CONFORMAL TRANSFORMATIONS ON THE BOUNDARIES OF THEIR REGIONS OF DEFINITION*

BY

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Riemann's problem of mapping a simply connected plane region whose boundary consists of more than a single point conformally on a circle as normal region may be divided into two parts: (a) the internal problem; namely, the map of the interior points,† and (b) the boundary problem; namely, the behavior of the map on the boundary.

The first of these problems was treated by RIEMANN in his dissertation with the aid of Dirichlet's principle, and rigorous proofs were supplied for regions with restricted boundaries by SCHWARZ and NEUMANN, the general case being established by OSGOOD through methods due to POINCARÉ.

The second problem was solved for analytic boundaries by Schwarz, the case of vertices being disposed of by PICARD.‡ PAINLEVÉ§ treated the case of arcs of boundaries that are convex regular curves, and his results can be extended by mere linear transformations to any boundaries consisting of arcs of regular curves with continuous curvature. For wholly arbitrary boundaries the problem of the continuity of the map on the boundary was considered by OSGOOD|| in a series of theorems. It is the purpose of the present paper to give proofs of these theorems and to supplement them.

The propositions of § 5,—more particularly, Lemmas 2 and 3, for in Lemma 1 it is only the uniqueness of the function that is new,—were suggested by intuition, a harmonic function being thought of as the temperature in a flow of heat or the potential in a flow of electricity.

In the comprehensiveness of the results here obtained, applying as they do to the most general boundaries that consist of a single piece and more than one point, lies, it is believed, an intrinsic interest. So many problems of

* Presented to the Society December 30, 1902 and April 30, 1910.

† For an account of the first problem and the second in the case of analytic boundaries, with references to the literature, cf. OSGOOD, *Lehrbuch der Funktionentheorie*, vol. 1, 2nd ed., 1912, chap. 14.

‡ PICARD, *Traité d'analyse*, vol. 2, chap. 10, § 7.

§ Comptes Rendus, vol. 112 (1891), p. 653.

|| Bulletin of the American Mathematical Society, vol. 9 (1903), p. 233.

mathematics must be restricted in order to be treated at all, that it is a source of satisfaction when the most general result of its class can be established.

§ 1. *Line and Surface Integrals.*

Let the functions u, u_1, u_2 be harmonic in a finite region S of the (x, y)-plane, whose boundary is denoted by C. Under suitable further restrictions pertaining to the boundary and the behavior of the functions there, the following equations are familiar.*

$$(A) \qquad \iint\limits_S \left(\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial y}\right) dS = -\int\limits_C u_2 \frac{\partial u_1}{\partial n} ds;$$

$$(B) \qquad \iint\limits_S \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dS = -\int\limits_C u \frac{\partial u}{\partial n} ds;$$

$$(C) \qquad \int\limits_C u_1 \frac{\partial u_2}{\partial n} ds = \int\limits_C u_2 \frac{\partial u_1}{\partial n} ds;$$

$$(D) \qquad \int\limits_C \frac{\partial u}{\partial n} ds = 0.$$

We shall need these equations for a somewhat more general case than the one ordinarily treated. It is sufficient for our purposes to assume that C consists of a finite number of arcs, each analytic inclusive of its extremities. Let P be a vertex of C, and denote the angle at P by α; then shall $0 < \alpha < 2\pi$.

The functions u, u_1, u_2 shall satisfy the following conditions:

1) u, u_1, u_2 are finite and harmonic within S;

2) u, u_1, u_2 are continuous on the boundary;

3) the first partial derivatives of u, u_1, u_2 are continuous on the boundary except at one or more of the vertices. In the neighborhood of such a vertex, P, each of these derivatives remains less numerically than $Cr^{-\mu}$, where r denotes the distance of (x, y) from P, where C and μ are constants, and where $0 < \mu < 1$:

$$\left|\frac{\partial u_1}{\partial x}\right| < \frac{C}{r^\mu}, \text{ etc.}$$

Under these conditions it is readily shown that each of the above integrals which is an improper integral converges. Let S' be the part of S that remains after a small neighborhood of each vertex P has been cut off by an arc γ of a circle of radius h and center P. All four equations hold for the region S'.

