THE SOLUTIONS OF NON-HOMOGENEOUS LINEAR DIFFERENCE EQUATIONS AND THEIR ASYMPTOTIC FORM*

BY

K. P. WILLIAMS

Introduction.

Very important progress has recently been made in the analytic theory of homogeneous linear difference equations.

GALBRUN † has used the Laplace transformation to derive important existence theorems, and has investigated the nature of certain *principal solutions* for large values of the variable.

At about the same time NÖRLUND ‡ applied the theory of factorial series to linear difference equations, showed the existence of the same solutions, and gave their asymptotic form. In addition to the case of polynomial coefficients he has considered the case where the coefficients can be expressed in factorial series. In a more recent paper he has investigated an equation with still more general coefficients.§

By means of a method of successive approximation, and a suitable extension of a contour integral due to GUICHARD, CARMICHAEL ‖ has independently shown the existence of solutions, and has found the bounds of increase and decrease of the solutions in a direction parallel to the real axis.

The latest important contribution to the general theory is by BIRKHOFF.¶ He employs a matrix notation, and shows in a direct manner the existence of certain *intermediate solutions* and of the principal solutions. The asymptotic form of these solutions is determined throughout the complex plane. A modification of the integral used by CARMICHAEL plays an important rôle.

He finds also the form of the periodic functions which connect the principal solutions, and formulates for the first time the properties which are sufficient to characterize a given system.

While the analytic theory of homogeneous linear difference equations has thus been extensively treated, no general theory has been developed for non-homogeneous equations,* although a number of equations of particular form have been considered (see CARMICHAEL, loc. cit., Introduction).

The purpose of the present paper is to prove the existence of certain *principal solutions* of the system of non-homogeneous equations

$$g_i(x+1) = \sum_{j=1}^{n} a_{ij}(x) g_j(x) \qquad (i = 1, 2, \cdots, n-1),$$

(1)

$$g_n(x+1) = \sum_{j=1}^{n} a_{nj}(x) g_j(x) + b(x+1),$$

and to develop their asymptotic form. The functions $a_{ij}(x)$ and $b(x)$ will be assumed to be rational,† with poles of order at most μ and ν respectively at infinity, so that we can write

$$a_{ij}(x) = x^\mu \left(a_{ij} + \frac{a_{ij}^{(1)}}{x} + \cdots \right),$$

$$b(x) = x^\nu \left(b + \frac{b^{(1)}}{x} + \cdots \right),$$

for $|x| > R$.

On account of the gain in simplicity of the formulæ we have taken a system where all but one of the equations are homogeneous. It is, furthermore, evident that the system (1) would be no more general if all the equations were non-homogeneous, since the solution of such a system could be obtained by adding together the solutions of the n systems, of the above type, which would be obtained by striking out the non-homogeneous term from all the equations except the first, second, . . . , last, in turn. Thus the theorems for (1) can be extended readily to a system of n equations, all non-homogeneous.

The method employed is based on a method analogous to that of variation of constants in the theory of linear differential equations. This method has previously been supposed to yield only formal results.‡ It will appear,

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* It is possible to reduce a non-homogeneous equation to a homogeneous equation. The application of the general results for a homogeneous equation will show the existence of solutions, but gives no direct means of studying their properties. This method is suggested by WALLENBERG and GULDBERG, *Theorie der linearen Differenzengleichungen*, p. 87.

† Most of the results which will be obtained apply with slight modification to the case where $a_{ij}(x)$ and $b(x)$ have the form of rational functions only at infinity.

‡ WALLENBERG und GULDBERG, p. vii, ". . . die Methode der Variation der Konstanten, die wegen der mit ihr verbundenen Summationen meist nur formale Bedeutung hat . . . ."

however, that in general, if not in all cases, the sum formulæ that arise can be interpreted so as to give solutions analytic in the finite plane except for poles.

There is a difference of treatment according as $\mu > 0$, $\mu < 0$, or $\mu = 0$.

In § 2 the reduction of the problem to that of making valid one of four formal types of sums is effected.

In § 3, under the hypothesis that $\mu > 0$, a direct summation to the right is used to derive a *first principal solution* of (1), which we denote by $g_{11}(x)$, $g_{21}(x)$, $\cdots$, $g_{n1}(x)$. This solution is analytic in the finite plane save for poles at determinate points near the negative real axis. In § 4 it is proved that in the right half plane this solution is asymptotic to the series formally satisfying (1).

It is shown in § 5 that the formal solution of (1) obtained by a direct summation to the right, such as is used in § 3, is independent of the fundamental set of solutions of the homogeneous system associated with (1) which is used to determine it. This principle is used in § 6 to determine the asymptotic form of the first solution in the left half plane.

A summation by means of a contour integral to the left is employed in § 7 to determine a *second principal solution*, $g_{12}(x)$, $g_{22}(x)$, $\cdots$, $g_{n2}(x)$, analytic save for poles at determinate points near the positive real axis. This solution is asymptotic to the formal series in the left half plane.

In § 8 the relation between the principal solutions is determined.

It is seen in § 9 that, if $\mu < 0$, the same theory holds except that the rôles of the right and left half planes are interchanged. Similar results are proved in § 10 for the intermediate case $\mu = 0$, except in certain extreme cases.

An application of the general theory is made in § 11 to a single equation of the nth order.

The writer wishes to express his appreciation and thanks to Professor BIRKHOFF for the valuable suggestions and aid that have been received from him.

§ 1. *The fundamental existence theorems for a homogeneous system.*

The system

$$(2) \qquad g_i(x+1) = \sum_{j=1}^{n} a_{ij}(x) g_j(x) \qquad (i = 1, 2, \cdots, n)$$

is the homogeneous system associated with (1). As we shall make use of its solutions, it will be convenient to enumerate some of their properties. We state below the existence theorems for (2) as they are given by Birkhoff.

There exist in general n sets of series:

$$s_{1j}(x) = x^{\mu x}(\rho_j e^{-\mu})^x x^{r_j}\left(s_{1j} + \frac{s_{1j}^{(1)}}{x} + \cdots\right),$$

(3)
$$s_{2j}(x) = x^{\mu x}(\rho_j e^{-\mu})^x x^{r_j}\left(s_{2j} + \frac{s_{2j}^{(1)}}{x} + \cdots\right), \quad (j = 1, 2, \cdots, n),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$s_{nj}(x) = x^{\mu x}(\rho_j e^{-\mu})^x x^{r_j}\left(s_{nj} + \frac{s_{nj}^{(1)}}{x} + \cdots\right),$$

each of which will formally satisfy (2), but which in general diverge.

The quantities ρ_j are the roots of the *determinant characteristic equation*

(4)
$$|a_{ij} - \delta_{ij}\rho| = 0,$$

where $\delta_{ij} = 0$ for $i \neq j$, and $\delta_{ii} = 1$, and they are so ordered that

(5)
$$|\rho_1| \gtreqless |\rho_2| \gtreqless \cdots \gtreqless |\rho_n|.$$

Furthermore, they are all supposed different from zero, that is

(6)
$$A = |a_{ij}| = \rho_1 \rho_2 \cdots \rho_n \neq 0.$$

It is assumed that such series exist, and also that

(7)
$$S = |s_{ij}| \neq 0.$$

We can now state the theorems we shall use.

There are two fundamental sets of solutions of (2), *the first and second principal solutions;* [*] *they may be denoted by* $h_{1j}(x)$, $h_{2j}(x)$, $\cdots$, $h_{nj}(x)$ $(j = 1, 2, \cdots, n)$, *and* $\bar{h}_{1j}(x)$, $\bar{h}_{2j}(x)$, $\cdots$, $\bar{h}_{nj}(x)$ $(j = 1, 2, \cdots, n)$, *respectively. They have the defining property that in any right half plane* $h_{ij}(x) \sim s_{ij}(x)$, [†] *while in any left half plane* $\bar{h}_{ij}(x) \sim s_{ij}(x)$ $(i, j = 1, 2, \cdots, n)$. *The functions* $h_{ij}(x)$ *are analytic in the finite plane save for poles at the poles of* $a_{ij}(x)$, *and the zeros of the determinant* $|a_{ij}(x)|$, *or points congruent* [‡] *to these points on the left; while the functions* $\bar{h}_{ij}(x)$ *are analytic save for poles at the poles of* $a_{ij}(x)$, *and points congruent on the right.*

While all the results which will be derived could be obtained by making use of only the principal solutions of (2), it will simplify the work to make

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* Our first and second principal solutions are the second and first principal solutions, respectively, as given by BIRKHOFF. For the present purpose it is convenient to take them as above.

† This means that for any c the difference between $h_{ij}(x)$ and $k + 1$ terms of $s_{ij}(x)$ can be written $x^{\mu x}(\rho_j e^{-\mu})^x x^{r_j}\zeta_k(x)/x^k$, where $\lim_{x=\infty}\zeta_k(x) = 0$, uniformly, for $-\frac{1}{2}\pi \leqq \arg(x-c) \leqq \frac{1}{2}\pi$ or, in a special case, for $-\frac{1}{2}\pi - \epsilon \leqq \arg(x-c) \leqq \frac{1}{2}\pi - \epsilon$, ϵ arbitrarily small (see GALBRUN, loc. cit.). We say then that $h_{ij}(x)$ is *asymptotic* to $s_{ij}(x)$, or in symbols $h_{ij}(x) \sim s_{ij}(x)$, and call $\zeta_k(x)$ the remainder coefficient.

‡ The points $\cdots x-2, x-1, x, x+1, x+2, \cdots$ are called a congruent set of points.

use of the properties of one of the two sets of intermediate solutions that are given by Birkhoff (loc. cit. § 3).

There exists a set of intermediate solutions $\bar{h}'_{1j}(x)$, $\bar{h}'_{2j}(x)$, $\cdots$, $\bar{h}'_{nj}(x)$ $(j = 1, 2, \cdots, n)$, *defined only at a sufficient distance from the axis of reals. The elements of these solutions are analytic in the region where they are defined, and are such that* $\bar{h}'_{ij}(x) \sim s_{ij}(x)$ *with regard to x in the left half plane,* and with regard to v* $(x = u + v \sqrt{-1})$, *in any right half plane.†*

Although these intermediate solutions are not defined near the real axis, they will be of service to us on account of the fact that they maintain their asymptotic form with regard to v in any right half plane, and because the periodic functions which connect them and the second principal solutions are of a simple nature. The relation between these intermediate and the second principal solutions is given by the formula

$$
(8) \quad \begin{aligned}
\bar{h}_{ij}(x) &= \varphi_{1j}(x)\,\bar{h}'_{i1}(x) + \varphi_{2j}(x)\,\bar{h}'_{i2}(x) + \cdots \\
&\quad + \varphi_{j-1,\,j}(x)\,\bar{h}'_{i,\,j-1}(x) + \bar{h}'_{ij}(x) \quad (i, j = 1, 2, \cdots, n),
\end{aligned}
$$

where $\varphi_{1j}(x)$, $\varphi_{2j}(x)$, $\cdots$, $\varphi_{j-1,\,j}(x)$ are periodic of period 1, and are analytic at a sufficient distance from the real axis.‡

When we put $j = 1$ in (8) we see that the solution $\bar{h}_{11}(x)$, $\cdots$, $\bar{h}_{n1}(x)$ (i. e., the one corresponding to the largest $|\rho|$) maintains its asymptotic form in the right half plane with regard to v. These functions form the elements of the first column of the determinant $|\bar{h}_{ij}(x)|$. It is also proved in Birkhoff's paper that $|\bar{h}_{ij}(x)|$, or any minor formed from its first k columns $(k < n)$, is asymptotic to $|s_{ij}(x)|$, or its corresponding minor, with regard to v in the right half plane.

