DECOMPOSITION OF AN N-SPACE BY A POLYHEDRON*

BY

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1. The following pages contain a proof of the theorem that an $(n-1)$-dimensional polyhedron decomposes an n-dimensional space into two regions. The proof has an essentially combinatorial character and involves few geometrical ideas beyond the principle than an n-dimensional convex region is decomposed into two n-dimensional convex regions by an $(n-1)$-space which contains one of its points and lies in the same n-space.

The theorem has been proved for the case $n = 2$ by H. HAHN † and for $n = 2, 3$ by N. J. LENNES.‡ The n-dimensional case, for polyhedra without singularities (cf. § 5 below), can be taken as a corollary under the theorem on the decomposition of an n-space by an $(n-1)$-dimensional manifold as proved by L. E. J. BROUWER.§ Our proof however allows a more general class both of spaces ‖ and of polyhedra within a given space.

2. The n-dimensional space, S_n, in which we are working may be defined as a "number space," or it may be taken as defined by a set of axioms. We shall adopt the latter point of view as being more convenient and general. For analytic purposes it is only necessary to verify the obvious fact that the axioms are satisfied by a number space.

We shall take as axioms, I–VIII of my thesis ¶ together with the further assumption (necessary for n-dimensional geometry) that there exists a set of $n + 1$ points not all in the same $(n-1)$-dimensional space. These axioms do not include an Archimedean axiom or any assumption about continuity.

3. We take for granted the definition and properties of the n-dimensional

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† Monatshefte für Mathematik und Physik, vol. 19 (1908), pp. 289–303. In this article HAHN points out that the proof in my thesis, these Transactions, vol. 5 (1904), p. 365 (Th. 28), is incorrect. The same mistake occurs in the proof of Theorem 9, p. 93 of my article, *Theory of Plane Curves in Non-Metrical Analysis Situs*, these Transactions, vol. 6 (1905), p. 93. Theorem 9 is a corollary of the theorem that a simple polygon decomposes the plane into two regions, so that, a correct proof of the polygon theorem having been supplied, this mistake does not affect the rest of the paper.

‡ American Journal of Mathematics, vol. 33 (1911), pp. 37–62.

§ Mathematische Annalen, vol. 71 (1911), p. 314.

‖ We do not assume continuity.

¶ These Transactions, vol. 5 (1904), pp. 343–384.

simplex, the generalization of the triangle and tetrahedron. This is a set of $n+1$ points (not all in the same S_{n-1}) together with the $(n+1)n/2$ segments which they determine by pairs, the $(n+1)n(n-1)/2.3$ triangular regions which they determine by trios, $\cdots$, the $(n+1)n\cdots 2/(n)!$ interiors of $(n-1)$-dimensional simplexes which they determine by sets of n. The simplex separates the points of S_n into two sets, its interior and exterior, such that any broken line joining a point of one set to a point of the other meets the simplex, such that any segment joining two points of the interior consists entirely of points of the interior, and such that the exterior contains an S_{n-1} consisting entirely of points of the exterior.

The details of this discussion are analogous to those given for the triangle and tetrahedron in my thesis. A like remark applies to the contents of §§ 4, 5, 6 below.

4. An *n-dimensional region* is defined as a set of points all in the same S_n such that any two points of the set are joined by a broken line consisting entirely of points of the set and such that any point of the set is interior to an n-dimensional simplex consisting entirely of points of the set. A region is *convex* if the segment joining any two of its points consists of points of the region. The *boundary* of an n-dimensional region is a set of points $[B]$ such that (1) any broken line lying entirely in the S_n which contains the region and joining a point of the region to a point not of the region must contain a point B, and (2) any subset of $[B]$ does not have the property (1). According to these definitions a region cannot contain any point of its boundary.

5. A one-dimensional region is either a segment,* a ray (half-line) not including its end point, or a complete line. A two-dimensional region whose boundary consists of a finite number of points and one-dimensional regions is called a two-dimensional *polygonal region*.†

An n-dimensional region whose boundary consists of a finite number of points, one-dimensional regions, two-dimensional polygonal regions, $\cdots$, $(n-1)$-dimensional polyhedral regions is called an n-dimensional *polyhedral region*.‡

An *n-dimensional polyhedron* is the set of points in the interiors and boundaries of a finite set of n-dimensional polyhedral regions $F_1, F_2, \cdots, F_k$ no two of which have a point in common but such that (1) every $(n-1)$-dimensional polyhedral region on the boundary of one n-dimensional region F is on the boundary of an even number of n-dimensional polyhedral regions

* A segment does not include its end-points; an interval does.

