit $\sin z_0$ for $n = \infty$, where z_0 is the smallest positive root of the equation

$$z_0 = \tan \tfrac{1}{2} z_0.$$

Furthermore it will be shown that the maximum in question is the absolute maximum of $V_n(x)$.

First supposing $0 \overline{\overline{<}} x < \pi/n$, we have

$$(52)\qquad \begin{aligned} V_n(x) &= \frac{|\sin x| + |\sin 2x| + \cdots + |\sin nx|}{n+1} \\ &= \frac{\sin x + \sin 2x + \cdots + \sin nx}{n+1} \\ &= \frac{1}{2(n+1)}\left(\cot\frac{x}{2} - \frac{\cos\frac{2n+1}{2}x}{\sin\frac{x}{2}}\right), \end{aligned}$$

whence

$$\begin{aligned} \frac{dV_n(x)}{dx} &= \frac{-1 + (2n+1)\sin\frac{2n+1}{2}x\sin\frac{x}{2} + \cos\frac{2n+1}{2}x\cos\frac{x}{2}}{4(n+1)\sin^2\frac{x}{2}} \\ &= \frac{-1 + (n+1)\cos nx - n\cos(n+1)x}{4(n+1)\sin^2\frac{x}{2}} \\ &= \frac{n\sin^2\frac{n+1}{2}x - (n+1)\sin^2\frac{n}{2}x}{2(n+1)\sin^2\frac{x}{2}}. \end{aligned}$$

If x_n'' is the value of x, in the interval considered, which makes $V_n(x)$ a maximum, we therefore have

$$\frac{\sin\frac{n+1}{2}x_n''}{\sin\frac{n}{2}x_n''} = \sqrt{\frac{n+1}{n}},$$

where the square root is positive, as

$$\frac{n}{2}x_n'' < \frac{n+1}{2}x_n'' \leqq \frac{n+1}{2n}\pi \leqq \pi$$

for $n \gtreqless 1$. From the above equation we obtain

$$\frac{\sin\frac{n+1}{2}x_n'' + \sin\frac{n}{2}x_n''}{\sin\frac{n+1}{2}x_n'' - \sin\frac{n}{2}x_n''} = \frac{\sqrt{1+\frac{1}{n}}+1}{\sqrt{1+\frac{1}{n}}-1},$$

or

$$\tan\frac{2n+1}{4}x_n'' = \frac{\sqrt{1+\frac{1}{n}}+1}{\sqrt{1+\frac{1}{n}}-1}\tan\frac{x_n''}{4}.$$

The right hand member being positive, it follows that

$$\frac{2n+1}{4}x_n'' < \frac{\pi}{2}.$$

Introducing the notations

$$(53)\qquad \frac{2n+1}{2}x_n'' = z < \pi, \qquad \sqrt{1+\frac{1}{n}} - 1 = h, \qquad \omega = \frac{h(2+h)}{2(h^2+2h+2)},$$

whence

$$\frac{x_n''}{4} = \omega z < \frac{\pi}{4}, \qquad n = \frac{1}{h(2+h)},$$

the above equation becomes

$$(54)\qquad \tan\frac{z}{2} = \frac{h+2}{h}\tan\omega z \qquad (z < \pi),$$

and from (52) we obtain

$$V_n(x_n'') = \frac{1}{2(n+1)}\left(\frac{1-\tan^2\frac{x_n''}{4}}{2\tan\frac{x_n''}{4}} - \frac{1+\tan^2\frac{x_n''}{4}}{2\tan\frac{x_n''}{4}}\cdot\frac{1-\tan^2\frac{2n+1}{4}x_n''}{1+\tan^2\frac{2n+1}{4}x_n''}\right)$$

$$= \frac{1}{\frac{2}{h(2+h)}+2}\left(\frac{1-\tan^2\omega z}{2\tan\omega z} - \frac{1+\tan^2\omega z}{2\tan\omega z}\cdot\frac{1-\tan^2\frac{z}{2}}{1+\tan^2\frac{z}{2}}\right)$$

$$= \frac{1}{\frac{2}{h(2+h)}+2}\left(\frac{1-\frac{h^2}{(2+h)^2}\tan^2\frac{z}{2}}{\frac{2h}{2+h}\tan\frac{z}{2}} - \frac{1+\frac{h^2}{(2+h)^2}\tan^2\frac{z}{2}}{\frac{2h}{2+h}\tan\frac{z}{2}}\cdot\frac{1-\tan^2\frac{z}{2}}{1+\tan^2\frac{z}{2}}\right)$$