There exists a second set of intermediate solutions $h'_{ij}(x)$, such that $h'_{ij}(x) \sim s_{ij}(x)$ with regard to x in the right and with regard to v in the left half plane.

§ 2. *The formal solution and the operation of summation.*

We are thus assured of the existence of solutions of a homogeneous system. This will enable us to determine easily a formal solution of (1). For this purpose let $l_{1j}(x)$, $l_{2j}(x)$, $\cdots$, $l_{nj}(x)$ $(j = 1, 2, \cdots, n)$ be any fundamental system of solutions of (2). Write

$$
(9) \qquad g_i(x) = \sum_{j=1}^{n} \omega_j(x)\, l_{ij}(x) \qquad (i = 1, 2, \cdots, n),
$$

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* That is, asymptotic in the sense already defined.

† This means that the difference between $\bar{h}'_{ij}(x)$ and $k + 1$ terms of $s_{ij}(x)$ can be written $x^{\mu x}(\rho_j e^{-\mu})^x x^{r_j} \zeta_k(x) v^{-k}$ where $\lim_{v=\infty} \zeta_k(x) = 0$, $-\frac{1}{2}\pi \leqq \arg(x - c) \leqq \frac{1}{2}\pi$.

‡ BIRKHOFF, loc. cit., p. 266.

and let us determine the quantities $\omega_1(x)$, $\omega_2(x)$, $\cdots$, $\omega_n(x)$ so that $g_1(x)$, $g_2(x)$, $\cdots$, $g_n(x)$ will be a solution of the non-homogeneous system. If we substitute the latter expressions in (1), after first putting

$$\omega_j(x+1) = \omega_j(x) + \Delta\omega_j(x),$$

we find in virtue of (2)

$$\sum_{j=1}^{n} \Delta\omega_j(x) \, l_{ij}(x+1) = 0 \qquad\qquad (i = 1, 2, \cdots, n-1),$$

(10)

$$\sum_{j=1}^{n} \Delta\omega_j(x) \, l_{nj}(x+1) = b(x+1).$$

This is a system of n linear non-homogeneous equations in the n quantities $\Delta\omega_1(x)$, $\Delta\omega_2(x)$, $\cdots$, $\Delta\omega_n(x)$. Let $L(x)$ represent the determinant $|\,l_{ij}(x)\,|$. Further, let $L_j(x)$ be the minor of the element in the last row and jth column of $L(x)$. From (10) we have

(11) $$\Delta\omega_j(x) = (-1)^{n+j} \frac{b(x+1) \, L_j(x+1)}{L(x+1)}.$$

We are thus naturally led to the consideration of the simple non-homogeneous equation

(12) $$\Delta f(x) \equiv f(x+1) - f(x) = \theta(x).^*$$

There are two series which formally satisfy (12), namely,

(13) $$f(x) = -\theta(x) - \theta(x+1) - \cdots,$$

(13') $$f(x) = \theta(x-1) + \theta(x-2) + \cdots,$$

which, if they converge, will be actual solutions. These will be called the *direct sums* to the *right* and *left* respectively.

Besides the direct sums, a formal solution of (12) exists in the form of a contour integral due to Guichard.† We shall use it in the form adopted by Carmichael.‡ Let $\theta(x)$ be analytic outside of the region D, extending to the right and lying within a finite distance of the positive half of the real axis (see fig. 1); then if the contour $L = \infty\,AB\,\infty$ is drawn so as to enclose D, and passes between $x-1$ and x in the manner shown in the figure, the function

(14) $$f(x) = \int_{L} \frac{\theta(t)\,dt}{e^{2\pi(x-t)\sqrt{-1}} - 1}$$

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* The solution written in the form $\sum_x \theta(x)$ is called a sum formula where the operation $\sum$ is such that $\sum_{x+1} \theta(x) - \sum_x \theta(x) = \theta(x)$.

† WALLENBERG und GULDBERG, loc. cit., p. 12.

‡ Loc. cit., p. 119.

formally satisfies (12) by Cauchy's formula, since $f(x+1) - f(x)$ reduces to the integral around $ABB'A'$, and the only singularity of the integrand within this region is at $t = x$.*

If now $\theta(t)$ decreases along $A\infty$ and $B\infty$ in such a way that the integral converges, (14) will furnish an actual solution of (12).

It is obvious that in case the region D extends to the left, we can take a contour L extending to the left so as to again enclose D. The sign of the integral must in this case be changed if we integrate so as to keep the area within L on our left.
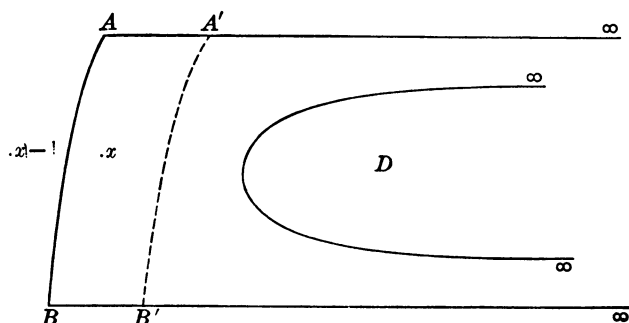
Fig. 1.

We therefore have four formal solutions of an equation of the type (12), the *direct sums* to the *right* and *left* respectively, and *contour integrals* about contours extending to the *right* or *left*.

When one of the above sums is applied to (11) we see that (9) will furnish a formal solution of (1). *Our main problem therefore is to determine whether we can so choose the system of solutions $l_{1j}(x)$, $l_{2j}(x)$, $\cdots$, $l_{nj}(x)$ $(j = 1, 2, \cdots, n)$, of (2), that the application of one or more of the four sums to (11) will lead to an analytic solution of that equation and therefore of (1).*

§ 3. *The first principal solution for $\mu > 0$.*

We choose first as the solutions of (2) the first principal solutions, $h_{1j}(x)$, $h_{2j}(x)$, $\cdots$, $h_{nj}(x)$ $(j = 1, 2, \cdots, n)$. Take, further, as the solution of (11) the direct sum to the right. Thus we have formally

$$(15) \quad \omega_j(x) = (-1)^{n+j+1} \left[\frac{b(x+1)H_j(x+1)}{H(x+1)} + \frac{b(x+2)H_j(x+2)}{H(x+2)} + \cdots \right],$$

where $H(x)$ is the determinant $| h_{1j}(x) |$, and $H_j(x)$ is the minor of this with reference to the element in the last row and jth column.

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* In case $\theta(x)$ is analytic within D, the integral in (14) reduces to (13) by a familiar theorem in the theory of residues, since the only singularities of the integrand are at x, $x+1$, $x+2$, $\cdots$. That is, we are led to (14) on writing (13) as a series of residues.

Each of the terms of (15) will be analytic save at the zeros of $H(x)$, and the poles of $b\,(x+1)$ and $H_j\,(x+1)$, or points congruent to these points on the left.

As has been stated, $h_{ij}\,(x)$ is asymptotic to $s_{ij}\,(x)$ in the right half plane, so that $H\,(x)$ is asymptotically represented by the determinant $|s_{ij}\,(x)|$. From (7) it follows that $H\,(x)$ will have no zeros in the right half plane at a sufficient distance from the origin. Since the quantities $h_{ij}\,(x)$ form a system of solutions of the homogeneous system, we see easily that $H\,(x+1)$ is equal to $A\,(x)\,H\,(x)$, where $A\,(x)$ is the determinant $|a_{ij}\,(x)|$. Therefore we have $H\,(x-1)=H\,(x)/A\,(x-1)$, and since $A\,(x)$ has no singularities in the finite part of the plane exterior to a circle of radius R, it follows that $H\,(x)$ will not vanish in the finite plane at a sufficient distance from the negative real axis, say at a distance greater than R'.

From the situation of the singularities of $h_{ij}\,(x)$ it is apparent that $H_j\,(x)$ is analytic in the finite plane at a distance greater than some quantity $\bar{R}$ from the negative real axis, while $b\,(x+1)$ is analytic if $x+1$ lies outside a circle of radius R.

Let T be the greatest of the quantities R, R', and $\bar{R}$. About the origin draw a circle of radius T, and the half-lines tangent to it and parallel to the neg-
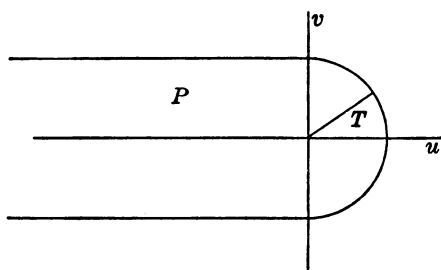

FIG. 2.

ative real axis (fig. 2). Denote by P the region enclosed by these lines and the semi-circumference in the right half plane. Then if x is any point in the finite plane exterior to P, all the terms of (15) are analytic. If x is within the region P, some of the first terms may have singularities, but all the terms from the mth remain analytic, if m be so chosen that $x+m$ is exterior to P.

It remains now to discuss the convergence of (15). Consider the mth term. We will show that it is less in absolute value than the mth term of a rapidly converging series. To do this, replace the elements of $H_j\,(x+m)$ and $H(x+m)$ by their asymptotic forms. When this is done it is evident that we can remove the factor $(x+m)^{\mu(x+m)+r_k}\,(\rho_k\,e^{-\mu})^{x+m}$ from the kth column $(k=1,\,2,\,\cdots,\,n)$ of $H\,(x+m)$. All these factors can be removed from $H_j(x+m)$

except the one for $k = j$, so that we have[*]

$$(16) \quad \frac{b(x+m)H_j(x+m)}{H(x+m)} \sim \frac{(x+m)^{\nu}}{(x+m)^{\mu(x+m)+r_j}(\rho_j e^{-\mu})^{x+m}} \left(d_j + \frac{d_j^{(1)}}{x+m} + \cdots \right),$$

where

$$(16') \qquad\qquad d_j = \frac{bS_j}{S},$$

S_j being the minor of the element in the last row and jth column of $S = |s_{ij}|$.