* In all line integrals involving ds, which appear in this paper, ds is taken as a positive quantity. Cf. the definition laid down by Osgood, *Funktionentheorie*, vol. 1, chap. 4.

Moreover, it is seen that the contributions of the above line integrals due to the arcs γ all approach 0 simultaneously with h, and this establishes the four equations under the given conditions.

§ 2. *A Lemma.*

LEMMA. *Let u satisfy the following conditions:*

1) *u is finite and harmonic within S;*

2) $$u\,|_L = 0,$$

where L denotes an arc of the boundary of S abutting on a vertex P, the relation holding at all points of L except P;

3) $$\left.\frac{\partial u}{\partial n}\right|_{L'} = 0,$$

*the relation holding at all points of L' except P.**

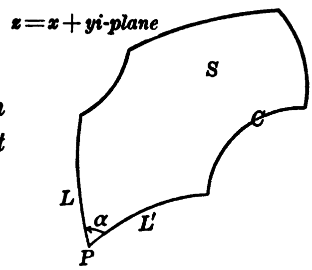

FIG. 1.

Then

$$\left|\frac{\partial u}{\partial x}\right| < \frac{C}{r^\mu}, \qquad \left|\frac{\partial u}{\partial y}\right| < \frac{C}{r^\mu},$$

where r denotes the distance of (x, y) from P, where C is a positive constant, and where $\mu = 1 - \frac{1}{2}\nu$, $\nu = \pi/\alpha$. Moreover, u is harmonic along L'.

Let the interior of S be transformed conformally on the upper half of the $w = \xi + \eta i = \rho e^{\theta i}$-plane by means of the function

$$(1) \qquad w = f(z),$$

the point P going over into the point O: $w = 0$. Then u, considered as a function of ξ, η, is finite and harmonic throughout the interior of a semicircle K_1: $|w| < R_0$, $0 < \eta$; and if R_0 be suitably restricted, u vanishes along the radius OQ_1: $\theta = \pi$, $0 < \rho \leqq R_0$, while $\frac{\partial u}{\partial n} = \frac{1}{\rho}\frac{\partial u}{\partial \theta}$ vanishes along the radius $OQ_2: \theta = 0$, $0 < \rho \leqq R_0$. Hence u is capable of harmonic continuation by reflection across OQ_1 into the lower half-plane, and u, thus extended, is seen to be harmonic throughout the domain K: $0 < \rho < R_0$, $0 < \theta < 2\pi$.

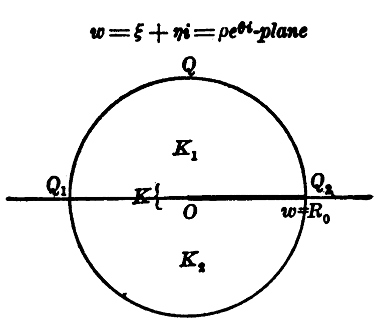

FIG. 2.

Consider the function

$$v = \frac{\partial u}{\partial \theta}.$$

* It is assumed throughout the paper, when the condition is imposed that a normal derivative vanish, that the first partial derivatives of the function are continuous along the piece of the boundary in question except at the extremities of the arc.

This function is harmonic in K, as follows by substitution in Laplace's equation. Moreover, v vanishes on each bank of the cut $OQ_2 : \theta = 0$, since along the upper bank

$$0 = \frac{\partial u}{\partial n} = \frac{1}{\rho}\left(\frac{\partial u}{\partial \theta}\right)_{\theta=0} \qquad (0 < \rho),$$

and along the lower bank

$$\left(\frac{\partial u}{\partial \theta}\right)_{\theta=2\pi-} = \left(\frac{\partial u}{\partial \theta}\right)_{\theta=0+}.$$

Subject the w-plane to the transformation

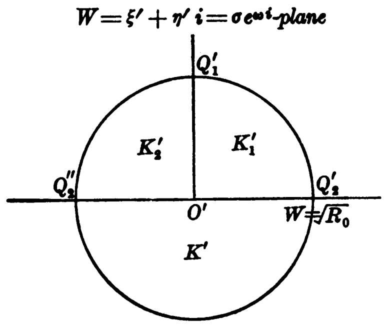

FIG. 3.