† For example, a complete plane or the two sides of a line in a plane are polygonal regions according to this definition. Throughout this paper there is no restriction as to figures lying in "a finite part of space."

‡ It is an immediate corollary of this definition that the points, if any, common to two polyhedral regions constitute one or more polyhedral regions.

F and (2) there is no subset of the points constituting the polyhedron which can be arranged in n-dimensional polyhedral regions so as to have the property (1).* The regions $F_1, \cdots, F_k$ are called the *faces* of the polyhedron.

According to this definition a polyhedron can have rather complicated singularities in all cases where $n > 2$. An n-dimensional polyhedron without singularities is obtained by requiring, in addition to conditions (1) and (2), that a complete neighborhood of any point on it be capable of being set in one-to-one correspondence, preserving order relations, with the interior of an $(n-1)$-dimensional simplex. In this paper, however, we are concerned with the general polyhedron, allowing singularities.

6. Let us now consider an $(n-1)$-dimensional polyhedron in an n-space S_n. Let us denote the faces of the polyhedron by $F_1, F_2, \cdots, F_k$. Let us denote the $(n-1)$-spaces which contain them by $S_{n-1}^1, S_{n-1}^2, \cdots, S_{n-1}^j$. There are not necessarily as many of these as of the F's since two F's may be in the same S_{n-1}.

By an argument which is a mere generalization of that in the proof of Theorem 26 of my thesis, the j S_{n-1}'s decompose S_n into a certain number, N, of mutually exclusive convex polyhedral regions

$$C_1, C_2, \cdots, C_N.$$

Each of the S_{n-1}'s is met by the other S_{n-1}'s in a number of S_{n-2}'s which decompose it into a number of convex $(n-1)$-dimensional polyhedral regions. Let us denote these by

$$a_1, a_2, \cdots, a_M,$$

M being the number of a's on all the S_{n-1}'s. The boundaries of the C's may be denoted by

$$b_1, b_2, \cdots, b_N$$

respectively. The b's are polyhedra. Any one of the a's is in two and only two b's. Moreover a single b has at most one region a in common with a given S_{n-1}.

The foregoing statements hold good for any finite number of S_{n-1}'s in the same S_n, except that, in general, each b is a set of polyhedra instead of a single polyhedron.

7. *If π is any polyhedron composed entirely of points of $S_{n-1}^1, S_{n-1}^2, \cdots, S_{n-1}^j$, and a_k is any one of the convex $(n-1)$-dimensional regions a, then if π contains one point of a_k it contains all points of a_k.*

In proving this we may assume that a_k is on S_{n-1}^j. No point of a_k is on $S_{n-1}^1, \cdots, S_{n-1}^{j-1}$. Hence if π contains a point of a_k, this point (which we shall

* This definition of a polyhedron is a generalization of that used by LENNES (*loc. cit.*) in the one- and two-dimensional cases.

call P) may be taken as interior to an $(n-1)$-dimensional polygonal region F of π which is contained entirely in S_{n-1}^j. If π does not contain all points of a_k there must be an $(n-1)$-dimensional simplex whose interior consists of points of a_k not in π. Let Q be one of these points such that the linear segment PQ does not meet any one of the finite number of polyhedral regions of dimensionality $(n-3)$ or less which bound the $(n-1)$-dimensional polyhedral regions of π. The linear interval PQ meets the $(n-1)$-dimensional regions of π and their boundaries in a finite number of points and segments. Let N be the last one of the finite set composed of these points and of the end points of these segments in the sense from P to Q. The point N must be on the boundary of an $(n-1)$-dimensional region F' of π; and since Q was so chosen that N must be interior to an $(n-2)$-dimensional region of the boundary of F', any other $(n-1)$-dimensional region of π in S_{n-1}^j which had N on its boundary would contain points of the segment NQ, contrary to the definition of N. Since N is in a it is not on any of $S_{n-1}^1, \cdots, S_{n-1}^{j-1}$, and hence cannot be interior to any $(n-1)$-dimensional region of π which is not in S_{n-1}^j. Hence N is on the boundary of only one $(n-1)$-dimensional region of π, contrary to the definition of a polyhedron. Hence π contains all points of a_k.