If we put $x + m = m_1$, we see that the typical term of our series can be written

$$\frac{m_1^c C^{m_1}}{m_1^{\mu m_1}} M(m_1),$$

where c and C are constants, and $M(m_1)$ is a uniformly limited function for m large and x in the vicinity of any point x_1. Since m_1 increases indefinitely in the successive terms of the series, it is evident that the above expression is the general term of a uniformly and rapidly converging series, irrespective of the values of c and C, provided only that $\mu > 0$, which we are assuming for the present.

By Weierstrass's theorem the series (15) will therefore represent an analytic function, provided x is exterior to P; for this makes all the terms analytic. If x is within P, the first terms of (15) may, as seen above, have poles, the remaining part of the series converging uniformly, and accordingly $\omega_j(x)$ may have a pole.

The values of $\omega_1(x), \cdots, \omega_n(x)$ when used in (9) will furnish a solution of (1) which is analytic in the finite part of the plane save for poles within P.

The solution which is obtained in this manner can be extended to the left as follows: let $A_{ji}(x)$ be the minor of the element in the jth row and ith column of $A(x)$, and solve (1) for $g_1(x), \cdots, g_n(x)$, as we may do since $A(x) \not\equiv 0$. This changes the equations to the form

$$(17) \quad g_i(x) = \sum_{j=1}^{n} (-1)^{j+i} \frac{A_{ji}(x)}{A(x)} g_j(x+1) + (-1)^{n+i+1} \frac{A_{ni}(x)}{A(x)} b(x+1)$$
$$(i = 1, 2, \cdots, n).$$

The solution can now by (17) be analytically continued to the left.

From this form of (1) we can also determine the location of the singularities of the solution without reference to those of $h_{ij}(x)$. Since the g's are analytic in the finite plane exterior to P, it follows that, in their continuation to the left, singularities will be introduced only within P at the singularities of $b(x+1)$, at the minors of $A(x)$, and at the zeros of $A(x)$. A singularity will in general recur at points congruent on the left.

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[*] The series asymptotically representing the sum, product, or quotient of any number of functions is found by adding, multiplying or dividing their respective asymptotic series according to the ordinary rules for convergent series, BOREL, *Les Séries Divergentes*, p. 30.

By using the first principal solutions of the homogeneous system, and the direct sum to the right, we have thus shown the existence of a first principal solution $g_{11}(x)$, $g_{21}(x)$, $\cdots$, $g_{n1}(x)$, *of* (1), *which is analytic in the finite plane, save for poles near the negative real axis, congruent on the left to the singularities of* $a_{ij}(x)$, $b(x+1)$, *and the zeros of* $A(x)$.

§4.   *The asymptotic form of the solution in the right half plane.*

We shall now investigate the asymptotic form of the solution $g_{11}(x)$, $g_{21}(x)$, $\cdots$, $g_{n1}(x)$. Manifestly for this purpose it is merely necessary to consider more closely the terms of (15). The asymptotic form of the typical term has already been determined, and is given in (16).

Before proceeding it will be convenient to introduce a simplification in notation. When $f(x)$ is asymptotically represented by a power series $s(x)$ of negative integral powers of x, instead of replacing $f(x)$ by the expression which is obtained on breaking off the series $s(x)$ at some point and inserting the remainder coefficient, (see footnote, §1), we shall put

$$f(x) = [a_0; x],$$

giving in each case the constant term a_0 and the argument x in the series.*

The leading factor of (16) can be written

$$(18) \qquad \frac{x^\nu}{x^{\mu x}(\rho_j e^{-\mu})^x x^{r_j+\mu}} \cdot \frac{\left(1+\dfrac{m}{x}\right)^\nu}{x^{\mu(m-1)}\left(1+\dfrac{m}{x}\right)^{r_j+\mu(x+m)}(\rho_j e^{-\mu})^m},$$

the first factor of which is independent of m. We shall omit, for the present, this factor, so that we have to consider the series

$$(19) \qquad \sum_{m=1}^{\infty} \frac{(e^\mu/\rho_j)^m}{x^{\mu(m-1)}\left(1+\dfrac{m}{x}\right)^{r_j-\nu+\mu(x+m)}}[d_j; x+m].$$

Consider in the first place the first k terms of (19), $k < |x|$, that is, the finite series

$$(20) \qquad \frac{e^\mu/\rho_j}{\left(1+\dfrac{1}{x}\right)^{r_j-\nu+\mu(x+1)}}[d_j; x+1] + \frac{(e^\mu/\rho_j)^2}{x^\mu\left(1+\dfrac{2}{x}\right)^{r_j-\nu+\mu(x+2)}}[d_j; x+2]$$

$$+ \cdots + \frac{(e^\mu/\rho_j)^k}{x^{\mu(k-1)}\left(1+\dfrac{k}{x}\right)^{r_j-\nu+\mu(x+k)}}[d_j; x+k],$$

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*This notation is similar to that used by BIRKHOFF, these T r a n s a c t i o n s, vol. 9 (1908), pp. 373–395.

and let us examine the first term in detail.   We have

$$\frac{e^{\mu}/\rho_j}{\left(1+\dfrac{1}{x}\right)^{r_j-\nu+\mu\,(x+1)}} = \frac{e^{\mu-\{r_j-\nu+\mu\,(x+1)\}\,\log\,(1+1/x)}}{\rho_j} = \frac{e^{-\frac{2(r_j-\nu)+\mu}{2x}+\cdots}}{\rho_j} = \frac{1}{\rho_j}+\frac{\lambda_1}{x}+\frac{\lambda_2}{x^2}\cdots.$$

If then, in the series $d_j + d_j^{(1)}/(x+1) + \cdots$, we put

$$\frac{1}{x+1} = \frac{1}{x}\left(1-\frac{1}{x}+\cdots\right),$$

the first term of (20) can be written

$$\frac{d_j}{\rho_j}+\frac{\xi_1}{x}+\cdots+\frac{\xi_{\mu k}+E_{\mu k}(x)}{x^{\mu k}},$$

where $E_{\mu k}(x)$ is a uniformly bounded function for $|x|$ sufficiently large and x in the right half plane, since it is composed of remainder terms in uniformly convergent series and the remainder coefficients in asymptotic series.[*]   The terms of (20) from the second on can be treated in the same way; but we notice that their expansions begin with $x^{-\mu}$, $x^{-2\mu}$, $\cdots$, $x^{-(k-1)\mu}$, respectively.   The first k terms can therefore be written

$$\frac{d_j}{\rho_j}+\frac{l_1}{x}+\frac{l_2}{x^2}+\cdots+\frac{l_{\mu k}+\eta_{\mu k}(x)}{x^{\mu k}},$$

where $\eta_{\mu k}(x)$ is limited for x large in the right half plane.

It remains now to consider the portion of (19) from the term $m = k+1$ on.   We will find an upper limit for the sum of these terms.   In the first place we notice that the factor

$$\left(1+\frac{m}{x}\right)^{r_j-\nu+\mu\,(x+m)},$$

which occurs in the denominator, is greater in absolute value than some quantity K, and in general increases with m, since x is in the right half plane. Also the expression $[d_j;\, x+m]$ can be replaced by a quantity larger in absolute value.   Consequently the portion of our series from the $(k+1)$th term on is less in absolute value than the series whose general term is

$$\frac{M}{K\,|\,x\,|^{\mu\,(m-1)}}\left|\frac{e^{\mu}}{\rho_j}\right|^m,$$

---

[*] To prove that $f(x) \sim a_0 + a_1/x + a_2/x^2 + \cdots$, we need show only that $\zeta_k(x)$ in $f(x) = a_0 + a_1/x + \cdots + [a_k + \zeta_k(x)]/x^k$ remains bounded for every k, since $\zeta_k(x) = [a_{k+1} + \zeta_{k+1}(x)]/x$, and hence $\lim\limits_{x=\infty}\zeta_k(x) = 0$. This principle will be used in what follows.

where M does not increase with $|x|$ and K does not decrease. This is a geometrical progression, whose ratio can be made as small as we please by taking $|x|$ large. The sum of the comparison series is therefore of the order of its first term, which is $1/|x|^{\mu k}$, since we are starting with m equal to $k+1$. This is of the same order as the last term of the expression found above for the first m terms, and consequently (19) can be written $[d_j/\rho_j;\, x]$, where the series can be taken to as many terms as desired, since k was arbitrarily large.

An expression for $\omega_j(x)$ will be obtained by multiplying $[d_j/\rho_j;\, x]$ by $(-1)^{n+j+1}$, (see (15)), and by the first factor in (18), which was removed from (19).

The quantity we wish to determine is $\omega_j(x)\, h_{ij}(x)$. Therefore, since $h_{ij}(x) = x^{\mu x}(\rho_j e^{-\mu})^x x^{r_j}[s_{ij};\, x]$, we have

$$(21) \qquad \omega_j(x)\, h_{ij}(x) = x^{\nu-\mu}[(-1)^{n+j+1} d_j s_{ij}/\rho_j;\, x].$$

Letting $j = 1, 2, \cdots, n$ and adding one finds

$$(22) \qquad g_{i1}(x) = x^{\nu-\mu}[h_i;\, x] \qquad (i = 1, 2, \cdots, n).$$

where

$$h_i = (-1)^n \left(\frac{d_1 s_{i1}}{\rho_1} - \frac{d_2 s_{i2}}{\rho_2} + \cdots + (-1)^{n+1} \frac{d_n s_{in}}{\rho_n} \right).$$

From (6) and (16′) we obtain

$$(23) \qquad h_i = \frac{(-1)^n b}{SA} \sum_{j=1}^{n} (-1)^{j+1} \rho_1 \rho_2 \cdots \rho_{j-1} \rho_{j+1} \cdots \rho_n s_{ij} S_j$$
$$(i = 1, 2, \cdots, n).$$

Let us now try to find series which will formally satisfy (1). Write

$$(24) \qquad g_i(x) = x^{\nu-\mu} \left(g_i + \frac{g_i^{(1)}}{x} + \cdots \right) \qquad (i = 1, 2, \cdots, n),$$

and substitute in (1). When we develop the resulting equations according to decreasing powers of x it is evident, on remembering that $\mu > 0$, that we can uniquely determine the successive constants g_i, $g_i^{(1)}$, $\cdots$ by comparing coefficients. The developments in all the equations start with x^ν; equating its coefficients we find

$$0 = \sum_{j=1}^{n} a_{ij} g_j \qquad (i = 1, 2, \cdots, n-1),$$

$$0 = \sum_{j=1}^{n} a_{nj} g_j + b.$$

From this we have for the first coefficient in $g_i(x)$

$$(25) \qquad g_i = (-1)^{n+i+1} \frac{bA_{ni}}{A} \qquad (i = 1, 2, \cdots, n),$$

where A_{ni} is the minor of the determinant $A = |a_{ij}|$ with reference to the element in the last row and ith column.

We call the series $g_1(x), \cdots, g_n(x)$ the formal solution of (1). These series are obtained by direct substitution, and in general diverge.