$$W = w^{\frac{1}{2}},$$

where

$$W = \sigma e^{\omega i}, \qquad \sigma = \sqrt{\rho}, \qquad \omega = \tfrac{1}{2}\theta.$$

Then v goes over into a function of σ, ω:

$$v = f(\sigma, \omega),$$

harmonic within the semicircle $Q''_2\, Q'_1\, Q'_2$ and vanishing along the diameter $Q''_2\, Q'_2$, except perhaps at O'. It is, therefore, possible to continue v by reflection harmonically across this diameter, and we are thus led to a function harmonic throughout the complete circle K': $0 \leqq \sigma < \sqrt{R_0}$ except at O'.

The function v can be developed into a series of the form*

$$v = k \log \sigma + \sum_{n=-\infty}^{\infty} \sigma^n\,(a_n \cos n\omega + b_n \sin n\omega),$$

where

$$a_n = \frac{1}{\pi a^n}\int_0^{2\pi} V \cos n\omega\, d\omega, \qquad n \neq 0; \qquad b_n = \frac{1}{\pi a^n}\int_0^{2\pi} V \sin n\omega\, d\omega,$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} V d\omega - k \log a, \qquad V = f(a, \omega) \qquad (0 < a < \sqrt{R_0}).$$

We proceed next to show that $a_n = 0$ and that $b_{2m} = 0$. In points of K' situated symmetrically with respect to the axis of ξ', v has values that are equal and opposite, while $\cos n\omega$ has equal values. Hence

$$\int_0^{\pi} V \cos n\omega\, d\omega = -\int_{\pi}^{2\pi} V \cos n\omega\, d\omega$$

and

$$a_n = 0, \qquad n \neq 0; \qquad a_0 = -k \log a.$$

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* OSGOOD, *Lehrbuch der Funktionentheorie*, vol. 1, 2nd ed., 1912, p. 660.

Furthermore, in two points of K_1' and K_2' which are situated symmetrically with respect to the axis of η', v has equal values, since such points are the images of points in the w-plane situated symmetrically with respect to the axis of reals, and in these latter points u has values which are equal and opposite. Hence

$$f\left(\sigma, \frac{\pi}{2} - \omega\right) = f\left(\sigma, \frac{\pi}{2} + \omega\right) \qquad (0 < \sigma < \sqrt{R_0}).$$

But

$$\sin n\left(\frac{\pi}{2} - \omega\right) = -\sin n\left(\frac{\pi}{2} + \omega\right)$$

if n is even. Hence, for such values of n,

$$\int_{-\pi/2}^{\pi/2} V \sin n\omega \, d\omega = -\int_{\pi/2}^{3\pi/2} V \sin n\omega \, d\omega,$$

or

$$b_{2m} = 0.$$

The result thus far obtained is, then, this:

$$v = k \log\frac{\sigma}{a} + \sum_{n=-\infty}^{\infty} b_{2n+1}\sigma^{2n+1} \sin(2n+1)\omega.$$

Since $v = 0$ when $\theta = 0$, and hence when $\omega = 0$, it follows that $k = 0$, $a_0 = 0$.

Consider next the function u carried over into the W-plane. Since

$$\frac{\partial u}{\partial \omega} = \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial \omega} = 2\frac{\partial u}{\partial \theta} = 2v,$$

it follows that

$$\frac{\partial u}{\partial \omega} = 2\sum_{n=-\infty}^{\infty} b_{2n+1}\sigma^{2n+1}\sin(2n+1)\omega.$$

The function u vanishes along the line $\omega = \pi/2$. Hence u and its harmonic continuation will be given at any point (σ, ω) of the circle K' distinct from the center by the integral

$$\int_{\pi/2}^{\omega} \frac{\partial u}{\partial \omega} d\omega,$$

σ being held fast. Along this path the above series converges uniformly. Hence

$$u = -2\sum_{n=-\infty}^{\infty} \frac{b_{2n+1}}{2n+1}\sigma^{2n+1}\cos(2n+1)\omega.$$

So much, independent of the condition that u be finite. Imposing this condition, we see that*

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* Osgood, *Funktionentheorie*, vol. 1, pp. 647 and 656.

$$b_{2n+1} = 0, \qquad 2n + 1 < 0.$$

For, u can have at most a removable singularity at O', and if u be defined there by its limiting value, it will be harmonic there and so admit a unique development of the above form, no negative powers of σ entering.