$$= \frac{1}{\frac{2}{h(2+h)}+2}\cdot\frac{1-\frac{h^2}{(2+h)^2}}{\frac{2h}{2+h}}\cdot\frac{2\tan\frac{z}{2}}{1+\tan^2\frac{z}{2}},$$

or finally

$$V_n(x_n'') = \frac{\sin z}{1+h}. \tag{55}$$

In order to prove that this expression increases monotonously with n, we differentiate and obtain

$$\frac{dV_n(x_n'')}{dh} = \frac{\cos z}{1+h}\frac{dz}{dh} - \frac{\sin z}{(1+h)^2},$$

or

$$\frac{dV_n(x_n'')}{dh} = \frac{2}{(1+h)\left(1+\tan^2\frac{z}{2}\right)}\left[\frac{1}{2}\left(1-\tan^2\frac{z}{2}\right)\frac{dz}{dh} - \frac{\tan\frac{z}{2}}{1+h}\right]. \tag{56}$$

Now (54) gives by differentiation

$$\left[\frac{1}{2}+\frac{1}{2}\tan^2\frac{z}{2} - \frac{2+h}{h}\,\omega\,(1+\tan^2\omega z)\right]\frac{dz}{dh}$$

$$= -\frac{2}{h^2}\tan\omega z + \frac{2+h}{h}(1+\tan^2\omega z)\,z\,\frac{d\omega}{dh},$$

or on substituting the value of $\tan \omega z$ given by (54) in the first member,

$$\left[\frac{1}{2}+\frac{1}{2}\tan^2\frac{z}{2}-\frac{2+h}{h}\omega-\frac{h}{2+h}\omega\tan^2\frac{z}{2}\right]\frac{dz}{dh}$$

$$=-\frac{2}{h^2}\tan\omega z+\frac{2+h}{h}(1+\tan^2\omega z)z\frac{d\omega}{dh},$$

or finally, introducing the value of ω given by (53) in the left hand member and in the expression $d\omega/dh$, and then making the necessary algebraic reductions,

$$\frac{1+h}{h^2+2h+2}\left(\tan^2\frac{z}{2}-1\right)\frac{dz}{dh}=-\frac{2}{h^2}\tan\omega z+\frac{2(1+h)(2+h)}{h(h^2+2h+2)^2}z(1+\tan^2\omega z),$$

whence

$$\frac{1}{2}\left(1-\tan^2\frac{z}{2}\right)\frac{dz}{dh}-\frac{\tan\frac{z}{2}}{1+h}=\left(\frac{h^2+2h+2}{h^2(1+h)}-\frac{2+h}{h(1+h)}\right)\tan\omega z-\frac{2\omega}{h^2}z(1+\tan^2\omega z)$$

$$=\frac{2}{h^2}\left[\frac{\tan\omega z}{1+h}-\omega z(1+\tan^2\omega z)\right]$$

$$=\frac{2\cos\omega z\sin\omega z-2(1+h)\omega z}{h^2(1+h)\cos^2\omega z}$$

$$=\frac{\sin 2\omega z-2\omega z(1+h)}{h^2(1+h)\cos^2\omega z}<0.$$

By comparison with (56) it then follows that

$$\frac{dV_n(x_n'')}{dh}<0,$$

so that the maximum of $V_n(x)$ for $0 \leqq nx < \pi$ increases monotonously with h decreasing, that is, with n increasing. We now have to find the limit of this maximum for $n=\infty$. To this purpose, let z_0 be one of the limit values toward which the root $z<\pi$ of (54) tends for h going towards zero by the values obtained by making $n=1, 2, 3, \cdots$ in (53). Extracting from this set of values an infinitely decreasing set such that for h passing through the latter values, z tends to the limit z_0, we have

$$\lim\tan\frac{z}{2}=\tan\frac{z_0}{2},\qquad \lim\frac{h+2}{h}\tan\omega z=z_0,$$

so that z_0 ($\leqq \pi$) satisfies the equation

$$\tan\frac{z_0}{2}=z_0. \tag{57}$$

This equation obviously has one and only one root in the interval $0 < z_0 \overline{\overline{<}} \pi$, whence it follows that the root $z < \pi$ of (54) tends towards one definite limit for $n = \infty$, viz. z_0. From (55) it then follows that