Since the expansions in the expression for $g_{i1}(x)$ and in the formal series $g_i(x)$ $(i = 1, 2, \cdots, n)$ are of the same form, the question naturally arises as to whether their coefficients are the same. We can show that this is the case in the following way. We know that $g_{11}(x), \cdots, g_{n1}(x)$ as given in (22) satisfy (1), and if we substitute this form of the solutions in (1) and compare coefficients of quantities of the same order we see that we must have precisely the same equations satisfied by h_i, $h_i^{(1)}$, $\cdots$ that were satisfied by g_i, $g_i^{(1)}$, $\cdots$ $(i = 1, 2, \cdots, n)$. Since these equations had unique solutions, this means that corresponding coefficients in the series $g_{i1}(x)$ and $g_i(x)$ $(i = 1, 2, \cdots, n)$ are equal.

Thus we see that the *first principal solution is asymptotically represented by the formal solution in the right half plane.*

Although we have shown easily that the constants in the series for $g_{i1}(x)$ must be the same as those in the formal series $g_i(x)$, it is interesting to note that we can show by direct computation that h_i is identical with g_i. To do this we must eliminate the quantities ρ_j and s_{ij} from (23).

When we substitute the formal series (3) in (2) and compare coefficients of $x^{\mu+r_j}$ we find

$$\sum_{k=1}^{n} (a_{ik} - \delta_{ik}\rho_j) s_{kj} = 0 \qquad (i = 1, 2, \cdots, n),$$

where $\delta_{ik} = 0$ if $i \neq k$, and $\delta_{ii} = 1$. Since ρ_j is a solution of (4), the last equations have solutions other than $s_{kj} = 0$. The first $n - 1$ of them give

$$\rho_j s_{ij} = \sum_{k=1}^{n} a_{ik} s_{kj} \qquad (i = 1, 2, \cdots, n-1)!.$$

If now we introduce the quantities $\rho_1, \cdots, \rho_{j-1}, \rho_{j+1}, \cdots, \rho_n$ into the successive columns of the determinant S_{nj} and eliminate the ρ's by means of the above expressions, we obtain the relation

$$(26) \qquad \rho_1 \cdots \rho_{j-i} \rho_{j+1} \cdots \rho_n S_{nj} = A_{n1} S_{1j} + A_{n2} S_{2j} + \cdots + A_{nn} S_{nj},$$

where A_{ij} and S_{ij} are the minors of A and S respectively with reference to the element in the ith row and jth column, and where we have put $S_{nj} = S_j$, the two being identical by definition.

If in (26) we let $j = 1, 2, \cdots, n$, and substitute in (23), we find for $i = 1$

$$h_1 = \frac{(-1)^n b}{SA} \sum_{k=1}^{n} A_{nk} \sum_{j=1}^{n} (-1)^{j+1} s_{1j} S_{kj} = \frac{(-1)^n b A_{n1}}{A} = g_1,$$

since

$$\sum_{j=1}^{n} (-1)^{j+1} s_{1j} S_{kj} = \begin{cases} S \text{ if } k = 1, \\ 0 \text{ if } k \neq 1. \end{cases}$$

In a similar way we can show $h_i = g_i$ $(i = 2, 3, \cdots, n)$.

We summarize the results of this and the preceding section in the following:

THEOREM I. *If $\mu > 0$, the system of non-homogeneous linear difference equations (1) has a first principal solution $g_{11}(x), g_{21}(x), \cdots, g_{n1}(x)$. All the constituent elements of this solution are analytic throughout the finite plane, save for poles at the poles of the functions $a_{ij}(x), b(x+1)$, at the zeros of the determinant $|a_{ij}(x)|$, and at points congruent to these points on the left. There exist n series $g_1(x), g_2(x), \cdots, g_n(x)$ which formally satisfy the system (1), and whose coefficients can be determined by direct substitution. These series in general diverge, but when x approaches infinity in any right half plane, $g_{i1}(x)$ is represented asymptotically by $g_i(x)$.*

This theorem is supplemented by Theorem II of § 6.

§ 5.   *A formal property of the direct sum.*

The solution of (1) which we have found was determined by making use of one of the sets of principal solutions of (2). The question now arises as to whether a different solution of (1) would be obtained if we used a different fundamental system of solutions of (2) and employed again the direct sum to the right. The answer is in the negative as will be shown.

Let $l_{1j}(x), l_{2j}(x), \cdots, l_{nj}(x)$ $(j = 1, 2, \cdots, n)$ be any other fundamental system of solutions of (2). From the theory of homogeneous systems we have

$$l_{1j}(x) = \sum_{i=1}^{n} \varphi_{ij}(x) h_{1i}(x),$$

(27) $$l_{2j}(x) = \sum_{i=1}^{n} \varphi_{ij}(x) h_{2i}(x), \qquad (j = 1, 2, \cdots, n),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$l_{nj}(x) = \sum_{i=1}^{n} \varphi_{ij}(x) h_{ni}(x),$$

where the functions $\varphi_{ij}(x)$ are periodic of period 1, and the determinant $\Phi(x) = |\varphi_{ij}(x)| \neq 0$.

Denote by $\bar{g}_i(x)$ $(i = 1, 2, \cdots, n)$ the solution of (1) arising from (27). Then one obtains the formula analogous to (9),

(28) $$\bar{g}_i(x) = \sum_{j=1}^{n} \theta_j(x) l_{ij}(x) \qquad (i = 1, 2, \cdots, n),$$

where the functions $\theta_j(x)$ $(j = 1, 2, \cdots, n)$ are to be determined. As before, let $L(x)$ denote the determinant $|\,l_{ij}(x)\,|$, and $L_j(x)$ the minor of this with reference to the element in the last row and jth column; then we have by (15), since the work to that point was purely formal,

$$(29) \quad \theta_j(x) = (-1)^{n+j+1}\left[\frac{b(x+1)L_j(x+1)}{L(x+1)} + \cdots\right] \quad (j = 1, 2, \cdots, n).$$

From (27) it follows that $L(x) = H(x) \cdot \Phi(x)$. Similarly, we can show that

$$L_j(x) = \Phi_{1j}(x)H_1(x) + \Phi_{2j}(x)H_2(x) + \cdots + \Phi_{nj}(x)H_n(x)$$
$$(j = 1, 2, \cdots, n),$$

where $\Phi_{ij}(x)$ is the minor of $\Phi(x)$ with regard to the element in the ith row and jth column, and $H_j(x)$ is the function in § 3. Therefore, since $|\Phi(x)|$ and all its minors are periodic functions, we obtain

$$\theta_j(x) = \frac{(-1)^{n+j+1}}{\Phi(x)}\sum_{i=1}^{n}\Phi_{ij}(x)\left(\frac{b(x+1)H_i(x+1)}{H(x+1)} + \cdots\right) \quad (j=1, 2, \cdots, n),$$

or by virtue of (15),

$$\theta_j(x) = (-1)^{j-1}\frac{\Phi_{1j}(x)}{\Phi(x)}\omega_1(x) + (-1)^{j-2}\frac{\Phi_{2j}(x)}{\Phi(x)}\omega_2(x) + \cdots$$
$$+ \frac{\Phi_{jj}(x)}{\Phi(x)}\omega_j(x) + \cdots + (-1)^{n-j}\frac{\Phi_{nj}(x)}{\Phi(x)}\omega_n(x).$$

Substitute the values just obtained for $\theta_1(x), \cdots, \theta_n(x)$ in (28); we find

$$\bar{g}_i(x) = \frac{\omega_1(x)}{\Phi(x)}\sum_{k=1}^{n}(-1)^{k+1}\Phi_{1k}(x)l_{ik}(x)$$
$$- \frac{\omega_2(x)}{\Phi(x)}\sum_{k=1}^{n}(-1)^{k+1}\Phi_{2k}(x)l_{ik}(x) + \cdots$$
$$+ (-1)^{n-1}\frac{\omega_n(x)}{\Phi(x)}\sum_{k=1}^{n}(-1)^{k+1}\Phi_{nk}(x)l_{ik}(x).$$

If we make use now of the values of $l_{ij}(x)$ as given by (27), the last expression becomes, upon rearranging,

$$\bar{g}_i(x) = \frac{\omega_1(x)}{\Phi(x)}\sum_{j=1}^{n}h_{ij}(x)\sum_{k=1}^{n}(-1)^{k+1}\varphi_{jk}(x)\Phi_{1k}(x)$$
$$- \frac{\omega_2(x)}{\Phi(x)}\sum_{j=1}^{n}h_{ij}(x)\sum_{k=1}^{n}(-1)^{k+1}\varphi_{jk}(x)\Phi_{2k}(x)$$
$$+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$+ (-1)^{n-1}\frac{\omega_n(x)}{\Phi(x)}\sum_{j=1}^{n}h_{ij}(x)\sum_{k=1}^{n}(-1)^{k+1}\varphi_{jk}(x)\Phi_{nk}(x).$$

Since

$$\sum_{k=1}^{n} (-1)^{k+1} \varphi_{jk}(x) \Phi_{lk}(x) = \begin{cases} 0, & \text{if } j \neq l, \\ (-1)^{l+1} \Phi(x), & \text{if } j = l, \end{cases}$$

the above expression for $\bar{g}_i(x)$ easily simplifies, and finally we have

$$\bar{g}_i(x) = \omega_1(x) h_{i1}(x) + \omega_2(x) h_{i2}(x) + \cdots + \omega_n(x) h_{in}(x).$$

Therefore $\bar{g}_i(x) = g_{i1}(x)$ $(i = 1, 2, \cdots, n)$, so that we have the same solution obtained in § 3.

This result can be stated as follows: *When the equation* (11) *can be solved by means of a direct sum to the right, the solution of* (1) *obtained by the method of variation of constants is independent of the choice of the fundamental system of solutions of the associated homogeneous system, for in this case the series which occur are convergent, and therefore can be rearranged as above.*

§ 6.   *The asymptotic form of the solution in the left half plane.*

By an application of the results of the last section we can determine the asymptotic form of the first principal solution in the left half plane.

Take as the solutions of (2) the intermediate solutions $\bar{h}'_{ij}(x), \bar{h}'_{2j}(x), \cdots,$ $\bar{h}'_{nj}(x)$ $(j = 1, 2, \cdots, n)$, of § 1.   Put

$$(30) \qquad\qquad g'_{i1}(x) = \sum_{j=1}^{n} \bar{\omega}'_j(x) \bar{h}'_{ij}(x) \qquad\qquad (i = 1, 2, \cdots, n),$$

and apply the direct sum to the right in order to determine $\bar{\omega}'_1(x), \cdots,$ $\bar{\omega}'_n(x)$; a formal value of $\bar{\omega}'_j(x)$ will be

$$(31) \qquad\qquad \bar{\omega}'_j(x) = (-1)^{n+j+1} \left[\frac{b(x+1)\bar{H}'_j(x+1)}{\bar{H}'(x+1)} + \cdots \right],$$

where $H'(x) = |\bar{h}'_{ij}(x)|$, and $\bar{H}'_j(x)$ is the minor of $\bar{H}'(x)$ with respect to the element in the last row and jth column.   The same convergence proof used in § 3 applies here, provided x is sufficiently distant from the real axis, since we have $\bar{h}'_{ij}(x) \sim s_{ij}(x)$ with regard to v in the right half plane.

