AN APPLICATION OF FINITE GEOMETRY TO THE CHARACTERISTIC THEORY OF THE ODD AND EVEN THETA FUNCTIONS*

BY

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Ordinary projective geometry has long been recognized as an important instrument of investigation in other and apparently quite distinct fields, such as the theory of equations and the theory of functions. One of the objects of this paper is to show that a similar purpose is served by the more recently formulated† finite projective geometry. It may be said with some logical justification that this coördination between different subjects is due to their common use of a certain body of abstract theorems. But there are notions, such as that of projection and section, which are so essentially geometric in their origin and significance that their import can only be clouded by viewing them in another or more abstract light.

The characteristic theory of the odd and even theta functions has been the subject of numerous memoirs since the appearance of the original papers of Göpel and Rosenhain. An excellent account of the present state and a valuable history of the development of this theory is given by Krazer in his *Lehrbuch Der Thetafunktionen.*‡ As a matter of convenience Krazer's notation and formulæ are used in the following. For known theorems also reference is made to Krazer, since often they appear in the original in misleading or distorted form.

The following account is essentially geometrical, the principal notions used being those of linear and quadratic dependence, of the null system, and of projection and section. Practically all of the known theorems are reproved by short and direct methods, which in many cases suggest important generalizations. Two ideas, that of "projection and section of a null system" and

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† Veblen and Bussey, these Transactions, vol. 7 (1906), pp. 241–59. Much of Dickson's *Linear Groups* can be interpreted as finite analytic geometry. Cf. also the author's article, *A Configuration in Finite Geometry*, etc., Johns Hopkins University Circulars, No. 7, 1908; and Mitchell's *Determination of the Ordinary and Modular Ternary Linear Groups*, these Transactions, vol. 12 (1911), pp. 207–42.

‡ Leipzig (Teubner), 1903; particularly, pp. 239–304; cited hereafter as K.

that of "section of a null system," are constantly utilized. The grouping of characteristics due to the first of these has been employed to some extent by FROBENIUS; the grouping due to the second seems to have been noted only in a few particular cases.

§ 1 contains an elementary account of the finite geometry (modulo 2) in a linear space in which there may or may not be defined a null system. It is shown in § 2 that the period characteristics behave like the points of an S_{2p-1} with reference to a given null system C. Quadrics in S_{2p-1} are studied in § 3; in particular, those whose polar systems coincide with the null system C. In § 4 the theta characteristics are identified with the quadrics belonging to C. By mapping the quadrics belonging to C upon a space R_{2p}, the period and theta characteristics are shown in § 5 to lie in a linear system. Numerous theorems concerning Steiner and Kummer groups are proved here. § 6 is devoted to the so-called systems of theta characteristics. In the earlier paragraphs translation schemes for the transition from the geometry to the characteristic theory are exhibited.

I think that even a hasty comparison of the presentation here given with the arithmetical method followed by Krazer and others will show that the geometrical point of view is very valuable, not only for suggesting novel ideas but also for giving precision to ideas* with consequent generality of statement.

§ 1. *The Finite Geometry Modulo 2.*

Let $x_0, x_1, \cdots, x_k$ be homogeneous coördinates in the linear space S_k in which the coördinates of points and the coefficients of loci are restricted to the finite number field determined by the modulus 2. The coördinates can take either of the values 0 or 1 but cannot all be zero. If P_k is the number of points in S_k, then

$$P_k = 2^{k+1} - 1, \qquad P_k - P_l = 2^{l+1} P_{k-l-1} \qquad (k > l). \tag{1}$$

An S_{k-1} in S_k is defined by the equation (congruence)

$$u_0 x_0 + u_1 x_1 + \cdots + u_k x_k = 0, \tag{2}$$

or equally well by the coördinates $u_0, u_1, \cdots, u_k$. Thus (2) is the condition that, in S_k, the point x and the S_{k-1} u be incident. In this geometry the fundamental theorems of linear dependence and duality are true; and only those developments which differ essentially from the corresponding developments of ordinary geometry will be considered in detail.

(3) *Given n linearly independent S_{k-1}'s in S_k, the number of points on m of*

* See the closing remarks.

*these S_{k-1}'s and not on the remaining $n - m$ S_{k-1}'s is 2^{k-n+1} if $m < n$, $2^{k-n+1} - 1$ if $m = n$.**

Let $z_1 = 0, \cdots, z_m = 0$ be the set of m S_{k-1}'s; and $z_{m+1} = 0, \cdots, z_n = 0$ be the set of $n - m$ S_{k-1}'s. If $m < n$, the required points are those on the $n - 1$ S_{k-1}'s

$$z_1 = 0, \cdots, z_m = 0, z_{m+1} + z_n = 0, \cdots, z_{n-1} + z_n = 0,$$

which are not on $z_n = 0$ (since $z_{m+1} \equiv 1, \cdots, z_n \equiv 1$ modulo 2). The $n - 1$ spaces meet in an S_{k-n+1} which cuts $z_n = 0$ in an S_{k-n}. Thus the required number is $P_{k-n+1} - P_{k-n} = 2^{k-n+1}$. But if $m = n$ the required points are in an S_{k-n} and are P_{k-n} in number.

The $k + 1$ points whose equations are $u_i = 0$ are linearly independent and constitute a *point reference basis in S_k*. Also the $k + 1$ S_{k-1}'s whose equations are $x_i = 0$ constitute an *S_{k-1} reference basis in S_k*. Either basis determines the other and the two constitute a *self-dual reference basis* of S_k. A point reference basis in an S_{k-1} and a point not in the S_{k-1} determine a point reference basis in S_k and each basis is thus determined in $k + 1$ ways. If R_k is the number of reference bases in S_k we have, since the number of the S_{k-1} is P_k, the recursion formula

$$R_k = \frac{P_k (P_k - P_{k-1})}{k+1} R_{k-1} = \frac{2^k}{k+1} P_k R_{k-1}.$$

(4) *The number R_k of reference bases in S_k is*

$$R_k = \frac{2^{k(k+1)/2}}{(k+1)!} P_k P_{k-1} \cdots P_1.$$

An S_h in S_k, where $h < k$, is fixed by choosing $h + 1$ linearly independent points in S_k; but the same S_h can be thus fixed in R_h ways. The $h + 1$ points, when ordered, can be chosen in

$$P_k (P_k - P_0)(P_k - P_1) \cdots (P_k - P_{h-1})$$

ways. This number divided by $R_h \cdot (h + 1)!$ is the number of S_h's in S_k. Hence, from (1) and (4),

(5) *For $h < k$, the number $P_k^{(h)}$ of S_h's in S_k is*

$$P_k^{(h)} = \frac{P_k P_{k-1} \cdots P_{k-h}}{P_h P_{h-1} \cdots P_1}.$$

We shall be concerned mainly with an odd space S_{2p-1} and coördinates $x_1, x_2, \cdots, x_p, x_{p+1}, x_{p+2}, \cdots, x_{2p}$. A set of $2p + 1$ points must be linearly related. A set of $2p$ linearly independent points, e. g., $u_1 = 0, \cdots, u_{2p} = 0$,

* K., p. 247, III.

determines uniquely a $(2p+1)$th point, $u_1 + u_2 + \cdots + u_{2p} = 0$, such that any $2p$ of the $2p+1$ points are linearly independent. Call such a set of $2p+1$ points a *point basis in* S_{2p-1}. According to the dual of (3) there is a unique S_{2p-2} which is not on $2p$ points of the basis and therefore must be on the $(2p+1)$th point. If

$$\sigma = \sum_{i=1}^{2p} x_i,$$

the set of $2p+1$ such S_{2p-2}'s derived from the point basis is

$$\sigma - x_1 = 0, \quad \sigma - x_2 = 0, \quad \cdots, \quad \sigma - x_{2p} = 0, \quad \sigma = 0.$$

Evidently any $2p$ of these are linearly independent and the set constitutes *an* S_{2p-2} *basis in* S_{2p-1}. The relation between the point basis and the S_{2p-2} basis is mutual. The basis can be defined similarly in an even dimension, the peculiarity of the odd dimension being that corresponding point and S_{2p-2} are incident. The two bases constitute a *self dual basis* in S_{2p-1}.

(6) *In a self dual basis of* S_{2p-1} *there are* $2p+1$ *incident elements (point,* S_{2p-2}*). Any set of* $2p$ *points* [$2p$ S_{2p-2}'s] *of the basis is a point* [S_{2p-2}] *reference basis.*

A point basis is fixed by any one of the $2p+1$ point reference bases in it. Hence, from (4),

(7) *The number of reference bases in* S_{2p-1} *is*

$$N_R = \frac{2^{p(2p-1)}}{(2p)!} P_{2p-1} P_{2p-2} \cdots P_1.$$

The number of bases is

$$N_B = \frac{1}{2p+1} N_R = \frac{2^{p(2p-1)}}{(2p+1)!} P_{2p-1} P_{2p-2} \cdots P_1.$$

A collineation in S_{2p-1} is determined when two ordered point bases are made to correspond; a correlation is determined when an ordered point basis and an ordered S_{2p-2} basis are made to correspond. The totality of collineations or correlations is gotten by fixing one basis and allowing the other to vary, whence the number of each is the number of ordered bases.

(8) *The order of the collineation group in* S_{2p-1} *is*

$$N = (2p+1)!\, N_B = (2p)!\, N_R = 2^{p(2p-1)} P_{2p-1} P_{2p-2} \cdots P_1.$$

N *is also the number of correlations in* S_{2p-1} *and* $2N$ *is the order of the correlation group in* S_{2p-1}.

Of particular interest are those correlations for which corresponding point and S_{2p-2} are incident, the so-called *null systems*. If $y_i = 0$, $i = 1, 2, \cdots, 2p+1$, is an S_{2p-2} basis,

(9) $C = y_1 y'_1 + y_2 y'_2 + \cdots + y_{2p+1} y'_{2p+1} = 0, \qquad \Sigma y = 0, \qquad \Sigma y' = 0,$

is a proper correlation. Since $\Sigma y^2 \equiv (\Sigma y)^2 \pmod 2$, the correlation is a null system. For the ith point of the self dual basis, $y_i = 0$, and $y_k = 1$, $k \neq i$, whence the ith point of the basis corresponds to the ith S_{2p-2} or *the self dual basis y is invariant under C*. Moreover, given any null system C, there are self dual bases invariant under C which are determined as follows. Let z_1 be any point of S_{2p-1} (P_{2p-1} choices), and let w_1 be its null S_{2p-2}. Let z_2 be any point not on w_1 (2^{2p-1} choices), and w_2 be its null S_{2p-2}. On the S_1 $z_1 z_2$, w_1 and w_2 cut out a reference basis and one point of S_1 lies on neither w. Let then z_3 be any point not on w_1 or w_2 and not on $z_1 z_2$ (P_{2p-3} choices) and w_3 be its null S_{2p-2}. The S_2 $z_1 z_2 z_3$ is in $w_1 + w_2 + w_3$ since w_i is on z_i but not z_j and z_k. Thus w_1, w_2, w_3 cut S_2 in a pencil of S_1's and every point of S_2 is on a w. Let then z_4 be a point not on w_1, w_2, w_3 (2^{2p-3} choices), and w_4 be its null S_{2p-2}. The S_3 z_1, z_2, z_3, z_4 is not contained in an S_{2p-2} on the S_{2p-5} w_1, w_2, w_3, w_4, whence the w's cut the S_3 in 4 linearly independent S_2's containing all but one of the points of S_3. Let then z_5 be a point not on $w_1, \cdots, w_4$ and not in the S_3 (P_{2p-5} choices), and w_5 be its null S_{2p-2}. The S_4 $z_1, \cdots, z_5$ is contained in $w_1 + \cdots + w_5$, whence every point of S_4 is on a w.* Let then z_6 be point not on $w_1, \cdots, w_5$ (2^{2p-5} choices), and w_6 be its null S_{2p-2}. Proceeding thus we find, in

$$P_{2p-1} \cdot 2^{2p-1} \cdot P_{2p-3} \cdot 2^{2p-3} \cdot \cdots \cdot P_1 \cdot 2 \cdot P_0$$

ways, a set of $2p + 1$ points, $z_1, z_2, \cdots, z_{2p+1}$, and a set of $2p + 1$ S_{2p-2}'s, $w_1, w_2, \cdots, w_{2p+1}$, subject only to the relation $\Sigma z = 0$, $\Sigma w = 0$, and such that z_i and w_i are incident while z_i and w_k, $k \neq i$, are not incident. Thus the two sets form a self dual basis which by its mode of formation is invariant under C. Taking account of the order we see that

(10) *The number of self dual bases invariant under the null system C is*

$$N_{BC} = \frac{2p^2}{(2p+1)!} P_{2p-1} P_{2p-3} \cdots P_1 .$$

Referred to a basis, C takes the unique form (9). This form is unaltered by the group of order $(2p + 1)!$ determined by permutations of the basis, whence the order N_C of the group of C is $(2p + 1)! \, N_{BC}$. Since all bases are conjugate under G_N, all proper null systems C are conjugate under G_N and their number is N / N_C.

(11) *All proper null systems are conjugate under G_N. Each is unaltered by a group G_{NC} of order*

$$N_C = 2p^2 P_{2p-1} P_{2p-3} \cdots P_1 .$$

* In general, in an S_{2k} $2k + 1$ S_{2k-1}'s, whose sum is zero but subject to no other relation, contain all the points of S_{2k}; while in an S_{2k+1}, $2k + 2$ similarly related S_{2k}'s contain all but one of the points of S_{2k+1}. Cf. K., pp. 267–9.