Finally, returning to the w-plane, we have:

$$u = -2\sum_{n=0}^{\infty} \frac{b_{2n+1}}{2n+1}\rho^{\frac{2n+1}{2}} \cos\frac{2n+1}{2}\theta. \tag{2}$$

From this development it is seen that in the neighborhood of the point O

$$\left|\frac{\partial u}{\partial \xi}\right| < \frac{M}{\rho^{\frac{1}{2}}}, \qquad \left|\frac{\partial u}{\partial \eta}\right| < \frac{M}{\rho^{\frac{1}{2}}}, \tag{3}$$

where M denotes a positive constant. Furthermore, u is harmonic along L'.

To complete the proof we need the theorem of the next paragraph.

§ 3. *An Asymptotic Approximation for the Map near a Vertex of the Boundary.*

THEOREM. *If the function $f(z)$ of § 2 be written in the form*

$$f(z) = z^{\nu}\lambda(z), \qquad \nu = \frac{\pi}{\alpha}, \tag{4}$$

the point $z = 0$ being taken at P, then $\lambda(z)$ approaches a limit not 0 when z, remaining in S, approaches 0 as its limit.

A proof of this theorem may be given by means of the results obtained by KELLOGG.* Let the region S be transformed on a region Σ of the $Z = X + Yi$-plane by the equation

$$Z = z^{\nu}.$$

Through the above function $Z = z^{\nu}$ and the function (1): $w = f(z)$ the region Σ is transformed on the upper half of the $w = \xi + \eta i$-plane, $Z = 0$ going over into $w = 0$. Consider a finite portion T of the latter half-plane bounded by a curve having continuous curvature and including so much of the neighborhood of $w = 0$ as lies in that half-plane. Let T also be so restricted that the boundary Γ of its image Σ' in Σ will have continuous curvature except at $Z = 0$. It is readily shown that Γ satisfies Kellogg's Condition (A), p. 41, of the first paper cited.

Furthermore η, regarded as a function of X, Y, is harmonic within Σ' and is 0 in all the boundary points near $Z = 0$, while its boundary values in all points satisfy Kellogg's Condition (B), l. c., p. 42. It follows, then, from

* O. D. KELLOGG, these Transactions, vol. 9 (1908), p. 39 and p. 51; ibid., vol. 13 (1912), p. 109.

Kellogg's results that the first derivatives of η in Σ' approach continuous boundary values.

Finally, $\partial\eta / \partial X$ and $\partial\eta / \partial Y$ never vanish simultaneously on the boundary. The proof of this fact can be given by the transformation which Kellogg uses in his third paper, p. 122.

Hence it appears that $\Delta w / \Delta Z$, formed for the point $Z = 0$, approaches a limit as ΔZ approaches 0, and this limit is not 0. But

$$w = Z\lambda\,(Z^{1/\nu}), \qquad \text{and} \qquad \frac{\Delta w}{\Delta Z} = \lambda\,(\Delta Z^{1/\nu}).$$

Thus the theorem is established.