According to the result of the last section this new solution must be identical with the first principal solution.   As derived above it is known to exist only at a sufficient distance from the whole extent of the real axis.

The typical term of (31) will again have the asymptotic form (16), the representation holding with regard to x in the left, and with regard to v in the right half plane.   To find the asymptotic form of the series, divide it into two parts: the first k terms, where k is such that $x + k + 1 = x_1$ is a point in the right half plane near the imaginary axis, and secondly, the remaining terms.

Take first the case where x approaches infinity along any ray making an

acute angle with the negative real axis. The first k terms can again be written in the form (20). By the definition of k we have in this case $k < |x|$, so that the quantity

$$\left(1 + \frac{m}{x}\right)^{r_j - \nu + \mu (x+m)} \qquad (m < k)$$

which appears in the denominator of the typical term can be developed in the same way that the first term of (20) was treated, since all the expansions are valid if $m < |x|$. Thus we have immediately that the first k terms of (31) when multiplied by $\bar{h}'_{ij}(x)$ can be put in the form (21).

Consider now the terms of (31) from the $(k+1)$th on. They can be written

$$\sum_{m=0}^{\infty} \frac{b(x_1 + m)\,\bar{H}'_j(x_1 + m)}{\bar{H}'(x_1 + m)}$$

Since $x_1, x_1 + 1, \cdots$ are points in the right half plane, the asymptotic form of the typical term of this series is again given by (16), the representation being now with regard to v. The series therefore takes the form

$$(32) \qquad \sum_{m=0}^{\infty} \frac{(e^\mu / \rho_j)^{x_1 + m}}{(x_1 + m)^{\mu(x_1+m)+r_j-\nu}} M_m (x_1 + m),$$

where $M_m(x_1 + m)$ remains uniformly bounded for $|v|$ large.

Let $x_1 = u_1 + v\sqrt{-1}$, $r_j = \xi + \zeta\sqrt{-1}$; then we can so choose x_1 that $\mu u_1 + \xi - \nu > 0$. We can also choose x so that the sign of $\mu v + \zeta$ is the same as that of v. It can then easily be shown that the absolute value of the denominator of the $(m+1)$th term of (32) is greater than $|x_1^{\mu(x_1+m)+r_j-\nu}|$. The absolute value of (32) is therefore less than

$$\frac{M\,|\,(e^\mu / \rho_j)^{x_1}\,|}{|\,x_1^{\mu x_1 + r_j - \nu}\,|} \sum_{m=0}^{\infty} \left|\frac{e^\mu}{x_1^\mu\,\rho_j}\right|^m,$$

where M is a constant not depending on x. Since $|\,e^\mu / x^\mu\,\rho_j\,|$ can be made less than unity, the sum of the last series is

$$(33) \qquad \frac{M'\,e^{(\mu v + \zeta)\arg x_1}}{|x_1|^{\mu u_1 + \xi - \nu}} \left|\left(\frac{e^\mu}{\rho_j}\right)^{x_1}\right| < M'\,e^{(\mu v + \zeta)\arg x} \left|\left(\frac{e^\mu}{\rho_j}\right)^{x_1}\right|,$$

where M' is bounded.

The terms of (31) from the $(k+1)$th on are also to be multiplied by $\bar{h}_{ij}(x)$, the absolute value of the principal part of which is $|\,x\,|^{\mu u + \xi}\,|\,(\rho_j e^{-\mu})^x\,|\,e^{-(\mu v + \zeta)\arg x}$. From (33) the absolute value of the product is seen to be less than $M'\,|\,x\,|^{\mu u + \xi}\,|\,\rho_j\,e^{-\mu}\,|^{u-u_1}$. This vanishes more rapidly than any power of $1/x$ along the rays considered, inasmuch as $|\,\rho_j\,e^{-\mu}\,|$ is constant; for along these rays u increases negatively. Consequently the terms

of (31) from the $(k+1)$th do not affect the asymptotic form of $\bar{\omega}'_j(x)\,\bar{h}'_{ij}(x)$, which is therefore given by (21).*

It remains now to consider the case where x approaches infinity parallel to the negative real axis at a sufficient distance from it. A slight modification in the treatment of the first k terms must be made, for the quantity $(1+m/x)^{r_j-\nu+\mu(x+m)}$ cannot, in all the terms, be developed as before, for in the last terms $|\,m/x\,|$ will approach very close to unity. The modification to be made is apparent: we divide the first k terms into two parts; the first l of them, where, for instance, l is the smallest integer such that $|\,x+l\,|<\frac{1}{2}\,|\,x\,|$ (such a choice is always possible when x goes to infinity parallel to the negative real axis), and secondly the remaining terms.

The first l terms when multiplied by $\bar{h}'_{ij}(x)$ will again have the asymptotic form (21). We can show readily that the second group of terms when multiplied by the same quantity will contribute nothing to the asymptotic from of $\bar{\omega}'_j(x)\,\bar{h}'_{ij}(x)$. The details of the argument are similar to the treatment of the terms after the kth, and we will merely outline them in a note.†

For this case we therefore have again that $\bar{\omega}'_j(x)\,\bar{h}'_{ij}(x)$ is given by (21). From this it follows immediately that $g'_{i1}(x)\sim g_i(x)$ in the left half plane. Remembering that the new solution is identical with the first principal solution, we have:

THEOREM II. *If $\mu>0$, the first principal solution is asymptotic to the formal series in the left half plane at a sufficient distance from the real axis.*‡

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* If $f(x)=\varphi(x)+\theta(x)$, where $\varphi(x)\sim x^n(a_0+a_1/x+\cdots)$ along some ray, while $\theta(x)$ becomes small of order $1/a^{|x|}$ $(a>1)$, then $f(x)\sim x^n(a_0+a_1/x+\cdots)$ since $1/a^{|x|}$ is smaller than any power of $1/x$ for x large.

† The terms in question can be written in the form (32), $l\leqq m\leqq k$, and x replacing x_1. We can show first that the absolute value of the typical term would be increased if we changed $(x+m)^{\mu x+r_j-\nu}$ in the denominator to $(x+l)^{\mu x+r_j-\nu}$, and the factor $(x+m)^{\mu m}$ to $x_0^{\mu m}$, where x_0 is the smallest in absolute value of $x+l,\cdots,x+k$. We obtain then for a series greater term by term in absolute value, a geometrical progression, which can be summed, the sum being less than

$$\frac{M\,|\,(e^\mu/\rho_j)^{x+l}\,|}{|\,(x+l)^{\mu x+r_j-\nu}\,|\cdot|\,x_0\,|^{\mu l}}<M\,|\,(x/2)^{\nu-\mu x-r_j}(e^\mu/\rho_j)^x\,|,$$

since $\left|\dfrac{e^\mu}{x_0^\mu\rho_j}\right|<1$, and $|\,x+l\,|<|\,x/2\,|$. The product of this and the absolute value of the leading factor of the asymptotic form of $\bar{h}'_{ij}(x)$ is easily seen to be

$$M\,|\,(x/2)^\nu 2^{\mu x+r_j}\,|=\frac{M\,|\,x\,|^\nu 2^{\mu u+\xi}}{2^\nu}$$

$(x=u+v\sqrt{-1})$. Since u is practically equal to $-|\,x\,|$, and M is a fixed number, this decreases more rapidly than any power of $1/x$.

‡ It should be noted that the asymptotic character of this solution of the non-homogeneous equation is somewhat different from that of the solutions of a homogeneous equation. For instance, the asymptotic form of $\Gamma(x)$ holds only for $-\pi<\arg x<\pi$. That functions

It is interesting here to refer to (17). The quantities $A_{ji}(x)/A(x)$ $(j, i = 1, 2, \cdots, n)$, which occur there have zeros of order μ at infinity. Consequently, if we assume that there exists a constant λ such that $\lim x^{\lambda} g_i(x) = k_i$, k_i finite, as x approaches infinity parallel to the negative real axis, equation (17) gives

$$\lim x^{\lambda} g_i(x) = \sum_{j=1}^{n} (-1)^{j+1} \frac{A_{ji}}{A} \left(1 + \frac{\bar{a}_{ij}}{x} + \cdots\right) \lim \frac{x^{\lambda} g_j(x)}{x^{\mu}}$$

$$+ \lim (-1)^{n+i+1} \frac{bA_{ni}}{A} \left(1 + \frac{\bar{b}_i}{x} + \cdots\right) x^{\lambda+\nu-\mu}.$$

From this relation we see that we must have

$$\lambda = \mu - \nu, \qquad k_i = (-1)^{n+i+1} \frac{bA_{ni}}{A},$$

so that

$$g_{i1}(x) = x^{\nu-\mu} \left[(-1)^{n+i+1} \frac{bA_{ni}}{A} + \zeta_i(x)\right] \quad (i = 1, 2, \cdots, n),$$

where $\zeta_i(x)$ tends to zero as $|x|$ becomes infinite. From (24) and (25) it is clear this is obtained by breaking off the series $g_i(x)$ after the first term.

§7. *The second principal solution for $\mu > 0$.*

We shall now show the existence of a solution of (1) defined throughout the finite plane, and whose asymptotic form is known in the whole left half plane. It will of course differ from the first principal solution merely by a solution of the homogeneous system (2).

Take as solutions of (2) the principal solutions on the left, that is, $\bar{h}_{1j}(x)$, $\bar{h}_{2j}(x)$, $\cdots$, $\bar{h}_{nj}(x)$ $(j = 1, 2, \cdots, n)$, where $\bar{h}_{ij}(x) \sim s_{ij}(x)$ in the left half plane. Put

$$(34) \qquad \qquad g_{i2}(x) = \sum_{j=1}^{n} \bar{\omega}_j(x) \bar{h}_{ij}(x) \qquad (i = 1, 2, \cdots, n).$$

We find in precisely the same way as before

$$(35) \qquad \qquad \Delta \bar{\omega}_j(x) = (-1)^{n+j} \frac{b(x+1) \bar{H}_j(x+1)}{\bar{H}(x+1)},$$

where $\bar{H}(x)$ is the determinant $|\bar{h}_{ij}(x)|$, while $\bar{H}_j(x)$ is the minor of this with respect to the element in the last row and jth column.