The number of proper null systems is

$$\frac{N}{N_C} = 2^{p(p-1)} P_{2p-2} P_{2p-4} \cdots P_2 .$$

If the null S_{2p-2} of x contains x' it contains the line $\overline{xx'}$, and the three null S_{2p-2}'s of points on this line contain the line. Such a line will be called a *null line,* any other line an *ordinary line.*

From the canonical form (9) of C in terms of a self dual basis it is clear that the involution

$$y_1' = y_2, \qquad y_2' = y_1, \qquad y_i' = y_i \quad (i = 3, 4, \cdots, 2p+1)$$

is contained in G_{NC}. Every point of $y_1 + y_2 = 0$, including its null point $1, 1, 0, 0, \cdots, 0$, is a fixed point. Every point not on $y_1 + y_2 = 0$ with coördinates $i, j, k, \cdots (i \neq j)$ is interchanged with $j, i, k, \cdots$, the conjugate pair being on a line with the null point of $y_1 + y_2 = 0$. Since G_{NC} is transitive on the points of S_{2p-1}, it contains a conjugate set of P_{2p-1} such involutions which generate G_{NC}. For the transpositions of the y's generate the subgroup of G_{NC} which leaves a basis unaltered. We have then only to show that one basis $a_1, a_2, \cdots, a_{2p+1}$, self dual under C can be transformed by these involutions into any similar basis, $b_1, b_2, \cdots, b_{2p+1}$. Suppose that $k-1$ points a already coincide with points b. By means of the transpositions we adjust the case so that $a_1 = b_1, \cdots, a_{k-1} = b_{k-1}$ while $a_k \neq b_k$. If $\overline{a_k b_k}$ is an ordinary line the null S_{2p-2}'s of the first $k-1$ points all pass through $a_k + b_k$ and the involution determined by $a_k + b_k$ leaves the first $k-1$ points fixed and transforms a_k into b_k. If $\overline{a_k b_k}$ is a null line, let c_k be a point not on the null S_{2p-2}'s of $a_1, a_2, \cdots, a_k, b_k$. Then the two points $a_k + c_k$, $b_k + c_k$ are on the null S_{2p-2}'s of the first $k-1$ points and the involution of the first followed by that of the second leaves $a_1, a_2, \cdots, a_{k-1}$ unaltered and transforms a_k into b_k. The point c_k is subject to $k+1$ conditions and can always be determined according to (3) until $k = 2p$. But two self dual bases coincide if $2p-1$ of their points respectively coincide. Hence

(12) *The group G_{NC} of the null system C is generated by a conjugate set of P_{2p-1} involutions. Each involution is associated with a point and its null S_{2p-2} in such a way that every point on the S_{2p-2} is fixed and every ordinary line on the point contains a conjugate pair of points.**

§ 2. *Period Characteristics of the Thetas as Points in S_{2p-1} Modulo 2.*

The theta function of p variables $\vartheta(v) = \vartheta(v_1, v_2, \cdots, v_p)$ has $2p$ independent *periods,* each consisting of p quantities $\omega_{1\alpha}, \omega_{2\alpha}, \cdots, \omega_{p\alpha}$, where

* Cf. K, § 6, p. 276.

$\alpha = 1, 2, \cdots, 2p$. That is,

$$\vartheta(v + \omega_\alpha) = \vartheta(v_1 + \omega_{1\alpha}, \cdots, v_p + \omega_{p\alpha}) = E\vartheta(v),$$

E being an exponential factor. The value system

$$\tfrac{1}{2}\sum_{\alpha=1}^{2p} \epsilon_\alpha \omega_{1\alpha}, \cdots, \tfrac{1}{2}\sum_{\alpha=1}^{2p} \epsilon_\alpha \omega_{p\alpha}$$

is called a *half period*, whose *period characteristic* or *Per. Char.* is the set of $2p$ integers $\epsilon_1, \cdots, \epsilon_{2p}$. Two characteristics whose half periods differ by a period are looked upon as not essentially distinct, whence the integers ϵ are reducible (modulo 2). Under integral linear transformation of the periods, the half periods are transformed so that the value of the expression

$$\sum_{\mu=1}^{p} (\epsilon_\mu \eta_{p+\mu} - \epsilon_{p+\mu} \eta_\mu)$$

is unaltered, ϵ and η being any two distinct characteristics.* Naturally the coefficients of the transformation are also reducible modulo 2. The zero characteristic $\epsilon_i = 0$, $i = 1, 2, \cdots, 2p$, differs from the remaining *proper Per. Char.* in that it is unaltered by every transformation. Thus our first fundamental theorem is apparent:

(13) *Under integral linear transformation of the periods of the theta function in p variables, the proper Per. Char. are transformed like the points (or their null S_{2p-2}'s) of a finite space S_{2p-1} modulo 2 under the group G_{NC} of collineations which leaves unaltered the proper null system*

$$C = (x_1 x'_{p+1} - x_{p+1} x'_1) + (x_2 x'_{p+2} - x_{p+2} x'_2) + \cdots + (x_p x'_{2p} - x_{2p} x'_p).$$

Thus properties of sets of Per. Char. which are independent of integral period transformation—and these are the only properties of essential importance—can be inferred from the properties of sets of points in S_{2p-1} with reference to C. The translation proceeds as follows:

	Point in S_{2p-1}.	Proper Per. Char.†
	Two points on a null line.	Two syzygetic Per. Char.‡
	Two points on an ordinary line.	Two azygetic Per. Char.†
(14)	Sum of a number of points.	Sum of a number of Per. Char.†
	Points of a self dual basis of C.	Fundamental system (F. S.) of $2p + 1$ Per. Char.; such that any two are azygetic.§

* K., pp. 242–3.

† With Per. Char., the term proper will be understood hereafter.

‡ K., p. 244.

§ K., p. 267.

The duality in S_{2p-1} established by C permits of expressing an occurrence in several ways; thus two points are on a null line if either is on the null S_{2p-2} of the other. Since the null lines on a point x are in its null S_{2p-2}, we have

(15) *A given point lies on a null line with $2P_{2p-3}$ other points; on an ordinary line with 2^{2p-1} other points.*	*A given Per. Char. is syzygetic with $2P_{2p-3}$ other Per. Char.; azygetic with 2^{2p-1} other Per. Char.**

The translation of (3) for $k = 2p - 1$ is

(16) *The number of Per. Char. syzygetic with m and azygetic with $n - m$ of n given linearly independent Per. Char. is 2^{2p-n} if $m < n$, and P_{2p-n-1} if $m = n$.*†

From (10) the number‡ of F.S.'s is obtained. Some theorems, such as the first part of the following,§ are self-evident from the geometrical point of view:

(17) *A F.S. of Per. Char. is transformed by integral linear transformation into a F.S. All F.S.'s are conjugate under such transformation.*

Let $x^{(1)}, x^{(2)}, \cdots, x^{(r)}$ be r points of S_{2p-1} which are not linearly related. They lie in an S_{r-1} and form a point reference basis of the S_{r-1}. Any space of dimension equal to or less than $2p - r - 1$ which has no point in common with S_{r-1} will be called a *skew space of S_{r-1} in S_{2p-1}*. The r null S_{2p-2}'s of the points x meet in an S_{2p-r-1} called *the null space of S_{r-1}*. In general S_{2p-r-1} meets S_{r-1} in an S_{m-1} called *the null subspace of S_{r-1}*. The line joining any point of S_{r-1} to any point of its null space, S_{2p-r-1}, is a null line. If the null subspace of S_{r-1} coincides with S_{r-1}, it is called a *null S_{r-1}*. The null space of largest dimension is an S_{p-1} called a *Göpel space.* The translation to Per. Char. is made according to the following table.

	S_{r-1}.	Group E_r of Per. Char. of rank r.‖
	Skew space of S_{r-1} in S_{2p-1}.	Group H conjugate to the group E_r.¶
	Null subspace of S_{r-1}.	Syzygetic subgroup of E_r.**
(18)	Null space of S_{r-1}.	Group adjoint to E_r.‡
	Null S_{r-1}.	Syzygetic group E.‡
	Göpel space.	Göpel group.†

A null S_k is determined by any point reference basis in it. The $k + 1$ points can be chosen as follows. Let z_1 be the first point (P_{2p-1} choices), w_1 its null S_{2p-2}. Let z_2 be any point on w_1 other than z_1 ($P_{2p-2} - P_0$ choices), w_2 its null S_{2p-2}. Let z_3 be any point on the S_{2p-3} $w_1 w_2$ other than a point on the S_1 $z_1 z_2$ ($P_{2p-3} - P_1$ choices), w_3 its null S_{2p-2}; etc. The null S_k is finally determined by means of an ordered point reference basis in it.

* K., p. 244. † K., p. 247. ‡ K., p. 268.
§ K., p. 270. ‖ K., p. 291. ¶ K., p. 295.
** K., p. 292.

Since the ordering and the particular basis in S_k are not material we get

$$\frac{1}{(k+1)!}\frac{1}{R_k}P_{2p-1}(P_{2p-2}-P_0)(P_{2p-3}-P_1)\cdots(P_{2p-k-1}-P_{k-1})$$

null S_k's. Or, by making use of (4),

(19) *There are**

$$P_{Ck}=\frac{P_{2p-1}P_{2p-3}\cdots P_{2p-2k-1}}{P_1P_2\cdots P_k}$$

null S_k's belonging to C | *syzygetic groups of rank $k+1$.*

Given a space S_{r-1} with the null subspace S_{m-1}, let S_{r-m-1} be a space skew to S_{m-1} in S_{r-1}. Begin the construction of a self dual basis of C by choosing $r-m$ points $y^{(m+1)}, y^{(m+2)}, \cdots, y^{(r)}$ in the S_{r-m-1}. The last point $y^{(r)}$, for example, must be on the $2p-(r-m)$ S_{2p-2}'s which meet in S_{r-m-1} and outside the $r-m-1$ null S_{2p-2}'s of $y^{(m+1)}, \cdots, y^{(r-1)}$. According to (3) such points can be found. Let $x^{(1)}, \cdots, x^{(m)}$ be any point reference basis in S_{m-1}. If the number of points $y^{(i)}$ is odd, the point $\Sigma y^{(i)}$, which is in S_{r-m-1}, is also on the null S_{2p-2} of every point $y^{(i)}$ and of every point $x^{(i)}$ and is therefore in S_{m-1}. But S_{m-1} is skew to S_{r-m-1}. Hence $r-m$ is even.

(20) *The difference $r-m$ of dimension of S_{r-1} and its null subspace S_{m-1} is even. A reference basis of S_{r-1}, $x^{(1)}, \cdots, x^{(m)}, y^{(m+1)}, \cdots, y^{(r)}$, can be selected so that every line $\overline{x^{(i)}x^{(k)}}$ and $\overline{x^{(i)}y^{(k)}}$ is a null line while every line $\overline{y^{(i)}y^{(k)}}$ is an ordinary line.*

The difference $r-m$ of rank of E_r and its syzygetic subgroup E_m is even. E_r has reference bases of the form $(\alpha_1), \cdots, (\alpha_m), (\beta_{m+1}), \cdots, (\beta_r)$, where the pairs $(\alpha_i)(\alpha_k)$ and $(\alpha_i)(\beta_k)$ are syzygetic while the $(\beta_i)(\beta_k)$ are azygetic.†

Such a reference basis of S_{r-1} will be called a *normal reference basis.*

Further theorems in this paragraph will be stated in only one form, the translation being obvious.

Since S_{m-1} is part or all of the space common to r S_{2p-2}'s, $m+r \overline{\gtrless} 2p$. Hence a space S'_{m-1} can be found skew to the null space S_{2p-1-m} of S_{m-1} and therefore skew to both S_{m-1} and S_{r-m-1}. Then S'_{m-1} and S_{r-m-1} lie in an S'_{r-1} in which part of a self dual basis of C can be constructed beginning with $y^{(m+1)}, \cdots, y^{(r)}$, in S_{r-m-1}, and ending with $y^{(1)}, \cdots, y^{(m)}$. These m points determine a space S''_{m-1} which may coincide with S'_{m-1}, but which at all events is skew to S_{2p-1-m}, to S_{m-1}, and to S_{r-m-1}. The null S_{2p-2}'s of $y^{(1)}, \cdots, y^{(m)}$ cut S_{m-1} in S_{m-2}'s since these points are not found in S_{2p-1-m}. Thus an S_{m-2} reference basis in S_{m-1} is obtained which carries with it a point reference basis

* For $k=p-1$, see K., p. 296.

† K., p. 294.