Every point of π which is on S_{n-1}^j is either interior to, or on the boundary of one of the convex regions $a_{k_1}, a_{k_2}, \cdots$ into which S_{n-1}^j is decomposed by $S_{n-1}^1, \cdots, S_{n-1}^{j-1}$. Hence what we have just proved shows that the regions F of π on S_{n-1}^j, together with their boundaries, constitute the interiors and boundaries of a finite number of a_k's. We shall denote these a_k's by $a_{k_1}, a_{k_2}, \cdots, a_{k_\nu}$. Any $(n-2)$-dimensional convex portion of the boundary of one of $a_{k_1}, a_{k_2}, \cdots, a_{k_\nu}$ is either on one and only one other of $a_{k_1}, a_{k_2}, \cdots, a_{k_\nu}$, or else it is also a portion of the boundary of one of the F's in S_{n-1}^j. Hence if the F's in S_{n-1}^j are replaced by $a_{k_1}, a_{k_2}, \cdots, a_{k_\nu}$, the latter together with the F's in $S_{n-1}^{j-1}, S_{n-1}^{j-2}, \cdots, S_{n-1}^1$ form a set of $(n-1)$-dimensional polyhedral regions which satisfy the definition of a polyhedron.

In like manner, the $(n-1)$-dimensional regions of π in $S_{n-1}^{j-1}, \cdots, S_{n-1}^1$ can be replaced by convex regions. Hence *any polyhedron π composed entirely of points of $S_{n-1}^1, S_{n-1}^2, \cdots, S_{n-1}^j$ can be regarded as a polyhedron whose faces are a subset of the convex $(n-1)$-dimensional regions, $a_1, a_2, \cdots, a_M$.*

8. Given any set whatever of the convex $(n-1)$-dimensional regions, $a_1, a_2, \cdots, a_M$, let us introduce the symbol

$$(x_1, x_2, \cdots, x_M),$$

in which x_i $(i = 1, 2, \cdots, M)$ is 1 if a_i is in the set and 0 if x_i is not in the set. This symbol may be used to denote the totality of points in the set of a's and on their boundaries. According to § 7 any $(n-1)$-dimensional polyhedron composed entirely of points of $S_{n-1}^1, \cdots, S_{n-1}^j$, may be denoted by a symbol of this form. By the definition of a polyhedron the sum of the x's corresponding to the a's having a given $(n-2)$-dimensional region on their boundaries is even or zero.

Let us agree to combine the x's by addition and multiplication, reducing modulo two. The symbol,

$$x + y = (x_1 + y_1, \quad x_2 + y_2, \quad \cdots, \quad x_M + y_M),$$

(read, the *sum* of x and y) will then stand for the set of points within and on the boundaries of the a's obtained by combining the a's (represented by $x = (x_1, x_2, \cdots, x_M)$ with those represented by $y = (y_1, y_2, \cdots, y_M)$ and suppressing those which appear in both sets.

According to this definition, the sum of two polyhedra is always one or more polyhedra, as follows directly from the definition of a polyhedron. Moreover we always have

$$x + x = (0, 0, \cdots, 0).$$

The essential feature of our argument will be to show that the symbol for any polyhedron can be expressed as a sum of the symbols for a certain set of the b's. We shall use $b_1, b_2, \cdots, b_N$ as abbreviations for the expressions of the form $(x_1, x_2, \cdots, x_M)$ which represent the boundaries $b_1, b_2, \cdots, b_N$ respectively.

9. If S^j_{n-1} be removed, the remaining S_{n-1}'s, $S^1_{n-1}, \cdots, S^{j-1}_{n-1}$ decompose S_n into a number of convex regions some of which are identical with some of the C's and some of which are such that they are decomposed into pairs of C's by S^j_{n-1}. The boundary of any one of the latter regions is expressible, according to the above convention, in the form

$$b_\gamma + b_\delta,$$

where C_γ and C_δ are the two convex regions into which the given convex region is divided by S^j_{n-1}.

By a repetition of this reasoning it is established that if any subset of the S_{n-1}'s be removed, the boundaries of the n-dimensional convex regions into which S_n is decomposed by the remaining S_{n-1}'s are linearly expressible in terms of the b's.

10. Since every $(n-1)$-dimensional convex region a on each b is on one and only one other b,

$$b_1 + b_2 + \cdots + b_N = (0, 0, \cdots, 0), \tag{1}$$

where each b enters once and only once into the summation.