In particular, setting $w = \rho e^{\theta i}$ and $z = re^{\phi i}$, we infer that

$$\rho = r^{\nu}\,|\,\lambda\,(z)\,|, \tag{5}$$

where

$$0 < A < |\,\lambda\,(z)\,| < B,$$

A and B being constants. Furthermore, for points z within S,

$$\frac{dw}{dz} = \frac{dw}{dZ}\,\frac{dZ}{dz} = \frac{dw}{dZ}\cdot \nu z^{\nu-1}.$$

Since dw / dZ is finite near $Z = 0$ and remains in absolute value greater than a positive constant, and since the same is true of $\nu e^{(\nu-1)\phi i}$, it follows that

$$\frac{dw}{dz} = \frac{\Omega\,(z)}{r^{1-\nu}},$$

where

$$0 < A' < |\,\Omega\,(z)\,| < B'.$$

Hence

$$\left|\frac{\partial\xi}{\partial x}\right| = \left|\frac{\partial\eta}{\partial y}\right| < \frac{B'}{r^{1-\nu}}, \qquad \left|\frac{\partial\xi}{\partial y}\right| = \left|\frac{\partial\eta}{\partial x}\right| < \frac{B'}{r^{1-\nu}}, \tag{6}$$

and

$$\left|\frac{\partial\xi}{\partial x}\right| + \left|\frac{\partial\xi}{\partial y}\right| = \left|\frac{\partial\eta}{\partial x}\right| + \left|\frac{\partial\eta}{\partial y}\right| > \frac{A'}{r^{1-\nu}}. \tag{7}$$

A simpler and more direct proof of the foregoing theorem is desirable. Kellogg's solution introduces more than is needed, since his determination of the density of the double distribution contains *all* the properties of the function η, while we need only such as are sufficient for the study of the partial derivatives of the first order near the boundary.

§ 4. *Completion of the Proof of the Lemma.*

We desire an upper limit (Abschätzung) for the numerical values of $\partial u/\partial x$, $\partial u/\partial y$ near P. We have:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x},$$

$$\left|\frac{\partial u}{\partial x}\right| \leqq \left|\frac{\partial u}{\partial \xi}\right| \cdot \left|\frac{\partial \xi}{\partial x}\right| + \left|\frac{\partial u}{\partial \eta}\right| \cdot \left|\frac{\partial \eta}{\partial x}\right|,$$

with a similar relation for $\left|\frac{\partial u}{\partial y}\right|$. From (3) and (5) it follows that

$$\left|\frac{\partial u}{\partial \xi}\right|, \qquad \left|\frac{\partial u}{\partial \eta}\right| < \frac{M}{\sqrt{A}} \cdot \frac{1}{r^{\nu/2}}.$$

Furthermore, from (6):

$$\left|\frac{\partial \xi}{\partial x}\right|, \qquad \left|\frac{\partial \eta}{\partial x}\right| < \frac{B'}{r^{1-\nu}}.$$

Hence, finally:

$$(8) \qquad \left|\frac{\partial u}{\partial x}\right|, \qquad \left|\frac{\partial u}{\partial y}\right| < \frac{2MB'}{\sqrt{A}} \cdot \frac{1}{r^{1-\nu/2}}, \qquad \text{Q. E. D.}$$

Obviously Condition (2) of the lemma can be replaced by the condition: $u\,|_L = k$, where k is any constant, since the function $u - k$ would then satisfy all the conditions of the lemma.

§ 5. *Some Properties of Logarithmic Potential Functions.*

Analytic Quadrilaterals.—In the following pages we have frequently to do with simply connected finite regions bounded by four arcs, each of which is analytic inclusive of its extremities, the angles at the vertices being each greater than 0 and less than 2π. Such a region we will call an *analytic quadrilateral.*

LEMMA 1. *Let S be an analytic quadrilateral with sides C_1, C_1', C_2, C_2'. Then there exists a function u harmonic within S, equal to a_1 on C_1 and a_2 on C_2, a_1 and a_2 being constants, and having its inner normal derivative zero at all points of C_1' and C_2' except the extremities. If u is finite in S, u is unique.**

Let S be mapped conformally on the upper half-plane, and hence, by an elliptic integral of the first kind, on a rectangle R in the $w = \xi + \eta i$-plane with its vertices at $\pm K$, $\pm K + K'i$, C_2 going into a segment of the axis of reals. Then the function

$$v = \frac{a_1 - a_2}{K'}\eta + a_2$$

fulfills the conditions of the theorem in R.

Let u be the function in S into which v is carried by the inverse transformations. Then u fulfills the conditions of the theorem in S.