$x^{(1)}, \cdots, x^{(m)}$ in S_{m-1}. The $m + r$ points

$$y^{(m+1)}, \cdots, y^{(r)}, y^{(1)}, \cdots, y^{(m)}, z^{(1)} = y^{(1)} + x^{(1)}, \cdots, z^{(m)} = y^{(m)} + x^{(m)}$$

form part of a self dual basis of C. For $y^{(m+i)}$ is azygetic with $y^{(k)}$ by construction and, being syzygetic with $x^{(k)}$, is azygetic with $z^{(k)} = x^{(k)} + y^{(k)}$. Also $y^{(i)}$ is azygetic with $y^{(k)}$ and syzygetic with $x^{(k)}$ by construction hence azygetic with $z^{(k)}$. Also $z^{(i)}$ and $z^{(k)}$ are azygetic since $y^{(i)}$ and $y^{(k)}$ are azygetic while $y^{(i)}$ and $x^{(k)}$, $y^{(k)}$ and $x^{(i)}$, $x^{(i)}$ and $x^{(k)}$, are syzygetic. Since $x^{(i)} = y^{(i)} + z^{(i)}$, we have from (20),

(21) *An S_{r-1} with a null subspace S_{m-1} has a reference basis of the form $y^{(1)} + z^{(1)}, y^{(2)} + z^{(2)}, \cdots, y^{(m)} + z^{(m)}, y^{(m+1)}, \cdots, y^{(r)}$, such that the points y, z form part of a self dual basis of C.**

Two self dual bases are conjugate in any order under G_{NC}, whence*

(22) *Two spaces of the same dimension in S_{2p-1} are conjugate under G_{NC} if and only if their null subspaces have the same dimension.*

In order to determine the number of S_{r-1}'s in a conjugate set it is convenient to introduce the notions of a "section" and of a "projection and section" of the null system C—notions that can be used later with advantage. According to (20) a space which has no null subspace is of odd dimension, S_{2k-1}. The null S_{2p-2} of a point in S_{2k-1} cannot contain S_{2k-1} else the point is part of a null subspace. Then the null S_{2p-2}'s of the points in S_{2k-1} cut S_{2k-1} in S_{2k-2}'s and thus there is defined a null system C_k in S_{2k-1} which will be called the *section of C by S_{2k-1}*. $2k$ points of a self dual basis of C_k in S_{2k-1}, being azygetic in pairs, are $2k$ points of a self dual basis of C in S_{2p-1} and conversely. The number of such sets in S_{2k-1} is $(2k+1)!\,(N_{BC})_{p=k}$. On the other hand the number of such sets in S_{2p-1} [see the enumeration before (10)] is

$$\frac{1}{(2k)!} 2^{2p-1} \cdot 2^{2p-3} \cdot \ldots \cdot 2^{2p-2k+1} P_{2p-1} P_{2p-3} \cdots P_{2p-2k+1}.$$

Dividing this number by the first we find that

(23) *The number of S_{2k-1}'s without a null subspace is*

$$2^{2k(p-k)} \frac{P_{2p-1} P_{2p-3} \cdots P_{2p-2k+1}}{P_{2k-1} P_{2k-3} \cdots P_1}.$$

Each is unaltered by a subgroup of G_{NC} of order

$$2^{p^2-2pk+2k^2} P_{2p-2k-1} P_{2p-2k-3} \cdots P_1 \cdot P_{2k-1} \cdots P_3 P_1.$$

Let S_{r-1} have the null subspace S_{m-1}. The null space of S_{m-1} is S_{2p-m-1} which contains S_{m-1}. If $\pi = p - m$, there are within S_{2p-m-1} and on S_{m-1} precisely $P_{2\pi-1}$ spaces S_m. These we regard as "points Σ_0 in a space $\Sigma_{2\pi-1}$

* K., p. 295.

or Σ." More generally then, spaces S_{r-1} within S_{2p-m-1} and on S_{m-1} are "spaces Σ_{r-m-1} in Σ." The null space of an S_m within S_{2p-m-1} and on S_{m-1} is an S_{2p-m-2} within S_{2p-m-1} and on S_m and therefore on S_{m-1} also or "a point Σ_0 is on its null $\Sigma_{2\pi-2}$ with reference to the thus defined null system Γ_π in Σ." This null system Γ_π in the derived space $\Sigma_{2\pi-1}$ we call the *projection and section of C in S_{2p-1} from the null space S_{m-1} and by the null space S_{2p-m-1} of* S_{m-1}. Two points of Σ are syzygetic or azygetic according as the null space of one corresponding S_m does or does not contain the other corresponding S_m. A space S_{s-1} within S_{2p-m-1} and on S_{m-1} has a null subspace $S_{m'-1}$ which contains S_{m-1}. To S_{s-1} there corresponds in Σ a space S_{s-m-1} which has with reference to Γ_π a null subspace $\Sigma_{m'-m-1}$ and conversely. In particular for $s = r$ and $m' = m$ we see that an S_{r-1} with the null subspace S_{m-1} corresponds to a Σ_{r-m-1} in Σ without a null subspace with reference to Γ_π. The number of these Σ_{r-m-1}'s has been determined in (23) where $2k$ is to be replaced by $r - m$ and p by $\pi = p - m$. This is the number of S_{r-1}'s with a *given* null S_{m-1} as null subspace. The number of null S_{m-1}'s is furnished by (19), whence

(24) *The number of S_{r-1}'s with a null subspace S_{m-1}, where $r - m$ is even, is*

$$2^{\frac{r-m}{2}(2p-r-m)}\frac{P_{2p-1}P_{2p-3}\cdots P_{2p-m-r+1}}{P_{r-m-1}P_{r-m-3}\cdots P_1\cdot P_{m-1}P_{m-2}\cdots P_2P_1}.$$

They all are conjugate and each is unaltered by a subgroup of G_{NC} of order

$$2^{p^2-p(r-m)+\frac{1}{4}(r^2-m^2)}P_{2p-m-r-1}P_{2p-m-r-3}\cdots P_1\cdot P_{r-m-1}P_{r-m-3}\cdots P_1$$
$$P_{m-1}P_{m-2}\cdots P_1.$$

§ 3. *Quadrics in S_{2p-1} Modulo* 2.

A *quadric* in S_{2p-1} is defined by a congruence or equation of the form

(25) $$f(xx) = \Sigma a_{ik}x_ix_k = 0 \quad (i, k = 1, \cdots, 2p;\ i \leqq k).$$

Points whose coördinates do or do not satisfy this equation will be called *quadric* or *outside points* respectively. The quadric will be called *proper* or *degenerate* according as it cannot or can be transformed by a collineation of G_N into a form in less than $2p$ variables.

The point $x + y$ is on the quadric if

(26) $$f(xx) + f'(xy) + f(yy) = 0,$$

where

$$f'(xy) = \Sigma a_{ik}(x_iy_k + x_ky_i) \quad (i, k = 1, \cdots, 2p;\ i < k).$$

Since

$$x_iy_k + x_ky_i \equiv x_iy_k - x_ky_i = \pi_{ik},$$

we see* that the polar system of $f, f'(xy) = 0$, is a null system whatever be the values of the coefficients a_{ii}.

(27) *The 2^{2p} quadrics f, obtained by varying the coefficients a_{ii}, have as polar systems the same null system. Conversely, any null system determines 2^{2p} quadrics whose polar systems coincide with the given null system.*

An important relation between the quadric and its polar system is this:

(28) *A quadric is proper or degenerate according as its polar system is a proper or degenerate null system.*

For if the quadric is degenerate it can be transformed into a quadric in $2p - 1$ variables at most, say $y_2, \cdots, y_{2p}$, and the polar of $1, 0, \cdots, 0$ is indeterminate, i. e., the null system is degenerate. Conversely, if the polar of $1, 0, \cdots, 0$ is indeterminate the quadric either has the form $g(y_2, \cdots, y_{2p})$, and is degenerate or has the form $y_1^2 + g(y_2, \cdots, y_{2p})$. In the latter case, $g(y_2, \cdots, y_{2p})$ is a quadric in an even space whose polar system is a null system and therefore necessarily degenerate. Let $y_2 = 1$, $y_{2+k} = 0$, be a singular point. Then g has either the form $y_2^2 + h(y_3, \cdots, y_{2p})$ or the form $h(y_3, \cdots, y_{2p})$. The original quadric has either the form $z^2 + h(y_3, \cdots, y_{2p})$, where $y_1^2 + y_2^2 \equiv (y_1 + y_2)^2 = z^2$ or the form $y_1^2 + h(y_3, \cdots, y_{2p})$, and in either case is degenerate.

We shall say that a *quadric belongs to C* if its polar system is the null system C. A line will be called a *skew line, tangent, secant,* or *generator of the quadric,* according as it has $0, 1, 2$, or 3 points in common with the quadric. It is easily verified from (26) that

(29) *The 2^{2p} quadrics which belong to a given null system C have null lines for tangents and generators, ordinary lines for secants and skew lines.*

Thus if x is a point of the quadric, u its null, or polar, or tangent S_{2p-2}, all lines on x in u are generators or tangents of the quadric, and all lines on x and not in u are secants of the quadric; if however x is an outside point, all lines on x in u are tangents of the quadric, and all lines on x and not in u are secants or skew lines of the quadric.

(30) *Quadrics in S_k have real points if $k > 1$.*

For, if (25) contains no reference point, every $a_{ii} \neq 0$; if no point like $1, 1, 0, \cdots, 0$, every $a_{ik} \neq 0$, but then it must contain points like $1, 1, 1, 0, \cdots, 0$.

Let $C(x, x') = 0$ be the equation of the null system $C, f(x, x) = 0$ be a quadric belonging to C, y be a point on f, and I_y be the involution determined by y [see (12)]. $C(x, y) = 0$ is the tangent space of f at y and contains the tangents and generators of f which pass through y. I_y transforms f into a quadric $\varphi(x, x) = 0$, which has in common with f the point y and its tangent space. Any line through y and not in the tangent space has a further point

* Cf., DICKSON, *Linear Groups*, p. 201, footnote.

on f and a further distinct point on φ. Thus f, φ, and $[C(x, y)]^2$ constitute a pencil, or since

$$[C(x, y)]^2 \equiv C(x^2, y^2) \equiv C(x^2, y) \pmod 2,$$

we have

$$f(x, x) + C(x^2, y) = \varphi(x, x).$$

If z is another point on f,

$$f(x, x) + C(x^2, z) = \psi(x, x),$$

whence

$$\varphi(x, x) + \psi(x, x) = C(x^2, y + z).$$

If $\overline{yz}$ is a null line, the point $y + z$ is on f, φ, ψ; if $\overline{yz}$ is an ordinary line, $y + z$ is not on f nor $C(x^2, y) \equiv C(x, y)$ and therefore is on φ and similarly is on ψ. Then φ and ψ also are in a pencil with the square of their common tangent space, $C(x^2, y + z)$. If f contains n points, the $n + 1$ quadrics, f, φ, ψ, being conjugate with f each contain n points and each determines the set by means of its tangent spaces. If r is a point outside f, I_r leaves unaltered the contacts of tangents from r to f but interchanges the points on a secant line through r, i. e., leaves f unaltered. Hence the set of $n + 1$ quadrics is a complete conjugate set under G_{NC}.

Let y and z be two points not on f, i. e., let $f(y, y) = 1$ and $f(z, z) = 1$. Let

$$f(x, x) + C(x^2, y) = \varphi(x, x), \qquad f(x, x) + C(x^2, z) = \psi(x, x).$$

Then

$$\varphi(x, x) + \psi(x, x) = C(x^2, y + z).$$

But $y + z$ is a point of φ and ψ, since

$$\varphi(y + z, y + z) = f(yy) + C(y, z) + f(zz) + C(z, y) \equiv 0.$$

Hence the set of quadrics obtained by adding to f the squares of its secant spaces is such that any one of the set differs from the others by the squares of its own tangent spaces. Again this set is a complete conjugate set. But all the quadrics which belong to C differ from any one by the square of a tangent or secant space of the one. Hence there are only two distinct types of quadrics which belong to C. Since any proper quadric belongs to a proper null system and all proper null systems are conjugate under G_N there are only two distinct types of proper quadrics in S_{2p-1}.*

G_{NC} is doubly transitive on either conjugate set of quadrics. For two quadrics in the set conjugate to f are associated with points y and z on f. If $\overline{yz}$ is an ordinary line, I_{y+z} leaves f unaltered and interchanges y and z. If $\overline{yz}$

* Cf. Dickson, *Linear Groups*, p. 197.

is a null line there is a point t on f and not on the tangent spaces of y and z; e. g., an ordinary line on y and a point outside the tangent space of z meets the quadric again at a point t. The product $I_{y+t}\, I_{t+z}$ leaves f unaltered and sends y into z.

G_{NC} is simultaneously simply transitive on both sets of quadrics. For if y and z are two points not on f and if $\overline{yz}$ is an ordinary line, then $y + z$ is not on f and I_{y+z} leaves f unaltered and interchanges y and z. If $\overline{yz}$ is a null line touching f at $y + z$, let t be a point on the tangent space of $y + z$ but not on f nor $C(x, y)$ and therefore not on $C(x, z)$. Then $\overline{yt}$ and $\overline{zt}$ are skew to f and the product $I_{y+t}\, I_{z+t}$ leaves f unaltered and sends y into z. But y and z determine any two quadrics of the set which does not contain f.

(31) *The 2^{2p} proper quadrics which belong to C divide into two sets conjugate under G_{NC}. Quadrics from the same set have contact along an S_{2p-2} tangent to both; quadrics from different sets have contact along a space secant to both. Every proper quadric in S_{2p-1} is conjuagte under G_N with one or the other type. G_{NC} is doubly transitive on the quadrics in either set; and simultaneously simply transitive on the quadrics of both sets. The group of any quadric is simply transitive on the points of the quadric and also on the outside points of the quadric.*

Let there be τ points on, and $(P_{2p-1} - \tau)$ points outside, the proper quadric f which belongs to C. According to (31) the quadric points are each of the same type; the same is true of the outside points. Let there be on an outside point, ρ_0, ρ_1, ρ_2 lines skew, tangent, secant, respectively to f; on a quadric point, σ_1, σ_2, σ_3 lines respectively tangent to, secant to, on, f. By joining an outside point to the other outside points and to the quadric points we get the equations

$$2\rho_0 + \rho_1 = P_{2p-1} - \tau - 1, \qquad \rho_1 + 2\rho_2 = \tau.$$

n the same way from a quadric point we get

$$2\sigma_1 + \sigma_2 = P_{2p-1} - \tau, \qquad \sigma_2 + 2\sigma_3 = \tau - 1.$$

But $\rho_1 = \sigma_1 + \sigma_3$ is the number of null lines on a point, i. e., P_{2p-3}. Hence we have, in terms of τ,

$$2\rho_0 = P_{2p-1} - P_{2p-3} - \tau - 1, \qquad \rho_1 = P_{2p-3}, \qquad 2\rho_2 = \tau - P_{2p-3},$$

$$2\sigma_1 = P_{2p-1} - \tau - 2^{2p-2}, \qquad \sigma_2 = 2^{2p-2}, \qquad 2\sigma_3 = \tau - 1 - 2^{2p-2}.$$