(58) $$\lim_{n=\infty} V_n(x_n'') = \sin z_0 .$$

The numerical values are approximately

(59) $$z_0 = 2.33133 \cdots = 133^\circ\, 34' \cdots,$$
$$\sin z_0 = 0.72457 \cdots > \frac{2}{\pi} = 0.6366 \cdots .$$

We now proceed to the investigation of $V_n(x)$ in the interval $\pi / n \overline{\overline{<}} x \overline{\overline{<}} \pi / 2$. Assuming that

(60) $$m\pi \overline{\overline{<}} nx < (m+1)\pi, \qquad m \overline{\overline{>}} 1,$$

we determine integers $l_1, l_2, \cdots, l_m$ such that

$$l_1 x \overline{\overline{<}} \pi < (l_1 + 1) x,$$
$$l_2 x \overline{\overline{<}} 2\pi < (l_2 + 1) x,$$
$$\cdots\cdots\cdots$$
$$l_m x \overline{\overline{<}} m\pi < (l_m + 1) x,$$

and obtain

$$V_n(x) = \frac{1}{n+1}\left(\sum_{\nu=1}^{l_1} \sin x - \sum_{\nu=l_1+1}^{l_2} \sin x + \cdots + (-1)^m \sum_{\nu=l_m+1}^{n} \sin x\right)$$

$$= \frac{\cos\frac{x}{2} - 2\cos\frac{2l_1+1}{2}x + 2\cos\frac{2l_2+1}{2}x - \cdots + (-1)^m \cdot \cos\frac{2l_m+1}{2}x + (-1)^{m+1}\cos\frac{2n+1}{2}x}{2(n+1)\sin\frac{x}{2}},$$

whence

(61) $$V_n(x) \overline{\overline{<}} \frac{2m+1+(-1)^{m+1}\cos\frac{2n+1}{2}x}{2(n+1)\sin\frac{x}{2}}.$$

We have

$$2(n+1)\sin\frac{x}{2} > 2(n+1)\left(\frac{x}{2} - \frac{x^3}{48}\right) = \frac{2n+1}{2}x + \frac{x}{2}\left(1 - \frac{(n+1)x^2}{12}\right)$$

$$> \frac{2n+1}{2}x + \frac{x}{2}\left(1 - \frac{(n+1)(m+1)^2\pi^2}{12n^2}\right),$$

whence

(62) $$2(n+1)\sin\frac{x}{2} > \frac{2n+1}{2}x$$

for

(63) $$n \overline{\gtrless} (m+1)^2,$$

(except for $m = 1$, $n = 4$, in which case it is easily shown by numerical calculation that the absolute maximum of $V_4(x)$ occurs for an x in the interval $0 < x < \pi/4$). From (61) and (62) it follows that, as soon as (63) is satisfied,

$$V_n(x) < \frac{2m+1+(-1)^{m-1}\cos\frac{2n+1}{2}x}{\frac{2n+1}{2}x},$$

or making

(64) $$\frac{2n+1}{2}x = m\pi + z, \qquad 0 < z < \pi + \frac{m+1}{2n}\pi < \frac{3\pi}{2},$$

we find

(65) $$V_n(x) < \frac{2m+1-\cos z}{m\pi + z},$$

so that an upper limit for $V_n(x)$ is furnished by the maximum of the right hand expression. In order to obtain this maximum, we must have

(66) $$\varphi(z) = (m\pi + z)\sin z - (2m+1) + \cos z = 0.$$

Determining z_∞ by the conditions $\sin z_\infty = 2/\pi$, $\pi/2 < z_\infty < \pi$, we find

$$\varphi'(z) = (m\pi + z)\cos z \begin{cases} > 0, & \left(0 < z < \frac{\pi}{2}\right), \\ < 0, & \left(\frac{\pi}{2} < z < \frac{3\pi}{2}\right), \end{cases}$$

and

$$\varphi(0) = -2m < 0,$$

$$\varphi\left(\frac{\pi}{2}\right) = (2m+1)\left(\frac{\pi}{2} - 1\right) > 0,$$

$$\varphi(z_\infty) = \frac{2}{\pi}z_\infty - 1 + \cos z_\infty < 0,$$

$$\varphi(\pi) = -(2m+2) < 0,$$

$$\varphi\left(\frac{3\pi}{2}\right) = -(2m+1)\left(\frac{\pi}{2} + 1\right) - 1 < 0,$$