* The first part of this theorem is not new. A rigorous proof of the uniqueness is, however, less familiar.

To show that u is unique, let u_1 be a second such function, and form the function $w = u - u_1$. Here, w is finite and harmonic within S, and

$$w\,|_{C_1} = 0, \qquad w\,|_{C_2} = 0, \qquad \left.\frac{\partial w}{\partial n}\right|_{C_1'} = 0, \qquad \left.\frac{\partial w}{\partial n}\right|_{C_2'} = 0.$$

From the lemma of § 2 it follows that w is a function to which Formula (B) of § 1 applies:

$$\iint\limits_{S}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] dS = -\int\limits_{C} w\frac{\partial w}{\partial n}\,ds.$$

Since the line integral vanishes, the double integral must also vanish, and from this it follows that

$$w = \text{const.} = 0.$$

LEMMA 2. *Let S_1 be an analytic quadrilateral with the sides C_1, C_1', C_2, C_2', and let the function u_1 satisfy the following conditions:*

1) *u_1 is finite and harmonic within S_1;*

2) $u_1\,|_{C_1} = k_1$, $u_1\,|_{C_2} = k_2$, *k_1 and k_2 being constants, and $k_1 > k_2 \geqq 0$;*

3) $\left.\dfrac{\partial u_1}{\partial n}\right|_{C_1'} = 0, \qquad \left.\dfrac{\partial u_1}{\partial n}\right|_{C_2'} = 0.$

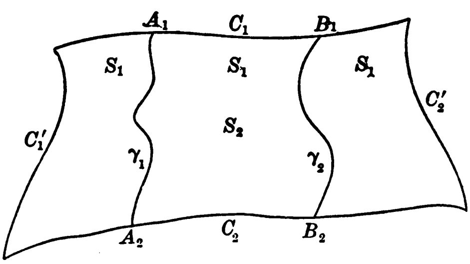

FIG. 4.

Let S_2 be a second analytic quadrilateral lying in S_1 and having two of its sides, A_1B_1 and A_2B_2, consisting of arcs of C_1 and C_2 respectively. If u_2 satisfies the same conditions in S_2 that u_1 does in S_1, then

$$-\int\limits_{A_1B_1}\frac{\partial u_2}{\partial n}\,ds < -\int\limits_{C_1}\frac{\partial u_1}{\partial n}\,ds.$$

From the lemma of § 2, and § 4, end, it follows that u_1 and u_2 are functions to which the formulas of § 1 are applicable. Hence

$$\iint\limits_{S_1}\left[\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_1}{\partial y}\right)^2\right] dS = -\int\limits_{\sigma_1} u_1\frac{\partial u_1}{\partial n}\,ds,$$

where σ_1 denotes the complete boundary of S_1. Moreover, by § 1, (D),

$$\int\limits_{\sigma_1}\frac{\partial u_1}{\partial n}\,ds = 0,$$

and since

$$\int_{\sigma_1} \frac{\partial u_1}{\partial n} ds = \int_{C_1} \frac{\partial u_1}{\partial n} ds + \int_{C_2} \frac{\partial u_1}{\partial n} ds,$$

it follows that

$$\int_{C_1} \frac{\partial u_1}{\partial n} ds = -\int_{C_2} \frac{\partial u_1}{\partial n} ds.$$

We thus obtain for the value of the double integral:

$$(9)\ \iint_{S_1} \left[\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_1}{\partial y}\right)^2\right] dS = -k_1 \int_{C_1} \frac{\partial u_1}{\partial n} ds - k_2 \int_{C_2} \frac{\partial u_1}{\partial n} ds = -(k_1 - k_2) \int_{C_1} \frac{\partial u_1}{\partial n} ds.$$

Similarly,

$$(10)\qquad \iint_{S_2} \left[\left(\frac{\partial u_2}{\partial x}\right)^2 + \left(\frac{\partial u_2}{\partial y}\right)^2\right] dS = -(k_1 - k_2) \int_{A_1B_1} \frac{\partial u_2}{\partial n} ds.$$