The total number of tangents of f is either $\sigma_1 \tau$ or $\frac{1}{2}\rho_1 (P_{2p-1} - \tau)$. Equating these values we get

$$\tau^2 - (2^{2p} - 2)\tau + (2^{2p} - 1)(2^{2p-2} - 1) = 0,$$

$$\tau_E = 2^{p-1}[2^p + 1] - 1, \qquad \tau_O = 2^{p-1}[2^p - 1] - 1.$$

Set

(32) $$E_p = 2^{p-1}[2^p + 1], \qquad O_p = 2^{p-1}[2^p - 1].$$

For a quadric E with $E_p - 1$ quadric points,

$$\rho_0 = 2^{p-2} P_{p-2} = O_{p-1}, \qquad \sigma_1 = 2^{p-2} P_{p-2} = O_{p-1},$$
$$(33) \qquad \rho_1 = P_{2p-3}, \qquad \sigma_2 = 2^{2p-2},$$
$$\rho_2 = E_{p-1}, \qquad \sigma_3 = E_{p-1} - 1.$$

For a quadric O with $O_p - 1$ quadric points,

$$\rho_0 = E_{p-1}, \qquad \sigma_1 = E_{p-1},$$
$$(34) \qquad \rho_1 = P_{2p-3}, \qquad \sigma_2 = 2^{2p-2},$$
$$\rho_2 = O_{p-1}, \qquad \sigma_3 = O_{p-1} - 1.$$

If $\pi_0, \pi_1, \pi_2, \pi_3$, denote respectively the total number of skew lines, tangents, secants, and generators, of a quadric, then

For a quadric E,

$$\pi_0 = \tfrac{1}{3} \rho_0 [P_{2p-1} - \tau_E] = \tfrac{1}{3} O_p O_{p-1},$$
$$(35) \qquad \pi_1 = \tfrac{1}{2} \rho_1 [P_{2p-1} - \tau_E] = \sigma_1 \tau_E = \tfrac{1}{2} P_{2p-3} O_p = O_{p-1} [E_p - 1],$$
$$\pi_2 = \rho_2 [P_{2p-1} - \tau_E] = \tfrac{1}{2} \sigma_2 \tau_E = O_p E_{p-1} = 2^{2p-3} (E_p - 1),$$
$$\pi_3 = \tfrac{1}{3} \sigma_3 \tau_E = \tfrac{1}{3} (E_p - 1)(E_{p-1} - 1).$$

For a quadric O,

$$\pi_0 = \tfrac{1}{3} \rho_0 [P_{2p-1} - \tau_O] = \tfrac{1}{3} E_p E_{p-1},$$
$$(36) \qquad \pi_1 = \tfrac{1}{2} \rho_1 [P_{2p-1} - \tau_O] = \sigma_1 \tau_O = \tfrac{1}{2} P_{2p-3} E_p = E_{p-1} (O_p - 1),$$
$$\pi_2 = \rho_2 [P_{2p-1} - \tau_O] = \tfrac{1}{2} \sigma_2 \tau_O = E_p O_{p-1} = 2^{2p-3} (O_p - 1),$$
$$\pi_3 = \tfrac{1}{3} \sigma_3 \tau_O = [O_p - 1][O_{p-1} - 1].$$

If $\phi = f + C(x^2, y)$, then φ contains the points of both or neither of f and $C(x^2, y)$. If f is an E quadric and y a point not on it, φ contains

$$P_{2p-3} + [O_p - (P_{2p-2} - P_{2p-3})] = O_p - 1$$

points, and φ is an O quadric. If f is an O quadric and y a point not on it, ϕ contains

$$P_{2p-3} + [E_p - (P_{2p-2} - P_{2p-3})] = E_p - 1$$

points, and is an E quadric. Hence

(37) *The 2^{2p} quadrics which belong to C divide into a set of $E_p = 2^{p-1}(2^p+1)$ quadrics E, each containing $E_p - 1$ points; and a set of $O_p = 2^{p-1}(2^p - 1)$ quadrics O each containing $O_p - 1$ points. The number of skew lines, tangents, secants, and generators, of these quadrics, and the number of similar lines on a quadric point or an outside point are furnished by the formulæ* (33), $\cdots$, (36).

It is convenient to use Q, $\bar{Q}$ to denote at the same time E, O and O, E. The following relations among the numbers defined above are sometimes useful:

$$(38)\qquad \begin{aligned} 2Q_{p-1}(\bar{Q}_p - 1) &= P_{2p-3}\,Q_p\,, & Q_p\,\bar{Q}_p &= 2^{2p-2}\,P_{2p-1}\,,\\ Q_p\,(Q_p - 1) &= 2P_{2p-1}\,Q_{p-1}\,, & Q_{p-1}\,\bar{Q}_p &= 2^{2p-3}\,(Q_p - 1)\,. \end{aligned}$$

A quadric associated with the null system C in (13) is

$$(39)\qquad q_p = x_1\,x_{p+1} + x_2\,x_{p+2} + \cdots + x_p\,x_{2p} = 0\,.$$

Of the p terms in q_p each takes the value 0 in 3 ways, namely, 00, 01, 10; the value 1 in one way, 11. But $q_p \equiv 1$ if $2k+1$ terms take the value 1 and $p-2k-1$ terms take the value 0 which occurs in $\binom{p}{2k+1}3^{p-2k-1}$ ways. Hence the number of points *not* on q_p is

$$\sum_k \binom{p}{2k+1}3^{p-2k-1} = \tfrac{1}{2}\left[(3+1)^p - (3-1)^p\right] = \tfrac{1}{2}\left[2^{2p} - 2^p\right] = O_p\,.$$

Hence q_p is an E quadric. The quadric

$$q_p + \sum_{i=1}^{2p} a_i\,x_i^2 \equiv q_p + \left[\sum_1^{2p} a_i\,x_i\right]^2$$

is of the same type as q_p provided $\Sigma a_i\,x_i$ is the tangent space of a point on q_p, i. e., if $\sum_{j=1}^{p} a_j\,a_{j+p} \equiv 0$. Hence

(40) *The* 2^{2p} *quadrics obtained by varying the a's in*

$$\sum_{j=1}^{p} x_j\,x_{j+p} + \sum_{i=1}^{2p} a_i\,x_i^2 = 0\,,$$

which belong to the null system

$$C = \sum_{j=1}^{p} (x_j\,x'_{j+p} - x_{j+p}\,x'_j) = 0\,,$$

are of the type E or O according as $\sum_{j=1}^{p} a_j\,a_{j+p} \equiv 0$ *or* $\equiv 1$.

Other canonical forms of a quadric are useful. Let

$$y_1\,y'_1 + y_2\,y'_2 + \cdots + y_{2p+1}\,y'_{2p+1} = 0$$

be the equation of C referred to one of its self dual bases. The quadric

$$\Sigma\, y_i\,y_k = 0 \qquad (i,\,k = 1,\,\cdots,\,2p+1;\; i < k)\,,$$

has the polar system C. Any two points of the basis are on a line with a third point all of whose coördinates except two are zero; call these *the residual* *points of the basis.* Evidently no residual point is on the quadric. Conversely if a quadric contains no residual point it is the above quadric. For, using $y_2, \cdots, y_{2p+1}$ as a reference basis, the residual points become the reference points and the points with only two coördinates not zero. Since the reference points are not on the quadric every square appears. Also every product term appears, else the corresponding residual point is on the quadric. But

$$(y_2^2 + \cdots + y_{2p+1}^2) = (y_2 + \cdots + y_{2p+1})^2 = y_1(y_2 + \cdots + y_{2p+1}),$$

and the quadric has the given form.

To find the number of points on the quadric and therefore its kind, let $2k$ be the number of coördinates of a point which are not zero. The point is on the quadric if $2k(2k-1)/2$ is even, i. e., if k is even. Then the total number of points on the quadric is $\sum_{l=1}^{g} \binom{2p+1}{4l}$, where g is the greatest integer for which $4g < 2p+1$. Since

$$4\sum_{l=0}^{g} \binom{2p+1}{4l} = (1+1)^{2p+1} + (1+i)^{2p+1} + (1-1)^{2p+1} + (1-i)^{2p+1},$$

we find that

$$\sum_{l=0}^{g} \binom{2p+1}{4l} = \tfrac{1}{4}\{2^{2p+1} + (1+i)^{2p+1} + (1-i)^{2p+1}\} - 1.$$

If $p = 4n + m$ this reduces to

$$2^{2p-1} + 2^{p-1}\left[\frac{(1+i)(2i)^m + (1-i)(-2i)^m}{2^{m+1}}\right] - 1,$$

i. e., to $E_p - 1$ if $p \equiv 0, 3 \pmod 4$; and to $O_p - 1$ if $p \equiv 1, 2 \pmod 4$. If p is even the basis points are on the quadric.

(41) *A basis, $y_1, \cdots, y_{2p+1}$, self dual under C, determines uniquely a quadric*

$$Q = \sum y_i y_k = 0 \quad (i, k = 1, \cdots, 2p+1;\ i < k),$$

belonging to C, which contains none of the residual points of the basis. If $p \equiv 0 \pmod 2$, Q contains the basis points; if $p \equiv 1 \pmod 2$, the lines of the basis are skew to Q. If $p \equiv 0, 3 \pmod 4$, Q is an E quadric; if $p \equiv 1, 2 \pmod 4$, an O quadric.

§4. *The Theta Characteristics as Quadrics in S_{2p-1} Modulo 2 Belonging to C.*

If ω_ϵ is the half period whose Per. Char. is ϵ, and E is a proper exponential factor,* then $E\vartheta(v + \omega_\epsilon)$, considered as a function of v, is a theta function

* K., p. 240. Formulæ (*a*), (*b*), (*c*) are given by K., p. 242; formula (*d*), p. 247.

$\vartheta[\epsilon]_2(v)$, whose *theta characteristic*, or *Th. Char.*, is ϵ. In particular the zero Per. Char. gives rise to the original theta function whose Th. Char. therefore is $\epsilon_i = 0$. Two functions whose Th. Char. are congruent modulo 2 arise from two similar Per. Char. and are not essentially distinct. There are then 2^{2p} Th. Char. including the zero Th. Char. The function with Th. Char. ϵ is *even* or *odd* according as

$$\sum_{\mu=1}^{p} \epsilon_\mu \epsilon_{p+\mu} \equiv 0 \text{ or } \equiv 1 \pmod 2).*$$

Under integral linear transformation of the periods,

$$(a) \qquad \bar{\omega}_{\mu\alpha} = \sum_{\beta=1}^{2p} c_{\alpha\beta}\, \omega_{\mu\beta} \quad (\mu = 1, 2, \cdots, p;\ \alpha = 1, 2, \cdots, 2p),$$

the Per. Char. are transformed as follows:

$$(b) \qquad \epsilon_\beta = \sum_{\alpha=1}^{2p} c_{\alpha\beta}\, \bar{\epsilon}_\alpha \qquad (\beta = 1, 2, \cdots, 2p).$$

The coefficients $c_{\alpha\beta}$ are such that $\sum_{\mu=1}^{p} (\epsilon_\mu \eta_{p+\mu} - \epsilon_{p+\mu} \eta_\beta)$ is invariant, i. e.,

$$(c) \qquad \sum_{\mu=1}^{p} (c_{\mu\beta}\, c_{p+\mu,\gamma} - c_{\mu\gamma}\, c_{p+\mu,\beta}) \equiv \begin{matrix} 1 \text{ if } \gamma = p + \beta, \\ 0 \text{ if } \gamma \neq p + \beta. \end{matrix}$$

The 2^{2p} functions $\vartheta[\epsilon]_2(v)$ are transformed, to within exponential factors, into a similar system $\bar{\vartheta}[\bar{\epsilon}]_2(\bar{v})$, the Th. Char. of the two systems being connected by the equations

$$(d) \qquad \begin{aligned} \bar{\epsilon}_\nu &= \sum_{\mu=1}^{p} (c_{\nu\mu}\, \epsilon_\mu - c_{\nu,p+\mu}\, \epsilon_{p+\mu} + c_{\nu\mu}\, c_{\nu,p+\mu}), \\ \bar{\epsilon}_{p+\nu} &= \sum_{\mu=1}^{p} (- c_{p+\nu,\mu}\, \epsilon_\mu + c_{p+\nu,p+\mu}\, \epsilon_{p+\mu} + c_{p+\nu,\mu}\, c_{p+\nu,p+\mu}) \end{aligned} \qquad (\nu = 1, 2, \cdots, p).$$

Since these equations hold as congruences modulo 2 and since $c_{\alpha\beta}^2 \equiv c_{\alpha\beta}$ and $-1 \equiv 1 \pmod 2$, we can modify them so as to read

$$(42) \qquad \bar{\epsilon}_\beta = \sum_{\mu=1}^{p} (c_{\beta,\mu}\, c_{\beta,p+\mu} + c_{\beta\mu}^2\, \epsilon_\mu + c_{\beta,p+\mu}^2\, \epsilon_{p+\mu}) \quad (\beta = 1, 2, \cdots, 2p).$$

In § 2 we have identified the points, $x_1, x_2, \cdots, x_{2p}$, of S_{2p-1} (mod 2) with the Per. Char. ϵ; and the collineations $x_\beta = \sum_{\alpha=1}^{2p} c_{\alpha\beta}\, \bar{x}_\alpha$ of S_{2p-1} which leave the null system, $C = \sum_{\mu=1}^{p} (x_\mu x'_{p+\mu} - x_{p+\mu} x'_\mu)$, unaltered with the transformations

* K., p. 240. Formulæ (*a*), (*b*), (*c*) are given by K., p. 242; formula (*d*), p. 247.

