ALGEBRAIC SURFACES INVARIANT UNDER AN INFINITE DISCONTINUOUS GROUP OF BIRATIONAL TRANSFORMATIONS:

(SECOND PAPER*)

BY

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In a recent paper Dr. ROSENBLATT gives two interesting examples of algebraic surfaces which are invariant under an infinite discontinuous group of birational transformations, and at the same time are not envelopes of quadric surfaces.†

In an earlier paper I mentioned ‡ that all surfaces belonging to such groups which have thus far been noticed were examples of surfaces defining an ordinary elliptic $(2, 2)$ correspondence. Practically all the memoirs bearing on the problem are mentioned in the articles just named. The surfaces discussed by DR. ROSENBLATT have a pencil of elliptic curves, and the transformations are expressed by a linear transformation of the parameters u, v of elliptic functions, in terms of which the coördinates of a point on either surface can be rationally expressed. The treatment is transcendental, the transformation is defined only for points on the surface, and no explanation of the geometric meaning of the transformation is given.

It is a curious fact that these transformations are birational for all space, that they have a simple geometric interpretation in terms of the $(2, 2)$ correspondence, and that surfaces of any order or of as high a geometric genus as may be desired can be constructed which are invariant under this group, or a larger group, under which that discussed by Dr. Rosenblatt is a subgroup of infinite index.

Consider the surface whose equation is of the form

$$\sum_{k=0}^{n} F^{n-k} x^k \varphi_{2k}(z, t) = 0,$$

* Presented to the Society January 1, 1913.

† A. ROSENBLATT: *Algebraische Flächen mit diskontinuierlich unendlich vielen birationellen Transformationen in sich,* Rendiconti . . . di Palermo, vol. 33 (1912), pp. 212–216.

‡ *Infinite discontinuous groups of birational transformations which leave certain surfaces invariant,* these Transactions, vol. 11 (1910), pp. 15–24.

wherein

$$F \equiv x^3 - ty\,(y - x)$$

and φ_{2k} is any binary form of order $2k$. The three lines

$$x = 0,\ t = 0;\quad x = 0,\ y = 0;\quad x = 0,\ y - z = 0$$

are each n-fold upon the surface, their common point $(0, 0, 0, 1)$ being a uniplanar point of order n. The line $z = 0, t = 0$ does not lie on the surface; any plane $t = mz$ through this line will cut the surface in n cubic curves belonging to a pencil, the equation of each being of the form

$$x^3 - mzy^2 + mz^2 y + l_i\, xz^2 = 0,$$

in which l_i is a function of m and of the coefficients in the various binary forms φ. The basis points of the pencil to which these curves belong are all on the line

$$x = 0,\quad t = mz,$$

being at the points at which the plane cuts the multiple lines of the surface.

At $S \equiv (0, 0, 1, m)$, and at $T \equiv (0, 1, 1, m)$ the curves have simple intersection, while at $U \equiv (0, 1, 0, 0)$ each component curve has a point of inflexion with a common inflexional tangent, and all have seven-point contact with each other.

Any point $P \equiv (x_1, y_1, z_1, t_1)$ on the surface will determine a plane of the pencil $t = mz$, and hence a value of m. In this plane there is one and only one cubic curve lying on the surface and passing through P. If the operation of projecting P upon this curve from S is denoted by S, then $S^2 = 1$, and the operation is birational for every point on the surface, since two curves in a plane $t = mz$ do not have any point in common except the basis points S, T, U. We may now define the operations of projecting each component cubic on itself from T and U by T, U respectively, so that $T^2 = 1$, $U^2 = 1$.

Any pair of involutions, as S, T, will now generate a group of infinite order and the larger group generated by S, T, U will have the former as a sub-group of infinite index.

The equations of these generators are found by the methods of analytic geometry to be

$$S:\quad \begin{aligned} \rho x' &= x\,(tz^2 - x^3), \\ \rho y' &= y\,(ty^2 - x^3), \\ \rho z' &= x^3 z, \\ \rho t' &= x^3 t. \end{aligned}$$

$$T: \begin{aligned} \rho x' &= x\,[t\,(y-z)^2 - x^3], \\ \rho y' &= t\,(y-z)^3 + 2x^3 z - x^3 y, \\ \rho z' &= x^3 z, \\ \rho t' &= x^3 t. \end{aligned}$$

$$U: \begin{aligned} \rho x' &= x, \\ \rho y' &= z - y, \\ \rho z' &= z, \\ \rho t' &= t. \end{aligned}$$

$$UTU = S.$$

In the preceding illustration, the order of the surface is $3n$ and the line $z = 0$, $t = 0$ does not lie on the surface. If F' is now defined as the product of F by any binary form in z, t, of order p then the equation

$$\sum_{k=0}^{n} F'^{\,n-k} x^k \varphi_{2k+p}(z, t) = 0$$

will define a surface of order $3n + p$, still invariant under the same transformations, and having $z = 0$, $t = 0$ as a singular line of multiplicity $n + p$.

In both cases the cubic curves are not uni-modular, but a plane of the pencil can always be found such that one component curve lying in it has a modulus which may be assigned at will.

Moreover, the form of the surface can be varied still further by replacing two of the straight line loci of basis points of the cubic curves by rational curves of order s, having a $(s - 1)$-fold point on the axis of the pencil of planes, which may also be a multiple line on the surface. Thus the equation may be written in the form

$$\sum_{k=0}^{n} H^{n-k} x^k f_{(2s-1)k}(z, t) = 0,$$

wherein

$$H \equiv x^3 t^{2s-3} - xzt^{2s-2} - y^2 t^{2s-2} + z^{2s},$$

and f_i is any binary form.

As in the preceding case, any plane $t = mz$ will cut from the surface a series of cubic curves all belonging to a pencil, the basis points being at the intersections of the line $x = 0$, $t = mz$ with the line $x = 0$, $z = 0$ and with the rational curves

$$yt^{s-1} + z^s = 0, \quad x = 0; \qquad yt^{s-1} - z^s = 0, \quad x = 0.$$

These points are respectively

$$N \equiv (0,\ 1,\ 0,\ 0), \qquad L \equiv (0,\ 1,\ m^{s-1},\ m^s),$$
$$M \equiv (0,\ 1,\ -m^{s-1},\ -m^s).$$

Using these same letters to denote the projections of the cubic curves upon themselves from the three collinear points, we obtain the following generating transformations.

$$L: \begin{aligned} \rho x' &= xt^{s-1}(z^s - t^{s-1}y)^2 - x^4 t^{3s-4}, \\ \rho y' &= 2x^3 z^s t^{2s-3} - (z^s - t^{s-1}y)^3 - x^3 yt^{3s-4}, \\ \rho z' &= x^3 zt^{3s-4}, \\ \rho t' &= x^3 t^{3s-3}. \end{aligned}$$

$$M: \begin{aligned} \rho x' &= xt^{s-1}(z^s + t^{s-1}y)^2 - x^4 t^{3s-4}, \\ \rho y' &= -2x^3 z^s t^{2s-3} + (z^s + t^{s-1}y)^3 - x^3 yt^{3s-4}, \\ \rho z' &= x^3 zt^{3s-4}, \\ \rho t' &= x^3 t^{3s-3}. \end{aligned}$$

$$N: \begin{aligned} \rho x' &= x, \\ \rho y' &= -y, \\ \rho z' &= z, \\ \rho t' &= t. \end{aligned}$$

$$NLN = M.$$

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