If we attempt to solve (35) by the direct sum to the left, which we would naturally try in this case, the series obtained is found to diverge. We can,

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actually exist with the property specified in Theorem II can be shown by considering the series $1/x + 1/(x+1)^2 + 1/(x+2)^3 + \cdots$. It is easy to prove that this function is asymptotic to its formal expansion in powers of $1/x$, with regard to x, for $-\pi < \arg x < \pi$, and also for $\arg x = \pm \pi$, provided that $|v| > a > 1$, where $x = u + v\sqrt{-1}$.

however, apply the integral formula (14). To prove this, we note first that the situation of the singularities of $\bar{H}_j(x)$, and the fact that $\bar{H}(x) \sim |s_{ij}(x)|$ with regard to v in the right half plane, show that the expression on the right in (35) is analytic outside some region D, such as was described before. Accordingly the expression

$$(36) \qquad \bar{\omega}_j(x) = (-1)^{n+j} \int_L \frac{b(t+1)\bar{H}_j(t+1)}{\bar{H}(t+1)} \frac{dt}{e^{2\pi(x-t)\sqrt{-1}} - 1}$$

will formally satisfy (35), and in case the integral converges suitably it will represent an analytic solution. The contour L is the contour $\infty AB \infty$ used earlier.

To examine the nature of the integrand we make use of the intermediate solutions; for although, as noted above, $\bar{H}(t)$ is asymptotic to the determinant $|s_{ij}(t)|$ with regard to v in the right half plane, the minor $\bar{H}_j(t)$ is not asymptotic to the corresponding minor of $|s_{ij}(t)|$, unless $j = n$, (see § 1). We therefore replace the elements of $\bar{H}_j(x)$ by their values as given by (8). When we do this, the resulting determinant can be simplified so as to give finally

$$(37) \quad \bar{H}_j(x) = \bar{H}_j'(x) + \varphi_{j+1}(x)\bar{H}_{j+1}'(x) + \cdots + \varphi_n(x)\bar{H}_n'(x),$$

where, as in the preceding section, $H_j'(x)$ is the minor of $\bar{H}'(x) = |\bar{h}_{ij}'(x)|$ with respect to the element in the last row and jth column, while $\varphi_{j+1}(x)$, $\cdots$, $\varphi_n(x)$ are periodic functions, and are analytic at a sufficient distance from the real axis.

The first factor of the integrand in (36) can now be written

$$(38) \quad \frac{b(t+1)\bar{H}_j(t+1)}{\bar{H}(t+1)} = \sum_{i=j}^{n} \varphi_i(t) \frac{b(t+1)\bar{H}_i'(t+1)}{\bar{H}(t+1)}, \qquad \varphi_j(t) = 1.$$

When we make use of the asymptotic form of the quantities involved, the first term of (38) becomes, (see (16)),

$$(39) \qquad \frac{(e^\mu / \rho_j)^{i+1}}{(t+1)^{\mu(i+1)+r_j-\nu}} \left(d_j + \frac{d_j^{(1)}}{t} + \frac{\eta_j(t)}{v} \right),$$

where $\eta_j(t)$ is limited, decreasing for large $|v|$ ($t = u + v\sqrt{-1}$). When $|v|$ is sufficiently great it is evident that (39) will decrease along $A\infty$ and $B\infty$ in approximately the manner of an exponential function, inasmuch as we are assuming $\mu > 0$. Since the functions $\varphi_{j+1}(t)$, $\cdots$, $\varphi_n(t)$ are periodic and limited along any line parallel to the real axis, like results hold for all the terms of (38). The quantity $(e^{2\pi(x-t)\sqrt{-1}} - 1)^{-1}$ remains finite along L; consequently the integral will converge in such a manner that $\bar{\omega}_j(x)$ is an analytic function outside of the region D.

With these values of $\bar{\omega}_j(x)$, equation (34) gives a solution of (1) analytic on the left outside of D. This solution can be extended to the right by means of the equations (1) themselves. From the fact that $a_{ij}(x)$ and $b(x+1)$ are analytic in the finite plane outside a certain circle, the extended solution can have singularities only at points congruent on the right to the singularities of $a_{ij}(x)$ and $b(x+1)$.

We now come to the asymptotic form of the solution. For this purpose we break up the contour L into two contours, L_1 and L_2 in the way that Birkhoff has done.* Here L_1 is a fixed contour $\infty AB\infty$, outside of which the integrand in (36) is analytic, and in case x lies between $A\infty$ and $B\infty$ extended to the left, L_2 is a loop circuit around x and the first point, $x+l$, to the left of AB, that is congruent to x; while if x is above $A\infty$ or below $B\infty$, L_2 is a loop circuit to infinity parallel to the positive real axis. In either case L_1 and L_2 are together equivalent to L, and L_1 can be deformed so that no point congruent to x falls on it.
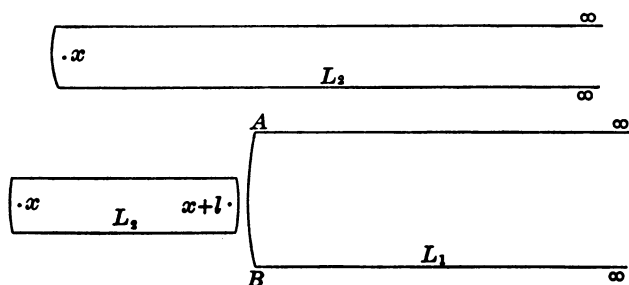
FIG. 3.

Denote the parts of $\bar{\omega}_j(x)$ arising from L_1 and L_2 by $\bar{\omega}_j^{(1)}(x)$ and $\bar{\omega}_j^{(2)}(x)$ respectively. Then

$$(40) \qquad g_{i2}(x) = \sum_{j=1}^{n} \bar{\omega}_j^{(1)}(x)\,\bar{h}_{ij}(x) + \sum_{j=1}^{n} \bar{\omega}_j^{(2)}(x)\,\bar{h}_{ij}(x) \quad (i = 1, 2, \cdots, n).$$

Consider first the case where x lies between $A\infty$ and $B\infty$; we have

$$(41) \qquad \bar{\omega}_j^{(2)}(x) = (-1)^{n+j+1} \sum_{m=1}^{l+1} \frac{b(x+m)\,\bar{H}_j(x+m)}{\bar{H}(x+m)}.$$

The asymptotic form of the typical term is again given by (16). Also we see

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* The reason for this is evident. The singularities of the integrand in (36) are at the congruent points x, $x+1$, $x+2$, $\cdots$, due to the zeros of $e^{2\pi(x-t)\sqrt{-1}} - 1$, and the fixed singularities of $\dfrac{b(t+1)\,\bar{H}_j(t+1)}{\bar{H}(t+1)}$, which are in the vicinity of the positive real axis. The residues at the first points will produce a series of the form (13), and the sum of the residues at the last points will be a periodic function. It is therefore natural to consider the two different groups of terms separately.

that precisely the treatment applied to the first k terms of (31) for the case where x approaches infinity parallel to the negative real axis, holds for (41), provided AB is taken far enough to the left. Therefore $\bar{\omega}_j^{(2)}(x)\,\bar{h}_{ij}(x)$ is again given by (21), so that we have

$$(42) \qquad \sum_{j=1}^{n} \bar{\omega}_j^{(2)}(x)\,\bar{h}_{ij}(x) \sim g_i(x) \qquad (i = [1, 2, \cdots, n),$$

for x between $A\infty$ and $B\infty$.

Next let x lie above $A\infty$ or below $B\infty$; then (41) is replaced by the infinite series

$$\bar{\omega}_j^{(2)}(x) = (-1)^{n+j+1} \sum_{m=1}^{\infty} \frac{b(x+m)\,\bar{H}_j(x+m)}{\bar{H}(x+m)}.$$

Since the fundamental system $\bar{h}_{1j}(x)$, $\bar{h}_{2j}(x)$, $\cdots$, $\bar{h}_{nj}(x)$ can be expressed in terms of the solutions $h_{1j}(x)$, $h_{2j}(x)$, $\cdots$, $h_{nj}(x)$ $(j = 1, 2, \cdots, n)$, it follows from the results of § 5 that

$$(43) \qquad \sum_{j=1}^{n} \bar{\omega}_j^{(2)}(x)\,\bar{h}_{ij}(x) = \sum_{j=1}^{n} \omega_j(x)\,h_{ij}(x) = g_{i1}(x) \qquad (i = 1, 2, \cdots, n).$$

But we have found that $g_{i1}(x) \sim g_i(x)$ along any ray making an angle with the negative real axis, or parallel to that axis at a sufficient distance from it. We see then that (42) is true throughout the left half plane.

We still have to examine the quantity $\bar{\omega}_j^{(1)}(x)$. L_1 is a fixed contour, so that $\bar{\omega}_j^{(1)}(x)$ is periodic, and, moreover, its absolute value will have a finite upper limit. Furthermore, we can write

$$\bar{\omega}_j^{(1)}(x) = \frac{(-1)^{n+j}}{e^{2\pi x\sqrt{-1}}} \int_{L_1} \frac{b(t+1)\,\bar{H}_j(t+1)}{\bar{H}(t+1)} \frac{dt}{e^{-2\pi t\sqrt{-1}} - e^{-2\pi x\sqrt{-1}}}.$$

The integral approaches a constant, C_j, when x approaches infinity along the negative imaginary axis, so that

$$(44) \qquad \lim_{x=\infty} e^{2\pi x\sqrt{-1}}\,\bar{\omega}_j^{(1)}(x) = C_j \qquad \left(\text{for arg } x = \frac{3\pi}{2}\right).$$

From (36) itself we see

$$(45) \qquad \lim_{x=\infty} \bar{\omega}_j^{(1)}(x) = c_j \qquad \left(\text{for arg } x = \frac{\pi}{2}\right),$$

where c_j is constant.*

Consider now the product $\bar{\omega}_j^{(1)}(x)\,\bar{h}_{ij}(x)$. It can easily be shown that the leading factor $x^{\mu_x}(\rho_j e^{-\mu})^x x^{r_j}$ in the asymptotic form of $\bar{h}_{ij}(x)$ vanishes more rapidly than any power of $1/x$ along any ray in the left half plane, including the positive axis of imaginaries. Then, since $\bar{\omega}_j^{(1)}(x)$ remains bounded, $\bar{\omega}_j^{(1)}(x)\,h_{ij}(x)$ will also vanish more rapidly than any power of $1/x$, and con-

*The inversion of order of the double limits employed here is easily justified.

sequently it will not effect the asymptotic form of $\bar{\omega}_j(x)\,\bar{h}_{ij}(x)$, so that

$$g_{i2}(x) \sim g_i(x) \qquad\qquad \left(\frac{\pi}{2} \leqq \arg x < \frac{3\pi}{2}\right).$$

By using the second principal solutions of the associated homogeneous system, and expressing the sum formulæ by contour integrals to the right, we have thus proved:

THEOREM III. *If $\mu > 0$, there exists for (1) a second principal solution $g_{12}(x)$, $g_{22}(x)$, $\cdots$, $g_{n2}(x)$ analytic throughout the finite plane, save for poles congruent on the right to the poles of $a_{ij}(x)$ and $b(x+1)$. This solution is such that $g_{i2}(x) \sim g_i(x)$ when x approaches infinity along any ray in the left half plane, inclusive of the positive imaginary axis.*

§ 8. *The relation between the principal solutions.*

The difference between any two solutions of (1) is a solution of the homogeneous system (2). To determine the relation between the principal solutions we make use again of the second principal solutions of the homogeneous system. We know that we must have

$$g_{i2}(x) - g_{i1}(x) = \sum_{j=1}^{n} \theta_j(x)\,\bar{h}_{ij}(x) \qquad (i = 1, 2, \cdots, n),$$

where the functions $\theta_1(x), \cdots, \theta_n(x)$ are periodic. We have really already determined these periodic functions in the form of contour integrals; for a comparison of (40) and (43) with the above expression shows immediately that we must have $\theta_j(x) = \bar{\omega}_j^{(1)}(x)$ when x is sufficiently distant from the real axis.