(*b*) of the Per. Char. Consider the effect of such a collineation upon the quadric

$$(43)\qquad \sum_{\mu=1}^{p} (x_\mu x_{p+\mu} + \epsilon_\mu x_\mu^2 + \epsilon_{p+\mu} x_{p+\mu}^2),$$

which belongs to C. It must be transformed into another of the same sort, say with coefficients $\bar{\epsilon}$. By effecting the collineation upon the quadric and making use of the relations (*c*) the coefficients $\bar{\epsilon}$ of the transformed quadric turn out to be those defined by (42). Hence by making use of (40), we obtain the second fundamental theorem:

(**44**) *Under integral linear transformation of the periods the 2^{2p} Th. Char. are permuted just as the 2^{2p} quadrics in S_{2p-1} modulo 2 which belong to C are permuted under the collineation group G_{NC} of C. The theta function with given Th. Char. is odd or even according as the corresponding quadric is an O or an E quadric. Thus the parity of the characteristic is invariant under such transformation.**

According to the formula

$$\vartheta[\epsilon]_2(v + \omega_\eta) = E\vartheta[\epsilon + \eta]_2(v),$$

the function $\vartheta[\epsilon]_2(v)$ vanishes when v is ω_η if $\vartheta[\epsilon + \eta]_2(v)$ is an odd function. Regarding $\vartheta[\epsilon]_2(v)$ as a quadric Q and ω_η, as a point P, then $\vartheta[\epsilon + \eta]_2(v)$ is the quadric Q' obtained by adding to Q the square of the null S_{2p-2} of P. If Q' is an O quadric then, either Q is an E quadric and P is not on Q or Q is an O quadric and P is on Q. We have therefore a further translation scheme:

(45)

An E (or O) quadric which belongs to C.	A theta function with an even (or odd) Th. Char.
An E quadric does (or does not) contain a given point.	An even theta function does not (or does) vanish for a half period with given Per. Char.
An O quadric does (or does not) contain a given point.	An odd theta function does (or does not) vanish for a half period with given Per. Char.

The number‡ of odd and of even thetas is gotten from (37), while the enumerations contained in (32), $\cdots$, (36) characterize very fully the behavior of a particular theta with regard to the sets of three syzygetic or three azygetic Per. Char.

* Cf. the proof of the invariance of parity given by K., pp. 247–50.

† K., p. 240 (VII).

‡ K., p. 252.

§ 5. *The Period and Theta Characteristics as a Linear System in R_{2p} with Reference to a Quadric. Projection and Section Applied to Quadrics. Steiner and Kummer Sets.*

We have already remarked that the sum of two quadrics belonging to C is the square of an S_{2p-2}. More generally, a sum of a number of quadrics and a number of squared S_{2p-2}'s is a quadric or a squared S_{2p-2} according as the number of quadrics in the sum is odd or even.* Thus the 2^{2p} quadrics and $2^{2p}-1$ squared S_{2p-2}'s lie in a linear system containing $2^{2p+1}-1$ elements, i. e., in a linear space R_{2p}. A concrete representation of this linear system is obtained by mapping the points of S_{2p-1} on the points of a quadric M in R_{2p} by means of $2p+1$ independent quadrics in S_{2p-1} belonging to C. Taking the convenient canonical form of (40), let

$$z_0 = \sum_{j=1}^{p} x_j \cdot x_{p+j}, \qquad z_i = x_i^2 \qquad (i = 1, 2, \cdots, 2p). \tag{46}$$

Then the points x of S_{2p-1} are mapped on the points z of

$$M = z_0^2 + \sum_{j=1}^{p} z_j z_{p+j} = 0. \tag{47}$$

M is the general type of non-degenerate quadric in R_{2p}. The collineation group which leaves it unaltered is simply isomorphic with G_{NC}.† The null system of M is necessarily degenerate whence there is one point, $z_0 = 1$, $z_i = 0$, whose polar R_{2p-1} as to M is evanescent. This point we shall call the *vertex* V of M. Since C connects a point of S_{2p-1} with its null S_{2p-2} or also with the square of its null S_{2p-2} we can identify the period characteristics with the squared S_{2p-2}'s and thus show that

(48) *The Per. Char. and Th. Char. can be represented as the linear system of $2^{2p+1}-1$ R_{2p-1}'s in a space R_{2p} with reference to a proper quadric M. The R_{2p-1}'s on the vertex V of M correspond to the Per. Char.; those not on V correspond to the Th. Char., which are odd or even according as the R_{2p-1} cuts M in an O or an E quadric.*

An R_{2p-1} on V cuts M in a quadric section with a double point z on M. The point z is the map of a point x in S_{2p-1} whose null S_{2p-2} corresponds to the R_{2p-1}. This is the trace in R_{2p} of the null system C.

The above representation with reference to M in R_{2p} will be retained only in the background for purposes of suggestion, the important feature being the linearity of the entire system of characteristics. As an instance of the usefulness of this feature let x be a given point in S_{2p-1}. It is a linear condition that a quadric or S_{2p-2} be on x, whence there are $2^{2p}-1$ quadrics and S_{2p-2}'s

* K, p. 254.

† Dickson, *Linear Groups*, p. 197, §§ 199, 200.

and $2^{2p-1} - 1$ S_{2p-2}'s on x. There are then 2^{2p-1} quadrics on x. These quadrics are paired by the involution, I_x, the members of a pair having the same section by the null S_{2p-2} of x. Such a set of $2 \cdot 2^{2(p-1)}$ quadrics will be called* *a first Steiner set* or *the Steiner set of x*. Again if $x^{(1)}$ and $x^{(2)}$ are two points of a null line a quadric or S_{2p-2} on both must contain the third point of the line. This imposes two linear conditions on the quadrics and S_{2p-2}'s whence there are $2^{2p-1} - 1$ quadrics and S_{2p-2}'s and $2^{2p-2} - 1$ S_{2p-2}'s on the line. Hence there are $4 \cdot 2^{2(p-2)}$ quadrics on the line which divide into $2^{2(p-2)}$ sets of 4, each set of 4 having the same section by the null S_{2p-3} of the null line $\overline{x^{(1)} x^{(2)}}$. The members of a set of 4 are permuted transitively by the Abelian G_4 generated by the involutions $I_{x^{(1)}}$ and $I_{x^{(2)}}$. Such a set of $2^2 \cdot 2^{2(p-2)}$ quadrics will be called a *second Steiner set* or *the Steiner set of the null line* $\overline{x^{(1)} x^{(2)}}$. Evidently the argument can be carried on to the limit set by the null S_{p-1} and we can say generally that

(49) *A null S_{m-1} ($m = 1, \cdots, p$) determines an mth Steiner set of $2^m \cdot 2^{2(p-m)}$ quadrics which contain S_{m-1}. They divide into $2^{2(p-m)}$ sets of 2^m quadrics. Each set of 2^m quadrics has the same section by the null S_{2p-m-1} of S_{m-1} and its members are permuted regularly by the Abelian G_{2^m} generated by the involutions I_x of the points x of a reference basis of S_{m-1}.*†

Further properties of the quadrics of an mth Steiner set can be derived from the projection and section of C from S_{m-1} and within S_{2p-m-1} as explained after (23). We have then a derived space $\Sigma_{2\pi-1}$, $\pi = p - m$, whose points correspond to S_m's on S_{m-1} and within S_{2p-m-1}. Evidently a set of 2^m quadrics on S_{m-1} which have the same section by S_{2p-m-1} determine in $\Sigma_{2\pi-1}$ a single quadric. Hence the above $2^{2\pi}$ sets of 2^m quadrics determine in $\Sigma_{2\pi-1}$ the $2^{2\pi}$ quadric belonging to the null ststem Γ_π. If a point of $\Sigma_{2\pi-1}$ lies on a quadric, the corresponding S_m on S_{m-1} lies on each of the corresponding set of 2^m quadrics in S_{2p-1}. For $m = 1$, we see from the values of σ_3 in (33) and (34) that the section of a quadric of type Q is a quadric of the same type. Since the general projection and section can be effected by projections and sections from successive points it is clear that the type of quadric is unaltered in the process. Thus from (37) for $p = \pi$ we find that

(50) *The $2^{2(p-m)}$ sets of 2^m quadrics in the mth Steiner set of a null S_{m-1} divide into O_{p-m} sets of 2^m O quadrics and E_{p-m} sets of 2^m E quadrics.*

For the important particular case, $m = 1$, the enumerations contained in (32) to (36) lead to the following results in S_{2p-1}:

(51) *Given a Q quadric and a line l tangent to Q at x; there are $\bar{Q}_{p-2}$ planes on l containing two further tangents at x, P_{2p-5} planes on l containing a further tangent and generator on x, and Q_{p-2} null planes on l containing two further*

* When $p = 3$, the odd quadrics of the set correspond to the well known Steiner complex of 12 double tangents of a plane quartic curve.

† For $m = 1, 2, 3$, cf. K., pp. 255–65.

generators on x; given a generator m of Q on x, there are $\bar{Q}_{p-2}$ planes on m containing two further tangents at x, 2^{2p-4} planes on m containing a further tangent and generator on x, and $Q_{p-2} - 1$ null planes on m containing two further generators on x. The number of planes on x containing respectively three tangents, two tangents and a generator, a tangent and two generators, and three generators, of Q on x is $\frac{1}{3}\bar{Q}_{p-2}\bar{Q}_{p-1}$, $\bar{Q}_{p-2}[Q_{p-1} - 1]$, $Q_{p-2}\bar{Q}_{p-1}$, and $\frac{1}{3}(Q_{p-1} - 1)(Q_{p-2} - 1)$.

Call a null S_{m-1} contained in Q *a generator S_{m-1} of Q*; and a null S_m which meets Q in a null S_{m-1} *a tangent S_m of Q*. Then after projection and section from S_{m-1} on Q, the earlier enumerations lead to the following results:

(**52**) *Given a generator S_{m-1} of Q, and a tangent S_m containing it; there are on S_m $\bar{Q}_{p-m-1}$ S_{m+1}'s containing two other S_m's tangent along S_{m-1}, $P_{2(p-m)-3}$ S_{m+1}'s containing another S_m tangent along S_{m-1} and a generator S_m, and Q_{p-m-1} null S_{m+1}'s containing two generator S_m's; given a generator S_m on S_{m-1}, there are on S_m, $\bar{Q}_{p-m-1}$ S_{m+1}'s containing two S_m's tangent along S_{m-1}, $2^{2(p-m-1)}$ S_{m+1}'s containing a generator S_m and a tangent S_m, and $Q_{p-m-1} - 1$ null S_{m+1}'s containing two other generator S_m's. On S_{m-1}, the number of S_{m+1}'s containing respectively three tangent S_m's, two tangents S_m's and a generator S_m, a tangent S_m and two generator S_m's, and there generator S_m's is $\frac{1}{3}\bar{Q}_{p-m}\bar{Q}_{p-m-1}$, $\bar{Q}_{p-m-1}(Q_{p-m} - 1)$, $\bar{Q}_{p-m}Q_{p-m-1}$, and $\frac{1}{3}[Q_{p-m} - 1][Q_{p-m-1} - 1]$.*

Some fairly obvious deductions from the above general theorems can now be drawn; e. g., from (50), for $m = 1$, it is clear that

(**53**) *An S_{2p-2} can be expressed in O_{p-1} ways as a sum of two O quadrics; in E_{p-1} ways as a sum of two E quadrics; and in 2^{2p-2} ways as a sum of an O and an E quadric. Or, if a squared S_{2p-2} be added to the 2^{2p} quadrics, then of the Q_p Q quadrics, $2Q_{p-1}$ become Q quadrics while the remaining 2^{2p-2} become $\bar{Q}$ quadrics.**

Two points determine a third on their join. If the join is a null line every quadric contains only one or all of the three points; those containing all constituting the second Steiner set of the line. If the join is an ordinary line a quadric on one point must contain a second point but cannot contain all three. In particular a pair on one of the points do not meet in another of the three. Calling the first Steiner sets of the points syzygetic or azygetic according as the points are syzygetic or azygetic [see (14)], we have

(**54**) *A first Steiner set is determined uniquely by any one of its Q_{p-1} pairs of Q quadrics. Two first Steiner sets determine another first Steiner set, the three being symmetrical. If the three are syzygetic, they together contain all the quadrics and have in common a second Steiner set; i. e., the sets have $4Q_{p-2}$ Q quadrics in common, while $3 \cdot 2^{2p-3}$ Q quadrics occur in only one set. If the three are azygetic, they have no common quadrics, any two have Q_{p-1} Q quadrics in com-*

* K., p. 258 (VI) and (VII).

*mon, but have no pairs in common. Together the three contain $3Q_{p-1}$ Q quadrics, and $\bar{Q}_{p-1}$ Q quadrics are found in none.**

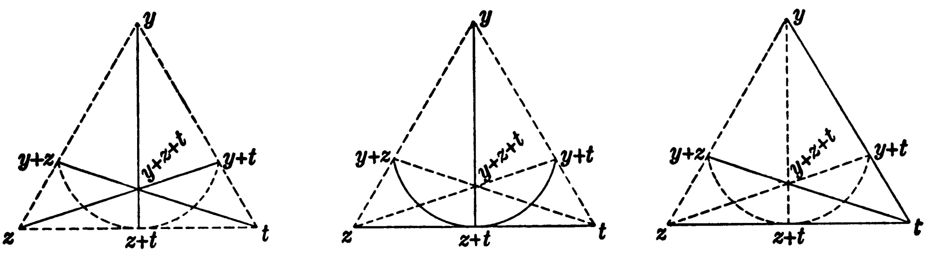
Applying (54) to the projection from S_{m-1} and translating the result from $\Sigma_{2\pi-1}$ back to S_{2p-1}, we obtain the theorem:

(55) *If three null S_m's on a null S_{m-1} are in an ordinary S_{m+1}, no quadric contains all three but any two are contained in $2^m Q_{p-m-1}$ Q quadrics; if the three are in a null S_{m+1} every quadric on S_{m-1} contains at least one, while $3 \cdot 2^m 2^{2(p-m)-3}$ Q quadrics contain only one and $4 \cdot 2^m \cdot Q_{p-m-2}$ Q quadrics contain all three.*

In (54) we have considered the Steiner sets of three points of a line. Let us suppose the three points form a triangle. To contain the points is three independent linear conditions on a quadric or S_{2p-2} whence there are $2^{2p-2} - 1$ quadrics and S_{2p-2}'s and $2^{2p-3} - 1$ S_{2p-2}'s on the three points or

(56) *The Steiner sets of three points which form a triangle have 2^{2p-3} quadrics in common.*

Four cases are possible according as 0, 1, 2, or 3, of the three sides of the triangle are null lines. Drawing null lines full, we see from the figure of Case I



that a quadric on y, z, t cannot contain $y + z + t$, else the three null lines are generators and therefore also the sides contrary to the hypothesis that they are ordinary lines. The null space S of $y + z + t$ contains the triangle and is a secant space of all quadrics on the triangle. If S be added to all these quadrics each Q quadric on the triangle becomes a $\bar{Q}$ quadric on the triangle whence the number of Q and $\bar{Q}$ quadrics is the same. Thus from (56) we find

(57) *The Steiner sets of three points which form a triangle with ordinary sides have in common 2^{2p-4} Q quadrics unpaired in each set. Each Q quadric meets a definite $\bar{Q}$ quadric on the null S_{2p-2} of the null subspace of the plane of the triangle.*†

In case IV the plane of the triangle is a null plane and every quadric on the triangle contains the plane whence

(58) *The first Steiner sets of three points which form a triangle with null sides have in common the third Steiner set of the plane of the triangle.*‡

* K., p. 260 (VIII), p. 261 (IX).