so that, in the interval $0 < z < 3\pi/2$, the equation (66) has one root between 0 and $\pi/2$, corresponding to a minimum, and one root z_m between $\pi/2$ and z_∞, corresponding to a maximum, which on account of (66) has the value

$$\frac{2m + 1 - \cos z_m}{m\pi + z_m} = \sin z_m . \tag{67}$$

On the other hand, this maximum decreases monotonously with m increasing, for we have by (66)

$$\begin{aligned}\frac{d}{dm}\frac{2m+1-\cos z_m}{m\pi+z_m} &= \frac{\partial}{\partial m}\frac{2m+1-\cos z_m}{m\pi+z_m} + \frac{\partial}{\partial z_m}\frac{2m+1-\cos z_m}{m\pi+z_m}\frac{dz_m}{dm} \\ &= \frac{\partial}{\partial m}\frac{2m+1-\cos z_m}{m\pi+z_m} \\ &= \frac{2(m\pi+z_m) - \pi(2m+1-\cos z_m)}{(m\pi+z_m)^2} \\ &= \frac{2-\pi\sin z_m}{m\pi+z_m} < 0, \text{ on account of } z_m > z_\infty .\end{aligned}$$

By numerical calculation for $m = 1$, we obtain $137° 35' < z_1 < 137° 36'$, $\sin z_1 < 0.6745 \cdots$, so that, for n satisfying (63) and any $m \geqq 1$,

$$V_n(x) < \sin z_1 < 0.6745. \tag{68}$$

Now turning our attention to the remaining case $n + 1 \leqq (m+1)^2$, we find from (25)

$$\begin{aligned}V_n(x) &< \frac{n}{n+1}\sqrt{\frac{2n+1}{4n}\,1 + \frac{1}{4m_1+3}} \\ &= \frac{1}{2}\sqrt{\left(1+\frac{1}{4m_1+3}\right)\left(2-\frac{1}{n+1}\right)\left(1-\frac{1}{n+1}\right)},\end{aligned}$$

where m_1 is determined by

$$\frac{2m_1\pi}{2n+1} \leqq x < \frac{2(m_1+1)\pi}{2n+1},$$

so that, by (60), $m_1 = m$ or $m_1 = m + 1$. In either case we have

$$\begin{aligned}V_n(x) &< \frac{1}{2}\sqrt{\left(1+\frac{1}{4m+3}\right)\left(2-\frac{1}{n+1}\right)\left(1-\frac{1}{n+1}\right)} \\ &\leqq \frac{1}{2}\sqrt{\left(1+\frac{1}{4m+3}\right)\left(2-\frac{1}{(m+1)^2}\right)\left(1-\frac{1}{(m+1)^2}\right)}.\end{aligned}$$

It is readily seen that for integral positive values of m, the last expression attains its greatest value $= 0.70904 \cdots$ for $m = 6$, and by comparison with (68) we finally find that for any n and any x in the interval $\pi / \leqq x \leqq \pi / 2$,

$$V_n(x) < 0.70904 \tag{69}$$

It is easily found that $V_n(x_n'')$, which we have shown to increase monotonously with n, exceeds the above value for $n \geqq 20$, and by numerical calculation (which may be very materially abbreviated by various artifices) it is shown that also for $n < 20$, $V_n(x_n'')$ is the absolute maximum of $V_n(x)$.

It is thus finally proved for all values of n that the absolute maximum of $V_n(x)$ occurs for an x in the interval $0 < x < \pi / n$ and increases monotonously with n toward the limit

$$\lim_{n=\infty} V_n(x_n'') = \sin z_0 = 0.72457 \cdots .$$

As we have $\sin z_0 > 2/\pi$, it follows that the limit values for $n = \infty$ of the maxima and minima of $V_n(x)$ cover a wider range (from 0 to $\sin z_0$) than the limit values of $V_n(x)$ for x independent of n, which all lie within the range from 0 to $2/\pi$; the analogy to Gibbs' phenomenon is obvious.

On account of (7), we further have, for $A = 0$,

$$| s_n(x) | < M + B \sin z_0 < M + \tfrac{3}{4} B,$$

which, in this particular case, gives a still closer limitation than (9).

CHICAGO, ILL.,
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