The functions $\theta_1(x), \cdots, \theta_n(x)$ are therefore analytic at a sufficient distance from the real axis, and as they have no essential singularities in a period strip, they are rational functions of $e^{2\pi x\sqrt{-1}}$.[*] Thus we can write

$$\theta_j(x) = \sum_{k=1}^{m_j} \sum_{l=1}^{q_k} \frac{D_{kl}^{(j)}}{(2^{2\pi x\sqrt{-1}} - e^{2\pi \alpha_{kj}\sqrt{-1}})^l} + \sum_{k=-n_j}^{\bar{n}_j} d_k^{(j)} e^{2\pi kx\sqrt{-1}} \qquad (j = 1, 2, \cdots, n),$$

where m_j, n_j, $\bar{n}_j$, and q_k are positive integers, and $D_{kl}^{(j)}$, $d_k^{(j)}$ are constants, while α_{kj} is a fixed point.

When we remember the identity of $\theta_j(x)$ and $\bar{\omega}_j^{(1)}(x)$, for x sufficiently distant from the real axis, we see from (44) that we must have

(46)
$$d_k^{(j)} = 0, \qquad k \geqq 0 \qquad\qquad (j = 1, 2, \cdots, n).$$
$$\sum_{k=1}^{m_j} D_{k1}^{(j)} = C_j.$$

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[*] OSGOOD: *Lehrbuch der Funktionentheorie*, I, 1st edition, p. 404, or 2d edition, p. 466.

Likewise from (45) we have

$$d_j^{(k)} = 0, \qquad k < 0,$$

(47)

$$\sum_{k=1}^{m_j} \sum_{l=1}^{q_k} \frac{D_{kl}^{(j)}}{(-e^{2\pi a_{kj}\sqrt{-1}})^l} = c_j, \qquad (j = 1, 2, \cdots, n).$$

The relation between the principal solutions is accordingly given by

(48) $$g_{i2}(x) - g_{i1}(x) = \sum_{j=1}^{n} \bar{h}_{ij}(x) \sum_{k=1}^{m_j} \sum_{l=1}^{q_k} \frac{D_{kl}^{(j)}}{(e^{2\pi x\sqrt{-1}} - e^{2\pi a_{kj}\sqrt{-1}})^l}$$

$$(i = 1, 2, \cdots, n),$$

with the conditions (46) and (47).

§ 9. *The case $\mu < 0$.*

We have thus far explicitly assumed that μ is positive. We shall now indicate the treatment for μ negative.

The series (15) diverges in this case, and so the direct sum to the right no longer furnishes a solution; but if we apply the direct sum to the left to (35), we can show, by using the asymptotic form of the terms, that the series so obtained will converge uniformly at a sufficient distance from the positive real axis.

By means of a direct sum to the left we thus establish the existence of a solution analytic in the finite plane save for poles at the poles of $a_{ij}(x)$ and $b(x+1)$ and points congruent on the right. This solution corresponds to what was called the second principal solution when μ is positive.

If now we take the first principal solutions as solutions of the homogeneous system, it can easily be shown that the resulting sum formulæ can be taken to be contour integrals about a contour to the left. This gives a second solution of the non-homogeneous system analytic in the finite plane at a sufficient distance from the negative real axis, with its singularities at those of $a_{ij}(x)$, $b(x+1)$, and the zeros of $A(x)$, or points congruent on the left. It thus corresponds to the first principal solution for $\mu > 0$.

These solutions will, as before, be asymptotic in the left and right half planes, respectively, to the series formally satisfying the equations, the first of them maintaining its asymptotic form in the right half plane at a sufficient distance from the positive real axis. The formal series have, however, a somewhat different form, and now are

$$g_i(x) = x^{\nu+\mu}\left(g_i + \frac{g_i^{(1)}}{x} + \cdots\right) \quad (i = 1, 2, \cdots, n-1),$$

$$g_n)(x = x^{\nu}\left(g_n + \frac{g_n^{(1)}}{x} + \cdots\right).$$

For if we substitute expressions of this form in the equations, and compare coefficients of like powers of x, it is evident that at each stage we shall have n equations in which there are n unknown coefficients and coefficients previously determined. Thus the coefficients can be calculated uniquely. In particular we have

$$g_i = a_{in}b \qquad (i = 1, 2, \cdots, n-1),$$

$$g = b.$$

A case of interest, and one which will be used later in the treatment of the equation of the nth order, is that where $a_{ij}(x) = 0$, for $j \neq i+1, i \neq n$. In that case the formal series become

(49)
$$g_i(x) = x^{\nu+(n-i)\mu}\left(g_i + \frac{g_i^{(1)}}{x} + \cdots\right) \quad (i = 1, 2, \cdots, n-1),$$

where

(50)
$$g_i = a_{i,\,i+1}\,a_{i+1,\,i+2}\cdots a_{n-1,\,n}\,b \qquad (i = 1, 2, \cdots, n),$$

$$g_n = b.$$

We can also show the existence of solutions for the case $\mu < 0$ in a second way by referring it to the case $\mu > 0$. When we remember the definition of $A(x)$, $A_{ji}(x)$, the system (17) can be written

$$g_i(x) = \sum_{j=1}^{n} \bar{a}_{ij}(x)\,g_j(x+1) + b_i(x+1) \quad (i = 1, 2, \cdots, n),$$

where

$$\bar{a}_{ij}(x) = x^{-\mu}\left(\bar{a}_{ij} + \frac{\bar{a}_{ij}^{(1)}}{x} + \cdots\right), \qquad \bar{a}_{ij} = (-1)^{j+1}\frac{A_{ji}}{A},$$

$$b_i(x) = x^{\nu-\mu}\left(b_i + \frac{b_i^{(1)}}{x} + \cdots\right), \qquad b_i = (-1)^{n+i+1}\frac{A_{ni}\,b}{A}.$$

If now we put $x = -(z+1)$, $g_i(x) = g_i(-(z+1)) = \bar{g}_i(z+1)$, and $\mu = -\lambda$ ($\lambda > 0$), this system can be written

$$\bar{g}_i(z+1) = \sum_{j=1}^{n} c_{ij}(z)\,\bar{g}_j(z) + \bar{b}_i(z+1) \quad (i = 1, 2, \cdots, n),$$

where, when $|z| > R$,

$$c_{ij}(z) = z^{\lambda}\left(c_{ij} + \frac{c_{ij}^{(1)}}{z} + \cdots\right), \qquad c_{ij} = (-1)^{j+1}\frac{A_{ji}}{A},$$

$$\bar{b}_i(z) = z^{\nu+\lambda}\left(\bar{b}_i + \frac{b_i^{(1)}}{z} + \cdots\right), \qquad \bar{b}_i = (-1)^{n+i+1}\frac{A_{ni}\,b}{A}.$$

Since λ is positive the previous existence theorems can be applied. The substitution $x = -(z+1)$ shows that the rôles of right and left half planes have been interchanged.

We can summarize the above results as follows:

When $\mu < 0$, first and second principal solutions exist and have their singularities at the same points as before. The first solution is asymptotic to the formal series in the right half plane; the second solution is asymptotic to the same series throughout the left half plane, and also in the right half plane at a sufficient distance from the real axis.

$$\S\ 10. \quad \textit{The case } \mu = 0.$$

It still remains to consider the question of solutions for the intermediate case $\mu = 0$. It will appear that at times this case is exceptional.

We shall first examine when the solutions, as found on the hypothesis that $\mu > 0$, are valid when we put $\mu = 0$. Consider the equation (15), and let us see under what conditions the series will still converge. The typical term can now by (16) be written

$$\frac{(x+m)^{\nu-r_j}}{\rho_j^{x+m}}\,[d_j;\ x+m].$$

If $|\rho_j| > 1$, we shall have a series in which the terms decrease in practically geometrical fashion, so that the series converges uniformly. Since $|\rho_s| \gtrless |\rho_i|\,(i < n)$, the series for $\omega_1(x), \cdots, \omega_n(x)$ will all converge if $|\rho_n| > 1$. The first principal solution, as found for $\mu > 0$ therefore still exists for $\mu = 0$, if $|\rho_n| > 1$. Under this same condition the second principal solution for $\mu > 0$, given by the contour integrals, is also valid. For in this case (39) can be written

$$\frac{(t+1)^{\nu-r_j}}{\rho_j^{t+1}}\left(d_j + \frac{d_j^{(1)}}{t} + \frac{\eta_j(t)}{v}\right),$$

and this expression decreases along $A\infty$ and $B\infty$ in such a way that the integral in (36) converges.

We can then say: *If every root of the characteristic equation is greater than unity in absolute value, we can consider the case $\mu = 0$ as a limiting case for $\mu > 0$.*

In a similar way we can show that: *When all the roots of the characteristic equation are less than unity in absolute value, the case $\mu = 0$ can be considered as a limiting case for $\mu < 0$.*

We still have to determine whether we can make the sum formulæ valid if (1) some of the roots of the characteristic equation are greater in absolute value than unity and others less, or (2) some of the roots of the characteristic equation are equal to unity in absolute value.

Consider the first case. Let $\rho_1, \cdots, \rho_k$ be greater than unity in absolute value, and $\rho_{k+1}, \cdots, \rho_n$ be less. Then, if we take as solutions of the homogeneous equations the first principal solutions, we can use for the sum formulæ for $\omega_1(x), \cdots, \omega_k(x)$ the direct sum to the right. We will next show how to express $\omega_{k+1}(x), \cdots, \omega_n(x)$ as contour integrals. As the work is similar for all of them, we shall consider the equation

$$(51) \qquad \Delta\omega_{k+1}(x) = (-1)^{n+k+1} \frac{b(x+1) H_{k+1}(x+1)}{H(x+1)}.$$

If x is in the right half plane and is sufficiently large, we know by (16) that the right-hand side of (51) can be written

$$\frac{(x+1)^{\nu-r_j}}{\rho_{k+1}^{x+1}} [d_{k+1}; x+1].$$

In § 1 it was stated that there is a set of intermediate solutions of (2) that are asymptotic to the formal series in the right half plane with regard to x, and with regard to v in the left half plane. Suppose these were the solutions of the homogeneous system that we were using; we could then write

$$(51') \qquad \Delta\omega_{k+1}(x) = (-1)^{n+k+1} \frac{(x+1)^{\nu-r_j}}{\rho_{k+1}^{x+1}} [d_{k+1}; x+1],$$

where $[d_{k+1}; x+1]$ is bounded for x large in the right half plane and for v large in the left half plane. Since $|\rho_{k+1}| < 1$, it is evident that for v large enough the above value of $\Delta\omega_{k+1}(x)$ decreases, when x goes to infinity in a direction parallel to the negative real axis, approximately as an exponential. The equation could then be solved by means of a contour integral sum to the left. Return now to the consideration of equation (51). The first principal solutions can be expressed in terms of these intermediate solutions by equations similar to (8), and reference to (37) and (38) shows that the right

hand side of (51) would be replaced by a group of terms such as the one on the right in (51') in which *every* ρ is less in absolute value than unity. From this we see that $\omega_{k+1}(x)$ can be expressed as a contour integral about a contour to the left, and similarly $\omega_{k+2}(x)$, $\cdots$, $\omega_n(x)$.