† Cf. K., p. 263 (X).

‡ Cf. K., p. 265 (XIII).

From the figure in case II we see that a quadric on y, z, t contains two null lines on $z + t$, but not the third, and conversely. Hence theorem (55), for $m = 1$, can be applied. In case III a quadric on y, z, t contains two null lines on t but not the third, and the same theorem applies. Hence

(59) *The Steiner sets of three points which form a triangle with one or with two null sides have in common* $2\,Q_{p-2}\,Q$ *quadrics. In the first case the quadrics are not paired in any set though they are paired in the Steiner set of the third point of the null side. In the second case the quadrics are paired in the Steiner set of the point on the two null sides.**

Some of these theorems are easily generalized. Thus (58) and the first case of (54) are special cases of the following:

(60) *If any number of null spaces lie in a null* S_{m-1} *their Steiner sets have in common the mth Steiner set of the null* S_{m-1}.

Following out (57), let $x^{(1)}, \cdots, x^{(2k+1)}$ be an odd number of points of a self dual basis of C. They lie in an S_{2k} which has an S_0, $x^{(1)} + \cdots + x^{(2k+1)}$, as a null subspace. As in (56), there are $2^{2p-2k-1}$ quadrics on the points which are interchanged in type by adding the null S_{2p-2} of S_0 which is secant to all. Hence

(61) *The Steiner sets of an odd number,* $2k + 1$, *of points of a self dual basis of* C *have in common* $2^{2p-2k-2}$ Q *quadrics which are unpaired in each set.*

If however we have an even number 2ρ of points of a self dual basis of C as in the second case of (54), their $S_{2\rho-1}$ has no null subspace and the above argument does not apply. $S_{2\rho-1}$ determines its null space $S_{2(p-\rho)-1}$, which also has no null subspace; in fact, each is the null space of the other and the two are skew. Let Γ_ρ be the section of C by $S_{2\rho-1}$ [as defined after (22)], $\Gamma_{p-\rho}$ the section of C by $S_{2(p-\rho)-1}$. These two sections define C. For if x is a point of $S_{2\rho-1}$, its null $S_{2\rho-2}$ under Γ_ρ together with $S_{2(p-\rho)-1}$ are contained in an S_{2p-2}, the null space of x under C; similarly, if y is a point of $S_{2(p-\rho)-1}$. If z is a point of neither, the $S_{2\rho}$ joining z to $S_{2\rho-1}$ meets $S_{2(p-\rho)-1}$ in a point y, the line yz meets $S_{2\rho-1}$ in a point x and the null S_{2p-2} of z is determined from those of x and y; the line yzx is of course a null line. If a reference basis of S_{2p-1} be chosen by taking a reference basis x of $S_{2\rho-1}$ and a reference basis y of $S_{2(p-\rho)-1}$, a quadric belonging to C can have only product terms in x and product terms in y. Since the squared terms also are separable the quadric is determined by its two sections. Hence

(62) *If* $S_{2\rho-1}$ *and* $S_{2(p-\rho)-1}$ *are skew null spaces of each other under* C, *and if* Γ_ρ *and* $\Gamma_{p-\rho}$ *are their sections of* C, *then* C *is determined by the two sections. If* Q' *is a quadric in* $S_{2\rho-1}$ *belonging to* Γ_ρ, Q'' *a quadric in* $S_{2(p-\rho)-1}$ *belonging to* $\Gamma_{p-\rho}$ *there is a single quadric* Q *belonging to* C *which contains the sections* Q' *and* Q''. Q *is an* E *quadric if* Q' *and* Q'' *are of the same type, an* O *quadric if* Q' *and* Q'' *are of different types.*

* K., p. 264 (XI), p. 265 (XII).

The last statement is proved as follows: Any line xy is a null line and therefore a tangent or generator of Q. If x is on Q', y on Q'' it is a generator and contains a point z of Q. If x is on Q' and y not on Q'' it must be a tangent to Q at x. If x is not on Q' and y is on Q'' it must be a tangent to Q'' at y. If x is not on Q' and y not on Q'' it must be a tangent to Q at z. Since every point z of Q is on one such line, the number of points on Q if Q' and Q'' are of the same sort is

$$(Q_\rho - 1)(Q_{p-\rho} - 1) + (Q_{p-\rho} - 1) + (Q_\rho - 1) + \bar{Q}_\rho \bar{Q}_{p-\rho} = E_p - 1$$

and Q is an E quadric. If Q' and Q'' are unlike the number of points on Q is

$$(Q_\rho - 1)(\bar{Q}_{p-\rho} - 1) + (Q_\rho - 1) + (\bar{Q}_{p-\rho} - 1) + \bar{Q}_\rho Q_{p-\rho} = O_p - 1$$

and Q is an O quadric.

The above suggests for $\rho = 1$ an obvious construction for the null system C and its quadrics in S_{2p-1} when a null system and its quadrics in S_{2p-3} are given.

According to (62), the 2^{2p} quadrics Q are determined by pairing the $2^{2\rho}$ quadrics Q' with the $2^{2(p-\rho)}$ quadrics Q''. If Q has the section Q' every quadric $Q + S^2$ where S is a null S_{2p-2} on $S_{2\rho-1}$ has the same section. S then must be the null S_{2p-2} of a point y on $S_{2(p-\rho)-1}$. If y lies on Q'', $Q + S^2$ is of the same type as Q, otherwise of a different type. Thus

(63) *If $S_{2\rho-1}$ has no null subspace and if $E(\rho)$ is an E quadric belonging to the section of C by $S_{2\rho-1}$, there are $E_{p-\rho}$ E quadrics and $O_{p-\rho}$ O quadrics containing E_ρ; if $O(\rho)$ is an O quadric belonging to the section of C by $S_{2\rho-1}$ there are $O_{p-\rho}$ E quadrics and $E_{p-\rho}$ O quadrics containing $O(\rho)$.*

For $\rho = 1$ we have the second case in (54), since then a quadric $E(\rho)$ is a pair of points and a quadric $O(\rho)$ has no real points.

We have therefore a method for determining the number and kind of quadrics on a group of points which lie in a space which has no null subspace. Taking up again the case of 2ρ points, azygetic in pairs, which lie in $S_{2\rho-1}$ we first find how many quadrics $Q(\rho)$ are on these points. Taking the points as a reference basis in $S_{2\rho-1}$ the quadric on them must be

$$\Sigma x_i x_k = 0 \qquad (i, k = 1, \cdots, 2\rho;\ i < k).$$

To find the type of this quadric we note that a point lies on it if $4r$ or $4r + 1$ of its coördinates are not zero. The number of its points is $\binom{2\rho}{1} + \binom{2\rho}{4} + \binom{2\rho}{5} + \cdots$. Adding and subtracting $\binom{2\rho}{0} = 1$, this number is obtained from the expansions of $(1 + i^s)^{2\rho}$, $s = 0, 1, 2, 3$, and is

$$\tfrac{1}{4}\{2(1+1)^{2\rho} + (1-i)(1+i)^{2\rho} + (1+i)(1-i)^{2\rho}\} - 1;$$

i. e., $E_\rho - 1$ if $\rho \equiv 0, 1 \pmod 4$ and $O_\rho - 1$ if $\rho \equiv 2, 3 \pmod 4$. Hence from (63),

(64) *The Steiner sets of* 2ρ *points of a self dual basis of* C *have in common* $Q_{p-\rho}$ Q *quadrics if* $\rho \equiv 0, 1$ (mod 4); *and* $\bar{Q}_{p-\rho}$ Q *quadrics if* $\rho \equiv 2, 3$ (*mod* 4).

When $\rho = 1$ we obtain again the second case of (54). Theorems (61) and (64) exhaust the cases arising from points of a self dual basis of C.

The above canonical form of a quadric is of especial interest for $\rho = p$. It is determined by $2p$ points of a basis and must coincide with the quadric of (41) when p is even.

(65) *If the* $2p$ *points of the reference basis* x *of* S_{2p-1} *belong to a self dual basis of* C, *the unique quadric belonging to* C *and on the* $2p$ *basis points*

$$\Sigma x_i x_k = 0$$

coincides with the quadric (41) *when* p *is even. When* p *is odd the quadric does not contain the* $(2p + 1)$*th basis point nor the residual points of the reference basis, but does contain the remaining residual points of the basis. It is an* E *or an* O *quadric according as* $p \equiv 0, 1$ *or* $p \equiv 2, 3$ (*mod* 4).

We define a *Kummer set of quadrics* to be all the quadrics which do *not* pass through a point. The Kummer and Steiner sets of a point exhaust the 2^{2p} quadrics. A *Caporali set of quadrics* consists of all the quadrics which are skew to an ordinary line.* By analogy with the Steiner sets, a Caporali set might be called a second Kummer set. But the analogy could be carried no further, since any S_k, $k > 1$, meets every quadric in real points.

Quadrics and squared S_{2p-2}'s constitute a linear system, R_{2p}; squared S_{2p-2}'s constitute a linear system S_{2p-1}. The quadrics common to the Kummer sets of r linearly independent points satisfy r inequalities and by the use of (3) we see that

(66) *The Kummer sets of* r *linearly independent points have* 2^{2p-r} *quadrics in common.*

An obvious argument from (54) shows that

(67) *The Kummer set of a point contains* 2^{2p-2} Q *quadrics. The Kummer sets of two syzygetic points have* 2^{2p-3} Q *quadrics in common, of three points on a null line no quadrics in common. The Kummer sets of two or three points on an ordinary line have* $\bar{Q}_{p-1}$ Q *quadrics in common which constitute a Caporali set.*

Let us consider the Kummer sets of three points which form a triangle taking up the Cases I, $\cdots$, IV defined above. In case I, a quadric common to the three sets must contain $y + z + t$ and that point only. By projection from the point we ask for the quadrics not on the three points of an ordinary line, i.e., a Caporali set of the projected space. We originally had therefore $2 \cdot \bar{Q}_{p-2}$ Q quadrics. In case IV, the common quadrics contain the null line $y + z$,

* Cf., for the nomenclature: TIMERDING, *Ueber die Gruppirungen der Doppeltangenten einer ebenen Curve vierter Ordnung*, Journal für Mathematik, vol. 122 (1900), p. 209, where the terms are analogously defined for odd quadrics, $p = 3$.

$z + t$, $t + y$, and are projected from this line into the Kummer set of a point whence originally there were

$$4 \cdot 2^{2(p-2)-2} = 2^{2p-4}$$

Q quadrics. In case II, the common quadrics contain $z + t$ and are projected from this point into a Caporali set so that originally there were $2 \cdot \bar{Q}_{p-2}$ Q quadrics. In case III, a quadric not on y, z, t must be on $y + t$, $z + t$, $y + z + t$ and we thus get again the case of (57) with the same result as our present case IV. Hence

(68) *The Kummer sets of the vertices of a triangle have in common* 2^{2p-3} *quadrics. These consist of* $2\ \bar{Q}_{p-2}$ Q *quadrics if none or one of the sides of the triangle are null lines, of* 2^{2p-4} Q *quadrics if two or three sides of the triangle are null lines.*

Case IV is easily generalized. Let $x^{(1)}, \cdots, x^{(m)}$ be a reference basis of a null S_{m-1}. A quadric common to their Kummer sets must contain a null S_{m-2} in S_{m-1} without containing S_{m-1}. Projected from S_{m-2}, S_{m-1} becomes a point and the quadric belongs to the Kummer set of the point; whence

(69) *The Kummer sets of the points of a reference basis of a null* S_{m-1} *have in common* 2^{2p-m-1} Q *quadrics which divide into sets of* 2^{m-1}.

To determine the quadrics common to the Kummer sets of a number of basis points we can utilize a new canonical form of the quadric. Take for reference point basis $2p$ points of a basis of C and let $x_1, \cdots, x_{2p}$ be the coördinates. Any quadric not on the $2p$ points must contain every squared term and, the pairs of points being azygetic, must also contain every product term and therefore is $q = \sum_{i=1}^{2p} x_i^2 + q'$, where

$$q' = \Sigma x_i x_k \qquad (i, k = 1, \cdots, 2p;\ i < k).$$

Since Σx_i is the polar S_{2p-2} of the unit point as to q' and this point lies on q' if p is even, q is of the same type as q' if p is even. Or, from (65), q is an E or an O quadric according as $p \equiv 0, 3$ or $\equiv 1, 2 \pmod 4$.

(70) *If the* $2p$ *points of the reference basis* x *of* S_{2p-1} *belong to a self dual basis of* C, *the unique quadric not on the* $2p$ *basis points*

$$\sum_{i=1}^{2p} \Sigma x_i^2 + \Sigma x_i x_k = 0 \qquad (i, k = 1, \cdots, 2p;\ i < k)$$

is the quadric (41) *when* p *is odd. When* p *is even, the quadric does not contain the residual points of the reference basis but does contain the* $(2p + 1)$*th basis point and the remaining residual points. It is an* E *or an* O *quadric according as* $p \equiv 0, 3$ *or* $\equiv 1, 2$ (*mod* 4).