Moreover there exists no relation of the form

$$b_\alpha + b_\beta + \cdots + b_\nu = (0, 0, \cdots, 0) \tag{2}$$

which does not include all the b's. For consider the corresponding C's and let P be a point in one of these C's and Q a point in one of the C's not in this set. Let P and Q be so chosen that the line PQ does not meet any of the finite number of convex regions of dimensionality $(n-2)$ or less which form parts of the b's. Then the segment PQ meets the $b_\alpha, b_\beta, \cdots, b_\nu$ in a finite number (>0) of points. Let K be the last of these in the sense from P to Q. The segment KQ is not in any of $C_\alpha, C_\beta, \cdots, C_\nu$, and hence the $(n-1)$-dimensional region of the boundary of the one of $b_\alpha, b_\beta, \cdots, b_\nu$ which contains K is on one and only one b. This contradicts Equation (2).

11. *Any $(n-1)$-dimensional polyhedron π composed entirely of points of $S^1_{n-1}, \cdots, S^j_{n-1}$ is expressible in the form*

$$\pi = b_{\nu_1} + b_{\nu_2} + \cdots + b_{\nu_r}. \tag{3}$$

This is proved by mathematical induction. In the first place, if there is only one S_{n-1} it constitutes the only $(n-1)$-dimensional polyhedron composed entirely of points of itself and is itself the boundary of a convex region. Hence by § 9 it is linearly expressible in terms of the b's.

Now supposing the theorem true for $S^1_{n-1}, \cdots, S^i_{n-1}$, any $(n-1)$-dimensional polyhedron composed of points from these S_{n-1}'s is expressible linearly in terms of the boundaries of the convex regions into which S_n is decomposed by $S^1_{n-1}, \cdots, S^i_{n-1}$, and hence by § 9 in terms of $b_1, b_2, \cdots, b_N$. Any $(n-1)$-dimensional polyhedron, π, composed of points of $S^1_{n-1}, \cdots, S^i_{n-1}, S^{i+1}_{n-1}$ is either of the sort considered in the last sentence or contains one or more of the convex $(n-1)$-dimensional polyhedral regions, R, into which S^{i+1}_{n-1} is decomposed by $S^1_{n-1}, \cdots, S^i_{n-1}$.

By § 7 these R's may be regarded as the faces of π in S^{i+1}_{n+1}. Any R is in the boundary of just two of the n-dimensional convex regions determined by $S^1_{n-1}, S^2_{n-1}, \cdots, S^{i+1}_{n-1}$ (§ 6). By § 9 the boundary of either of these regions is expressible in the form $b_{\lambda_1} + b_{\lambda_2} + \cdots + b_{\lambda_\rho}$. By § 6 the polyhedron or set of polyhedra represented by $b_{\lambda_1} + b_{\lambda_2} + \cdots + b_{\lambda_\rho}$ has no other face in common with S^{i+1}_{n+1}. Hence $\pi + b_{\lambda_1} + b_{\lambda_2} + \cdots + b_{\lambda_\rho}$ is a polyhedron or set of polyhedra which has in common with S^{i+1}_{n-1} one $(n-1)$-dimensional convex region fewer than π has. By repeating this process a finite number of times we finally arrive at a polyhedron or set of polyhedra,

$$\pi + b_1 + b_2 + \cdots + b_r,$$

which has no $(n-1)$-dimensional regions in common with S^{i+1}_{n-1}. Such polyhedra have already been seen to be linearly expressible in terms of the b's. But since the relation

$$\pi + b_1 + \cdots + b_r = b_{r+1} + \cdots + b_{r+s}$$

is equivalent (§ 8) to

$$\pi = b_1 + \cdots + b_r + b_{r+1} + \cdots + b_{r+s},$$

we have the result that every $(n-1)$-dimensional polyhedron composed entirely of points of $S_{n-1}^1, \cdots, S_{n-1}^{i+1}$ is expressible in this form.

12. By adding Equation (1) in § 8 to Equation (3) in § 10 we obtain a second equation for π,

$$(4) \qquad \pi = b_{\mu_1} + b_{\mu_2} + \cdots + b_{\mu_t}.$$

Every b appears once and only once in the two equations, (3) and (4). There is no other linear expression for π in terms of the b's. For if there were, on adding it to (3) the sum would give a relation of the form (2) which has been shown not to exist.

13. Consider now the set of n-dimensional convex regions $C_{\nu_1}, C_{\nu_2}, \cdots, C_{\nu_r}$ corresponding to the b's in (3), and let $[P]$ be the set of points interior to and on the boundaries of these C's but not on π. I say that any two points of $[P]$ can be joined by a broken line consisting of points P.