If, therefore, we use the *first principal solutions of the homogeneous system, and divide the resulting sum formulæ into two groups, according as they correspond to roots of the characteristic equation greater in absolute value than unity, or less, we can express the first group by a direct sum to the right, and the second by a contour integral to the left.* Thus we have the *first principal solution* of the non-homogeneous system.

In a similar way we can show that if we use the second principal solutions of the homogeneous system, and divide the sum formulæ as before, *we can solve the first group by a contour integral to the right, and the second by a direct sum to the left.* This gives a *second principal solution.*

The only remaining case is that in which some of the roots of the characteristic equation are equal to unity in absolute value. This case has not been completely investigated for the homogeneous system. Here, in certain cases, at least, a modified sum may give a convergent series. An example of this is the equation occurring in the Weierstrass development of the gamma function, namely the equation

$$g(x+1) - g(x) = \frac{1}{x}.$$

The direct sum to the right is

$$-\left(\frac{1}{x} + \frac{1}{x+1} + \cdots\right),$$

which diverges, while the modified sum

$$\frac{1}{1} - \frac{1}{x} + \left(\frac{1}{2} - \frac{1}{x+1}\right) + \cdots$$

is also a solution, and converges.

When $\mu = 0$ the asymptotic form of the solution requires a slightly different treatment. Take the case where all the roots of the characteristic equation are greater than unity in absolute value, and consider the asymptotic form of the first principal solution.

We need only find the asymptotic form of $\omega_j(x) h_{ij}(x)$, where

$$\omega_j(x) = \sum_{m=1}^{\infty} \frac{(x+m)^{r-r_j}}{\rho_j^{x+m}} [d_j; x+m].$$

Take the first k terms of $\omega_j(x)$, where $\frac{1}{2}|x|-1 < k \gtreqless \frac{1}{2}|x|$; we can write them in the form

$$\frac{x^{\nu-r_j}}{\rho_j^x} \sum_{m=1}^{k} \frac{1}{\rho_j^m} \left(1+\frac{m}{x}\right)^{\nu-r_j} [d_j;\ x+m].$$

In all these terms the expression $(1+m/x)^{\nu-r_j}$ can be developed in a series in $1/x$, since $|m/x| < 1$. There is a marked difference, however, between these developments and those for the case $\mu > 0$ as given in §4. In the present case each of the k terms of the summation when expanded will contain a constant term, while in the corresponding expression (20) in the former case, the first term alone when expanded contained a constant term, the others beginning with $x^{-\mu}, \cdots, x^{-(k-1)\mu}$. The final constant term in the present instance will evidently be $d_j \sum\limits_{m=1}^{k} \rho_j^{-m}$. When k increases this approaches $d_j/(\rho_j-1)$, since $|\rho_j| > 1$. The first k terms of $\omega_j(x)$, when multiplied by $h_{ij}(x)$, can therefore be written $x^\nu[\lambda_j^{(k)};\ x]$, where $\lambda_j^{(k)}$ approaches $d_j s_{ij}/(\rho_j-1)$ as k increases.

The terms of $\omega_j(x)$ after the kth behave ultimately like a geometrical progression, since the ratio of successive terms obviously approaches $1/\rho_j$ as the number of the terms becomes large. It follows then that the remainder after k terms in the series for $\omega_j(x)$, is, provided k is large enough, of essentially the order of the $(k+1)$th term, which is $M(x+k+1)^{\nu-r_j}/\rho_j^{x+k+1}$, where M is bounded. The product of this and $h_{ij}(x)$ can be written

$$\bar{M} \frac{x^\nu}{\rho_j^{k+1}} \left(1+\frac{k+1}{x}\right)^{\nu-r_j},$$

where $|\bar{M}|$ is less than some fixed constant. Since k increases in a constant ratio to x, this becomes smaller than any power of $1/x$, so that we have ultimately $\omega_j(x) h_{ij}(x) = x^\nu[\lambda_j^{(k)};\ x]$ where $\lambda_j^{(k)}$ approaches $d_j s_{ij}/(\rho_j-1)$ for k large.

In this way we can determine the asymptotic form of the solution, and we are led to try to find formal solutions of the form

$$g_i(x) = x^\nu\left(g_i + \frac{g_i^{(1)}}{x} + \cdots\right) \qquad (i=1,2,\cdots,n).$$

That the coefficients in these expressions can be uniquely determined so as to satisfy the equations can be shown easily by substitution. In particular we find

(52) $$g_i = (-1)^{n+i+1} \frac{b\bar{A}_i}{\bar{A}} \qquad (i=1,2,\cdots,n),$$

where $\bar{A}$ is the determinant $|\, a_{ij} - \delta_{ij}\,|$, $(\delta_{ij} = 0$, if $i \neq j$, $\delta_{ii} = 1)$, and $\bar{A}_i$ is the minor of $\bar{A}$ with reference to the element in the last row and ith column. Also we find again that *the first and second principal solutions are asymptotic, in the right and left half planes respectively, to the series formally satisfying the equations.*

§ 11.   *The equation of the nth order.*

We shall consider a single non-homogeneous equation of the nth order, which we write in the form

$$(53) \quad g\,(x+n) + a_1\,(x)\,g\,(x+n-1) + \cdots + a_n\,(x)\,g\,(x) = b\,(x).$$

The functions $a_1\,(x),\ \cdots,\ a_n\,(x),\ b\,(x)$ are assumed to be rational, so that we have

$$a_i\,(x) = x^{i\mu}\left(a_i + \frac{a_i^{(1)}}{x} + \cdots \right),$$

$$\text{(for } |x| > R\,).$$

$$b\,(x) = x^\nu\left(b + \frac{b^{(1)}}{x} + \cdots \right),$$

In place of the single equation (53), consider the system

$$g_1\,(x+1) = x^\mu\,g_2\,(x),$$

$$g_2\,(x+1) = x^\mu\,g_3\,(x),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$g_{n-1}\,(x+1) = x^\mu\,g_n\,(x),$$

$$(54) \quad g_n\,(x+1) = -\sum_{i=1}^{n} \frac{x^\mu\left(a_{n-i+1} + \dfrac{a_{n-i+1}^{(1)}}{x} + \cdots \right)}{\left\{\left(1+\dfrac{i-1}{x}\right)\left(1+\dfrac{i}{x}\right)\cdots\left(1+\dfrac{n-1}{x}\right)\right\}^\mu}\,g_i\,(x)$$

$$+ \frac{x^{\nu-(n-1)\mu}\left(b + \dfrac{b^{(1)}}{x} + \cdots \right)}{\left\{\left(1+\dfrac{1}{x}\right)\left(1+\dfrac{2}{x}\right)\cdots\left(1+\dfrac{n-1}{x}\right)\right\}^\mu}.$$

If from these equations we eliminate $g_2\,(x),\ \cdots,\ g_n\,(x)$, we shall find that $g_1\,(x)$ satisfies (53), so that the system is equivalent to the original equation of the nth order.   When we expand the quantities $(1+j/x)^\mu$, which occur in

the last equation above we find that that equation can be written

$$g_n(x+1) = \bar{a}_1(x)\,g_1(x) + \bar{a}_2(x)\,g_2(x) + \cdots + \bar{a}_n(x)\,g_n(x) + b(x+1),$$

where

$$\bar{a}_i(x) = -x^\mu\left(a_{n-i+1} + \frac{\bar{a}_i^{(1)}}{x} + \cdots\right),$$

$$\bar{b}(x) = x^{\nu-(n-1)\mu}\left(b + \frac{\bar{b}^{(1)}}{x} + \cdots\right),$$

so that the system is of precisely the form (1). The determinantal characteristic equation reduces to

$$(55) \qquad\qquad \rho^n + a_1\,\rho^{n-1} + \cdots + a_{n-1}\,\rho + a_n = 0,$$

which can therefore be found directly from (53).

We can now apply to the system the existence theorems already established, and we thus have solutions $g_1(x)$ of (53) in all cases except when μ is zero and some of the roots of the characteristic equation are equal to unity in absolute value.

The asymptotic form of the solutions of (54) can be determined by applying the general results for a system (for $\mu < 0$, apply (49)). In each case it is found that the solution $g_1(x)$ is asymptotic to the series formally satisfying the equation (53). This formal solution is now

$$g(x) = x^{\nu-n\mu}\left(g + \frac{g^{(1)}}{x} + \cdots\right) \qquad (\text{for } \mu > 0),$$

$$g(x) = x^{\nu}\left(g + \frac{g^{(1)}}{x} + \cdots\right) \qquad (\text{for } \mu \leqq 0),$$

where

$$g = \frac{b}{a_n} \qquad (\text{for } \mu > 0),$$

$$g = b \qquad (\text{for } \mu < 0)$$

$$g = \frac{b}{1 + a_1 + a_2 + \cdots + a_n} \qquad (\text{for } \mu = 0).$$

The constant term can in all cases be found either from the known form of the constant term of the solution of the corresponding system (for $\mu < 0$, apply (50); for $\mu = 0$, use (52)), or by direct substitution in equation (53).

We can now state:

THEOREM IV.   If $\mu \gtrless 0$, and all the roots of the characteristic equation (55) are different from zero, the equation (53) has two principal solutions $g_1(x)$ and $g_1'(x)$. The solution $g_1(x)$ is analytic in the finite plane save for possible poles at the poles of $a_1(x)$, $\cdots$, $a_{n-1}(x)$, $b(x)$, at the zeros of $a_n(x)$, or at points congruent on the left; while $g_1'(x)$ is analytic save for poles at the poles of $a_1(x)$, $\cdots$, $a_n(x)$, $b(x)$ or points congruent on the right.   If $\mu = 0$, the same is true provided in addition that none of the roots of the characteristic equation are equal to unity in absolute value.   There exists a series $g(x)$ formally satisfying (53), and in all cases in which we have shown the existence of solutions we have $g_1(x) \sim g(x)$ in the right half plane, while $g_1'(x) \sim g(x)$ in the left half plane.

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