Given then an even number 2ρ of basis points. They lie in an $S_{2\rho-1}$ without a null subspace. A quadric not on these points is cut by $S_{2\rho-1}$ in a quadric $Q(\rho)$ not on these points. According to (70) there is a unique quadric $Q(\rho)$ of this type. From $Q(\rho)$ we pass to the original quadric as in (63) and find that

(71) *The Kummer sets of 2ρ points of a self dual basis of C have in common $Q_{p-\rho}\,Q$ quadrics if $\rho \equiv 0, 3$ (mod 4), and $\bar{Q}_{p-\rho}\,Q$ quadrics if $\rho \equiv 1, 2$ (mod 4).*

$2\rho + 1$ points of a self dual basis lie in an $S_{2\rho}$ which has a null subspace S_0 and any quadric not on the $2\rho + 1$ points is on S_0. By projection and section from S_0 we obtain the case of (71) and have therefore shown that

(72) *The Kummer sets of $2\rho + 1$ points of a self dual basis of C have in common $2\;Q_{p-\rho-1}\,Q$ quadrics if $\rho \equiv 0, 3$ (mod 4), and $2\;\bar{Q}_{p-\rho-1}\,Q$ quadrics if $\rho \equiv 1, 2$ (mod 4).*

The Steiner and Kummer sets have received considerable attention in the particular case, $p = 3$. Numerous other sets are suggested by the smaller values of p and can be readily generalized and discussed by the foregoing methods. In the next paragraph a somewhat different point of view is emphasized and again the geometrical method of treatment seems most effective.

§ 6. *Systems of Quadrics.*

A linear system F_r of quadrics is determined by $r + 1$ linearly independent quadrics $Q, Q_1, \cdots, Q_r$. The sum of any even number of the quadrics is the square of an S_{2p-2} which has a null point y. Let $Q + Q_i$ have the null point $y^{(i)}$. Then F_r determines an S_{r-1} with the reference basis $y^{(1)}, \cdots, y^{(r)}$. Moreover any reference basis of S_{r-1} together with any quadric of F_r determines F_r. Let us call S_{r-1} the *allied space of* F_r. Two systems, F_r and F'_r, with the same allied space and a common quadric coincide. Thus the 2^{2p} quadrics can be divided in a single way into 2^{2p-r} systems F_r with a given common allied space S_{r-1}. These 2^{2p-r} systems F_r are called a ***complex allied with*** S_{r-1}. Two systems, F_r and F_s, are ***skew systems*** or ***null systems of each other*** if their allied spaces are respectively skew spaces or null spaces of each other. Two skew systems may or may not have one common quadric. The two cases can be distinguished by the respective terms *partially skew* or *completely skew*. A *Göpel system* F_r has for allied space a null S_{r-1}; a *Göpel system* is allied with a Göpel space S_{p-1}.*

There are two types of system F_0, namely, an E quadric and an O quadric. There are three types of system F_1, a pair of E quadrics, a pair of O quadrics, and an E and O quadric. The allied space S_0 is on the quadrics in the first two types but not in the last.

Let a system F_2 be determined by the quadric Q and the allied space S_1

* Cf., for these definitions, K., p. 296, § 9.

with points y, z, $y + z$. The system contains four quadrics,

$$Q, \quad Q + C(x^2, y), \quad Q + C(x^2, z), \quad Q + C(x^2, y + z),$$

whose sum is identically zero. Any three of the four determine the system. According as the line S_1 is skew to, tangent to, secant to, or on, Q, three, two, one, or none, of the quadrics are $\bar{Q}$ quadrics. That is, the set of four contains an even number of each kind if S_1 is a null line, an odd number of each kind if S_1 is an ordinary line. In the first case we say that any three of the four are syzygetic;* in the second case any three of the four are azygetic.* There are then five types of system F_2: three syzygetic types with $4, 2, 0, E$ quadrics respectively; and two azygetic types with $3, 1, E$ quadrics respectively. In the first and third types S_1 is a generator of the four quadrics; in the second type S_1 touches one pair at one point, the other pair at another point; in the fourth and fifth types S_1 is skew to the one quadric and cuts the other three in two out of three of its points.

Given two quadrics of the same type, Q, $Q' = Q + C(x^2, y)$, where y is on Q, the number of pairs which can be added to the given pair to form a syzygetic or azygetic system is determined by the numbers σ_1, σ_2, σ_3, [(33) and (34)]. If the given pair are of opposite type, Q, $\bar{Q}$, the numbers ρ_0, ρ_1, ρ_2, serve. Hence

(73) *Given two quadrics* Q, Q', *there are* $\bar{Q}_{p-1}$ *pairs* $\bar{Q}$, $\bar{Q}'$, *and* $Q_{p-1} - 1$ *pairs* Q'', Q''', *each syzygetic with the two and* 2^{2p-2} *pairs* Q'', $\bar{Q}$, *each agyzetic with the two; given two quadrics* Q, $\bar{Q}$, *there are* P_{2p-3} *pairs*, Q', $\bar{Q}'$, *each syzygetic with the two, and* $\bar{Q}_{p-1}$ *pairs* $\bar{Q}'$, $\bar{Q}''$, *and* Q_{p-1} *pairs* Q', Q'', *each azygetic with the two.*

Similarly, the numbers π_0, π_1, π_2, π_3 [(35) and (36)] serve to determine the number of sets of three quadrics syzygetic or azygetic with one given quadric.

(74) *A* Q *quadric is syzygetic with* $\bar{Q}_{p-1}(Q_p - 1)$ *triads* Q', $\bar{Q}$, $\bar{Q}'$ *and with* $\frac{1}{3}(Q_p - 1)(Q_{p-1} - 1)$ *triads* Q', Q'', Q''', *azygetic with* $\frac{1}{3}\bar{Q}_p \bar{Q}_{p-1}$ *triads* $\bar{Q}$, $\bar{Q}'$, $\bar{Q}''$, *and with* $2^{2p-3}(Q_p - 1)$ *triads* Q', Q'', $\bar{Q}$.

An obvious enumeration and the use of (38) leads to the following result:

(75) *There are* $\frac{1}{12} Q_p (Q_p - 1)(Q_{p-1} - 1) = \frac{1}{3} P_{2p-1} P_{2p-3} Q_{p-2}$ *syzygetic tetrads* Q, Q', Q'', Q'''; $\frac{1}{2} Q_p \bar{Q}_{p-1}(Q_p - 1) = 2^{2p-4} P_{2p-1} P_{2p-3}$ *syzygetic tetrads* Q, Q', $\bar{Q}$, $\bar{Q}'$; *and* $\frac{1}{3} 2^{2p-2} P_{2p-1} Q_{p-1}$ *azygetic tetrads* Q, Q', Q'', $\bar{Q}$.

Let $s^{(1)}, \cdots, s^{(2p+1)}$, be a self dual basis of C and let $Q^{(0)}$ be any quadric. $Q^{(0)}$ and the $2p + 1$ quadrics $Q^{(i)} = Q^{(0)} + S^{(i)2}$ are subject to the single relation $\sum_{i=0}^{2p+1} Q^{(i)} = 0$. Any three of the quadrics are azygetic if $S^{(i)} + S^{(k)}$, $S^{(k)} + S^{(l)}$, $S^{(l)} + S^{(i)}$ do not form a null pencil. $S^{(i)} + S^{(k)}$ and $S^{(k)} + S^{(l)}$

* K., p. 253.

are in an ordinary pencil if

$$(S^{(i)} + S^{(k)},\ S^{(k)} + S^{(l)}) = (S^{(i)},\ S^{(k)}) + (S^{(i)},\ S^{(l)}) + (S^{(k)},\ S^{(l)}) \neq 0.$$

But by the definition of the basis, $(S^{(i)}, S^{(k)}) \neq 0$. Such a set of $2p+2$ quadrics any three of which are azygetic is called a *fundamental set* and is denoted by F.S. If the set be given and one be added to the others the squares of $2p+1$ S_{2p-2}'s are obtained which must form a basis because of the azygetic property of the quadrics. Thus a basis determines 2^{2p} F.S.'s while an F.S. is determined from $2p+2$ bases. The number of sets then is $N_{BC} 2^{2p} / (2p+2)$.

(76) *A fundamental set, F.S., of $2p+2$ quadrics (a set such that all are connected by one linear relation and any three are azygetic) is obtained by adding one quadric to the squared S_{2p-2}'s of a basis of C. If any quadric of a F.S. be added to the others a self dual basis of C is obtained. The number N_F of F.S.'s is**

$$N_F = \frac{2^{2p+p^2}(2^{2p}-1)(2^{2p-2}-1)\cdots(2^2-1)}{(2p+2)!} = \frac{2^{p(p+2)}}{(2p+2)!} P_{2p-1} P_{2p-3} \cdots P_1 .$$

In (41) we showed that with a self dual basis $y_1, \cdots, y_{2p+1}$, there is associated a unique quadric

$$R_y = \Sigma\, y_i y_k \qquad (i,\ k = 1,\ \cdots,\ 2p+1,\ i < k).$$

From the basis and R_y we can construct a definite F.S. which has certain special properties and which will be called a *normal* F.S. Since R_y contains the basis points if p is even, the quadrics of the normal F.S. are then all of the same type. When p is odd R_y is the only one of its type.

(77) *The normal F.S. determined by the basis y of C and its quadric R_y contains only E quadrics if $p = 0$ (mod 4); contains only O quadrics if $p \equiv 2$ (mod 4); contains the O quadric R_y and $2p+1$ E quadrics if $p \equiv 1$ (mod 4); and contains the E quadric R_y and $2p+1$ O quadrics if $p \equiv 3$ (mod 4). Moreover a F.S. which contains $2p+1$ quadrics of the same type is a normal F.S. for which the remaining quadric is R_y.*†

The last statement is proven as follows: Let Q, $Q + y_i^2$, $i = 1, 2, \cdots, 2p+1$, be a F.S. such that the quadrics $Q + y_i^2$ are all of the same type. The null points of y_1 and y_2 are both on or both off Q according to the type of $Q + y_i^2$. Their line is an ordinary line since any three of the quadrics are azygetic and is either a secant or skew line of Q. In either case the residual point, the null point of $y_1 + y_2$ is not on Q, and Q is the quadric R_y associated with the basis.

Any F.S., Q, $Q + y_i^2$, determines not only the basis y but also the $2p+1$ bases gotten by varying k in y_k, $y_k + y_i$ $(i \neq k)$. This set of $2p+2$ bases

* Cf. K., pp. 283–5.

† Cf. K., pp. 274–6; also p. 288.

is a symmetrical set consisting of any one basis and its residual points. The set is transformed into itself by a $G_{(2p+2)!}$, the symmetric group on the $2p+2$ bases, which is generated by the $(p+1)(2p+1)$ involutions I_x on the point x belonging to the bases. Sample involutions are $y'_i = y_i$, $y'_k = y_k + y_i$ $(i \neq k)$ and $y'_i = y_k$, $y'_k = y_i$, $y'_l = y_l\,(l \neq i,\ l \neq k)$. The quadric R_y attached to one basis is also attached to the others if p is odd since according to (41) it contains neither the basis points nor the residual points. But if p is even, the basis points are on R_y and the residual points are not, whence there are $2p+2$ quadrics R_y.

(**78**) *The points and residual points of a basis form a basis configuration, i. e., a set of $(p+1)(2p+1)$ points which can be divided into a basis and its residual points in $2p+2$ ways, each point lying in two bases. The configuration is unaltered by a $G_{(2p+2)!}$ generated by the involutions on its points, which is symmetric on its bases. Any F.S. determines a basis configuration.*

If p is even, the basis configuration determined by a normal F.S. contains bases whose quadrics R_y make up the normal F.S. But if p is odd the basis configuration of the normal F. S. determines the unique quadric R_y, isolated in the normal F.S. R_y and the basis configuration determine $2p+2$ normal F.S.'s containing $(p+1)(2p+1)$ quadrics (each quadric in two F.S.'s) apart from R_y which occurs in each F.S.

(**79**) *If p is even a normal F.S. contains the $2p+2$ quadrics attached to the bases of the configuration determined by the F.S. and is invariant under the configuration $G_{(2p+2)!}$. If p is odd, a normal F.S. is one of a set of $2p+2$ normal F.S.'s each having the same isolated quadric and basis configuration. The configuration $G_{(2p+2)!}$ permutes the $2p+2$ F.S.'s symmetrically, each F.S. being invariant under the $G_{(2p+1)!}$ attached to its particular basis.*

For example, when $p = 2$ the 15 points of S_3 form a single basis configuration containing the 6 bases in S_3. There is but one normal F.S., consisting of the 6 O quadrics attached to the 6 bases. When $p = 3$, a normal F.S. containing one E quadric and 7 O quadrics determines a basis configuration. This configuration contains 8 bases and arises from 8 normal F.S.'s each having the same E quadric. That is, the Aronhold sets of seven can be grouped in 36 ways, corresponding to the even characteristics, into 8 sets, each odd characteristic occurring in two sets.

The following enumeration is immediate:

(**80**) *If $p \equiv 0$ or 3 (mod 4), for each E quadric there are N_{BC}/E_p self dual bases of C whose lines are secant or skew lines respectively of E. If $p \equiv 2$ or 1 (mod 4) there are N_{BC}/O_p self dual bases of C whose lines are secant or skew lines respectively of O. If p is odd the bases can be grouped into basis configurations.*

The normal F.S. affords a convenient method for studying the general F.S.