For let P_0 be an arbitrary point interior to C_{ν_1}. P_0 can be joined by an interval consisting entirely of points P to all points of the interior and boundary of C_{ν_1} which are not on π. Unless π includes all the $(n-1)$-dimensional convex regions, a, of the boundary of C_{ν_1} (in which case π would be identical with b_{ν_1}, by the definition of the polyhedron, and $[P]$ would be identical with C_{ν_1}) the boundary of b_{ν_1} has an a in common with the boundary of another C, say with b_{ν_2}. Let a_{ν_1} be the $(n-1)$-dimensional region common to b_{ν_1} and b_{ν_2} and let P_1 be any point of a_{ν_1}.

The broken line $P_0 P_1 Q$ will join P_0 to any point, Q, of the interior or boundary of C_{ν_2} which is not on π. This broken line consists entirely of points of $[P]$. Unless $\pi = b_{\nu_1} + b_{\nu_2}$, there is some $(n-1)$-dimensional convex region a_{ν_2} common to $b_{\nu_1} + b_{\nu_2}$ and one (say b_{ν_3}) of $b_{\nu_3}, \cdots, b_{\nu_r}$; for the equation $\pi = b_{\nu_1} + \cdots + b_{\nu_r}$ means that every $(n-1)$-dimensional region a of each b is either a region of π or of another b_ν. If P_2 is any point of a_{ν_2}, the broken line joining P_0 to P_2 together with the interval $P_2 Q$ forms a broken line joining P_0 to any point Q of C_{ν_3} and consisting entirely of points of $[P]$.

Since π is not linearly expressible in terms of any subset of $b_{\nu_1}, \cdots, b_{\nu_r}$, the process continues till it shows that P_0 can be joined by a broken line to any point of $[P]$.

14. In like manner, the set of points $[Q]$ interior to and on the boundaries of $C_{\mu_1}, \cdots, C_{\mu_t}$ but not on π, have the property that any two can be joined by a broken line consisting entirely of points Q. Consider now a broken line joining a point P to a point Q. This broken line meets $S_{n-1}^1, \cdots, S_{n-1}^j$ in a finite number of points and intervals. Let these points and the ends of

these intervals be denoted by $A_1, A_2, \cdots, A_\rho$, taken in the sense from P to Q. Since the broken line from P to A_1 is entirely in one of the regions $C_{\nu_1}, \cdots, C_{\nu_r}$, A_1 is either in $[P]$ or on π. If A_1 is in $[P]$, it follows in like manner that A_2 is in $[P]$ or on π. Unless one of $A_1, \cdots, A_\rho$ is on π we should thus have Q in $[P]$ or on π. *Hence every broken line joining a* P *to a* Q *meets* π.

15. Consider now any point not on π; by definition, it is either a P or a Q. Since π is composed of a finite number of convex $(n-1)$-dimensional regions and their boundaries, any point not on π is interior to an n-dimensional simplex containing no points of π. Any two points interior to such a simplex can be joined by a segment interior to the simplex. Hence all points interior to the simplex are either in $[Q]$ or in $[P]$. *Hence* $[P]$ *and* $[Q]$ *are regions and* π *is the boundary of each region.*

16. Thus we have proved the theorem. *Any* $(n-1)$*-dimensional polyhedron,* π*, in an* S_n *decomposes the* S_n *into two regions of each of which* π *is the boundary. Any broken line joining a point of one region to a point of the other contains a point of* π.

Incidentally, we have proved the following corollaries: (1) *Any* $(n-1)$*-dimensional polyhedron* π *in an n-space can be regarded as defined in terms of convex* $(n-1)$*-dimensional polyhedral regions.* (2) *The two regions determined by an* $(n-1)$*-dimensional polyhedron* π *in an n-dimensional space are composed each of a finite number of convex polyhedral regions.*

It requires only a slight modification of the argument in § 11 to prove that any $(n-\nu)$-dimensional polyhedron in an S_n is linearly expressible in terms of boundaries of $(n-\nu+1)$-dimensional convex regions. From this theorem it follows easily that the Betti numbers of an S_n are all equal to unity when the class of manifolds with respect to which they are defined consists of polyhedra. For the case in which the manifolds have perfect generality or even when they are composed of analytic pieces no rigorous determination of all the Betti numbers has yet been given.

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