Let $Q, Q_1, \cdots, Q_{2p+1}$, where $Q + Q_i = y_i$, be any F.S. Since $\sum_1^{2p+1} y_i^2 = 0$, any S_{2p-2} can be expressed as $\sum_{i=1}^{2k} y_i^2$, where $k = 1, \cdots, p$. Hence Q itself can be written as, $Q = R_y + \sum^{2k} y_i^2$, and the F.S. takes the form

$$Q = (R_y + \sum^{2k} y_i^2), \quad Q_1 = (R_y + \sum^{2k} y_i^2) + y_1^2, \quad \cdots, \quad Q_h = (R_y + \sum^{2k} y_i^2) + y_h^2, \quad \cdots.$$

Q and R_y are of the same or different type according as k is even or odd. The quadrics Q_h divide into two sets, according as y_h^2 does or does not occur in $\sum^{2k} y_i^2$. Hence we have

(α) 1 quadric Q of the form $R_y + \sum^{2k} y_i^2$,

(β) $2k$ quadrics of the form $R_y + \sum^{2k-1} y_i^2 = R_y + \sum^{2p-2k+2} y_j^2$,

(γ) $2p + 1 - 2k$ quadrics of the form $R_y + \sum^{2k+1} y_i^2 = R_y + \sum^{2p-2k} y_j^2$.

Quadrics (α), (β), (γ) are of the type R if respectively k is even, $p - k$ is odd, $p - k$ is even. Thus we have the table:

Residue of p modulo 4	R of type	Number of O quadrics for k even	Number of O quadrics for k odd
0	E	$2k$	$2(p-k)+2$
1	O	$2k+1$	$2(p-k)+1$
2	O	$2(p-k)+2$	$2k$
3	E	$2(p-k)+1$	$2k+1$

Hence if s is the number of O quadrics, $s \equiv p \pmod 4$. Moreover, for any such number s, a F.S. containing s O quadrics can be constructed by determining k in the table from the given s. The required F.S. can be written down in terms of the basis and $2k$ arbitrarily selected S_{2p-2}'s of it. Since any two bases are conjugate and within a basis any two sets of $2k$ spaces are conjugate, then any two F.S.'s with the same number s are conjugate, provided that the same F.S. can be determined from k even or k odd—a double possibility for the same s. Denote by y' the basis

$$y_1' = y_1, \qquad y_i' = y_1 + y_i \qquad (i = 2; \cdots, 2p+1).$$

If p is odd let y_1 be not contained in $\sum^{2k} y_i^2$. Then $R_y = R_{y'}$, $\sum^{2k} y_i^2 = \sum^{2k} y_i'^2$, and the above F.S. is

$$Q_1 = (R_{y'} + \sum^{2k+1} y_i'^2), \quad Q = (R_{y'} + \sum^{2k+1} y_i'^2) + y_1'^2, \quad Q_2 = (R_{y'} + \sum^{2k+1} y_i'^2) + y_2'^2, \quad \cdots.$$

But

$$\sum^{2k+1} y_i'^2 = \sum^{2(p-k)} y_j'^2 .$$

Hence, if p is odd, k can be replaced by $p - k$. If p is even, let y_1 be in $\sum^{2k} y_i^2$. Then

$$R_y = R_{y'} + y_1'^2, \qquad \sum^{2k} y_i^2 = \sum^{2k-1} y_i'^2,$$

and the original F.S. is

$$Q_1 = (R_{y'} + \sum^{2k-1} y_i'^2), \quad Q = (R_{y'} + \sum^{2k-1} y_i'^2) + y_1'^2, \quad Q_2 = (R_{y'} + \sum^{2k-1} y_i'^2) + y_2'^2, \quad \cdots .$$

Since $\sum^{2k-1} y_i'^2 = \sum^{2(p-k)+2} y_j'^2$, if p is even, k can be replaced by $p - k + 1$. Thus the same F.S. can be gotten from an even or an odd k. If Q and Q_i are of the same type, the involution, I_x, determined by $Q + Q_i$, interchanges Q and Q_i and leaves all the other quadrics unaltered. No collineation other than the identity can leave every quadric unaltered.

(81) *A F.S. contains s O quadrics, where $s \equiv p$ (mod 4). If $s \equiv p$ (mod 4), there are $N / \{s\,!\,(2p + 2 - s)\,!\}$ F.S.'s containing s O quadrics, all conjugate under G_N and each invariant under a subgroup $G_{s!\,(2p+2-s)!}$ of G_N. This subgroup is the product of the interchangeable groups $G_{s!}$ and $G_{(2p+2-s)!}$ the symmetric groups on the O and E quadrics respectively of the F.S. The subgroup has an invariant quadric or an invariant space (the sum of the O or E quadrics in F.S.), according as p is odd or even. It has also two invariant skew spaces S_{s-2} and S_{2p-s}, the allied spaces of the O and of the E quadrics. The F.S. can be obtained from a normal F.S. by adding a squared S_{2p-2}.**

Let S_{p-1} be a Göpel space, F_p a Göpel system. If F_p contains two quadrics, Q and $\bar{Q}$, of different types, any third quadric Q' of F_p is syzygetic with the two and therefore paired with a fourth $\bar{Q}'$ of different type, so that $Q + \bar{Q} + Q' + \bar{Q}' = 0$. Hence F_p contains 2^{p-1} E quadrics and 2^{p-1} O quadrics. If however the quadrics of F_p are all of the same type, they all must contain S_{p-1}. Since there are, on a null S_k, $2^{k+1}\, Q_{p-k-1}$ Q quadrics [60], and $E_0 = 1$ while $O_0 = 0$, S_{p-1} is on 2^p E quadrics and those only. Hence

(82) *In every complex of 2^p Göpel systems there is one Göpel system which contains only E quadrics; each of the $2^p - 1$ other Göpel systems contains 2^{p-1} Q quadrics. Each set of 2^{p-1} Q quadrics has an allied null space S_{p-2} which lies on every quadric of the set. Any three quadrics in a system are syzygetic.*†

A precisely similar argument applies to the case where the S_{m-1} allied to the system F_m is a null S_{m-1}. If F_m contains a quadric of each type half of the 2^m quadrics in F_m are Q quadrics. If all the quadrics are of one type they

* Cf. K., pp. 286–9; XXVIII–XXXI.

† K., p. 300, XXXIV.

all contain S_{m-1}. There are $2^m\, Q_{p-m}\, Q$ quadrics containing S_{m-1} and these divide in a unique way into sets of 2^m with the allied space S_{m-1}.

(83) *If S_{m-1} is a null space, the complex of $2^{2p-m}\, F_m$'s with the allied space S_{m-1} contains Q_{p-m} systems F_m containing only Q quadrics. The members of these systems contain S_{m-1}. The remaining $2^{2(p-m)}\, P_{m-1}$ systems F_m contain 2^{m-1} quadrics of each type.**

Denote by $\phi_{Q,m}$ the system F_m containing quadrics of the type Q only. Such a system becomes, after projection and section from S_{m-1}, a single quadric of type Q in the derived space $\Sigma_{2\pi-1}$, $\pi = p - m$. From the definition of syzygetic and azygetic quadrics in $\Sigma_{2\pi-1}$ we have

(84) *Three systems $\varphi_{Q,m}$ allied with the null S_{m-1} are contained in a system F_{m+2} allied with an S_{m+1} on S_{m-1}. The three systems are syzygetic or azygetic according as S_{m+1} is a null or an ordinary space. The entire theory here developed of the quadrics Q belonging to C in S_{2p-1}, when applied to the quadrics belonging to C_π in $\Sigma_{2\pi-1}$ yields an analogous theory of the systems $\varphi_{Q,m}$ in S_{2p-1}.*

This theorem is due essentially to FROBENIUS† though its remarkable utility, exemplified in many of the preceding theorems, is clearly apparent only when it is viewed as a result of the general process—projection and section from a null space.

Let us now investigate the "section C_ρ of C by the space $S_{2\rho-1}$ which has no null subspace"; i. e., by *the Rosenhain space $S_{2\rho-1}$*. Let us call a system of quadrics, $F_{2\rho}$, with the allied Rosenhain space, $S_{2\rho-1}$, *a Rosenhain system.*‡ A Rosenhain space, $S_{2\rho-1}$, determines its *complementary* skew Rosenhain space, $S_{2(p-\rho)-1}$, the two being null spaces of each other. According to (62) and (63),

(85) *The $2^{2\rho}$ quadrics of a Rosenhain system $F_{2\rho}$ allied with an $S_{2\rho-1}$ have the same section $Q\,(p-\rho)$ by the complementary Rosenhain space $S_{2(p-\rho)-1}$. They divide into E_ρ Q quadrics and O_ρ $\bar{Q}$ quadrics. The complex of $2^{2(p-\rho)}$ Rosenhain systems with the allied $S_{2\rho-1}$ contains $E_{p-\rho}$ systems with E_ρ E quadrics and O_ρ O quadrics and $O_{p-\rho}$ systems with E_ρ O quadrics and O_ρ E quadrics.*

Denoting by $\psi_{E,\rho}$ and $\psi_{O,\rho}$ these respective systems we obtain the theorem analogous to (84):

(86) *Three Rosenhain systems. $\psi_{Q,\rho}$ allied with the Rosenhain space $S_{2\rho-1}$ are contained in a system $F_{2\rho+2}$ allied with an $S_{2\rho+1}$ which cuts the complementary Rosenhain space $S_{2(p-\rho)-1}$ in an S_1. The three systems are syzygetic or azygetic according as $S_{2\rho+1}$ is not or is a Rosenhain space; or also according as S_1 is not or is an ordinary line, i. e., a Rosenhain S_1. The theory developed above of the quadrics Q belonging to C in S_{2p-1}, applied to the quadrics belonging to $C_{p-\rho}$ in $S_{2(p-\rho)-1}$, yields an analogous theory of the systems $\psi_{Q,\rho}$ in S_{2p-1}.*

* K., p. 303, XXXVI.

† Cf. K., pp. 302–5, where references are given.

‡ For $p = 2$, $\rho = 1$, cf. K., pp. 337–8.

Let us determine finally the number of E and O quadrics in the general system F_r with allied space S_{r-1}. Let S_{r-1} have the null subspace S_{m-1} with reference basis $x^{(1)}, \cdots, x^{(m)}$. Let S_{r-m-1} be a space skew to S_{m-1} in S_{r-1} with reference basis $y^{(m+1)}, \cdots, y^{(r)}$ which is part of a self dual basis of C. According to (20), $r - m$ is even and S_{r-m-1} is a Rosenhain space. F_r is fixed by means of S_{r-1} and any one of its quadrics Q. If Q contains S_{m-1} so also does every quadric of F_r. Then either all or none of the quadrics of F_r contain S_{m-1}. In the latter case Q and S_{m-1} determine a system F_m and by adding to F_m the null spaces of the $2^{r-m} - 1$ points in S_{r-m-1}, 2^{r-m} systems F_m are obtained each consisting [see (83)] of $2^{m-1}\,Q$ quadrics. The systems F_m exhaust the system F_r, whence F_r contains $2^{r-1}\,Q$ quadrics. In case, however, Q contains S_{m-1}, the system F_r becomes by projection and section from S_{m-1}, a system F_{r-m} with an allied Rosenhain space derived from the projection of S_{r-m-1}. In S_{2p-1} the complementary Rosenhain space of S_{r-m-1} is an $S_{2p-(r-m)-1}$ which has in common with the null S_{2p-m-1} of S_{m-1} an $S_{2p-m-(r-m)-1}$ which contains S_{m-1} and a skew space $S_{2(p-m)-(r-m)-1}$. This last space and S_{r-m-1} itself project from S_{m-1} into complementary Rosenhain spaces in $\Sigma_{2\pi-1}$. Then theorem (85) can be applied to the system F_{r-m} in $\Sigma_{2\pi-1}$ and the result translated to S_{2p-1}. Hence

(87) *A space S_{r-1} with null subspace S_{m-1} is determined by the Göpel space S_{m-1} and a skew Rosenhain space S_{r-m-1}. The complementary Rosenhain space cuts the null space of the Göpel space in an $S_{2p-m-(r-m)-1}$ which can be determined by the Göpel space and a skew $S_{2(p-m)-(r-m)-1}$. By projection and section from the Göpel space, S_{r-m-1} and $S_{2(p-m)-(r-m)-1}$ become complementary Rosenhain spaces.*

*A system F_r determined by Q' and S_{r-1} contains $2^{r-1}\,Q$ quadrics if Q' does not contain S_{m-1}. If Q' contains S_{m-1} and meets $S_{2(p-m)-(r-m)-1}$ in a quadric of type Q'', then F_r contains $2^m\,E_{(r-m)/2}$ quadrics of type Q'' and $2^m\,O_{(r-m)/2}$ quadrics of type $\bar{Q}''$.**

Krazer remarks (p. 266) that the Per. and Th. Char. have been confused by various writers. He notes (pp. 253–4) some points of difference between the two but does not call express attention to the fact—fundamental in the exposition here given—that the coefficients of the transformation occur *linearly* in the transformation of the Per. Char. and *quadratically* in the transformation of the Th. Char. Though we find (p. 254) that "die Summe einer geraden Anzahl von Th. Char. sich wie eine Per. Char., die Summe einer ungeraden Anzahl von Th. Char. aber wie eine Th. Char. transformiert," yet it is stated (p. 284) that "Die Summe der $2p + 2$ Th. Char. eines F.S. is [0]," i. e., is the zero Th. Char.; and (p. 305) that "Durch . . . der Addition einer beliebigen Th. Char. zu den sämtlichen Th. Char. eines Systems geht ein System von Th. Char. immer wieder in ein System von Th. Char. über."

* Cf. K., p. 301, XXXV.

Consider also the statement (p. 270): "Man fasse nun die (Per.) charakteristiken des F.S. als Th. Char. . . . auf." A Per. Char. can be regarded as a Th. Char. only through the intervention of some given Th. Char. which Krazer implicitly takes to be the zero Th. Char. However if entire accuracy is sought in the use of such a process one should state whether the resulting theorems are or are not independent of the given Th. Char. employed for the transition, i. e., whether the results are covariant under G_N or covariant only under the subgroup of G_N defined by the given Th. Char. Such distinctions or limitations are almost self evident from the geometrical point of view.

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