Let $w = u_1 - u_2$ in S_2; $u_1 = u_2 + w$. Then

$$\iint_{S_2} \left[\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_1}{\partial y}\right)^2\right] dS = \iint_{S_2} \left[\left(\frac{\partial u_2}{\partial x}\right)^2 + \left(\frac{\partial u_2}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] dS$$
$$+ 2 \iint_{S_2} \left(\frac{\partial u_2}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial u_2}{\partial y}\frac{\partial w}{\partial y}\right) dS.$$

The last integral, being equal to

$$-\int_{\sigma_2} w \frac{\partial u_2}{\partial n} ds,$$

is seen to vanish. Hence

$$\iint_{S_2} \left[\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_1}{\partial y}\right)^2\right] dS \geqq \iint_{S_2} \left[\left(\frac{\partial u_2}{\partial x}\right)^2 + \left(\frac{\partial u_2}{\partial y}\right)^2\right] dS.$$

But

$$(11)\ \iint_{S_1} \left[\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_1}{\partial y}\right)^2\right] dS > \iint_{S_2} \left[\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_1}{\partial y}\right)^2\right] dS.$$

Substituting for the integrals in (11) their values obtained from (9) and (10), we have

$$-(k_1 - k_2) \int_{C_1} \frac{\partial u_1}{\partial n} ds > -(k_1 - k_2) \int_{A_1B_1} \frac{\partial u_2}{\partial n} ds,$$

and the proposition is proved.*

* The proof here given is due to Professor B. O. PEIRCE.

LEMMA 3. *In the analytic quadrilateral S with sides C_1, C_1', C_2, C_2' the functions u_1, u_2 shall satisfy the following conditions:*

1) *u_i is finite and harmonic within S, $= 1, 2$;*

2) $$\left.\frac{\partial u_i}{\partial n}\right|_{C_1'} = 0, \qquad \left.\frac{\partial u_i}{\partial n}\right|_{C_2'} = 0, \qquad (i = 1, 2);$$

3₁) $$0 < u_1 |_{C_1} < k, \qquad 0 < u_1 |_{C_2};$$

where k is a positive constant, and u_1 is harmonic in all points of C_1 and C_2;†

3₂) $$u_2 |_{C_1} = k, \qquad u_2 |_{C_2} = 0.$$

Then
$$-\int_{C_1} \frac{\partial u_1}{\partial n} ds < -\int_{C_1} \frac{\partial u_2}{\partial n} ds.$$

Let S be mapped on a rectangle R of the (x', y')-plane, C_1 going over into a segment AB of the line $y' = 1$, and C_2 into a segment CD of the axis of x'. Let l be the length of either of these segments. Denote by u_1', u_2' the functions into which u_1, u_2 are thus transformed. Then
$$u_2' = ky'.$$
For, ky' is evidently one function having the properties of u_2', and by Lemma 1 there is only one such function. We wish, then, to prove that

$$-\int_{AB} \frac{\partial u_2'}{\partial n'} ds' > -\int_{AB} \frac{\partial u_1'}{\partial n'} ds'. \tag{12}$$

Since
$$\left.\frac{\partial u_2'}{\partial n'}\right|_{AB} = -k, \qquad \left.\frac{\partial u_2'}{\partial n'}\right|_{CD} = k,$$
the inequality (12) is equivalent to the following:

$$\int_{AB} \frac{\partial u_1'}{\partial n'} ds' > -kl. \tag{13}$$

From § 1, (C), we have, since u_1' and u_2' satisfy in R the requisite conditions:
$$\int_\sigma u_2' \frac{\partial u_1'}{\partial n'} ds' = \int_\sigma u_1' \frac{\partial u_2'}{\partial n'} ds',$$
where σ denotes the complete boundary of R. Here,
$$\int_\sigma u_2' \frac{\partial u_1'}{\partial n'} ds' = \int_{AB} u_2' \frac{\partial u_1'}{\partial n'} ds' = k \int_{AB} \frac{\partial u_1'}{\partial n'} ds';$$

† This latter condition can be replaced by a less restrictive one; for example, that the first partial derivatives of u_1 be continuous on the boundary.

and

$$\int_{\sigma} u_1' \frac{\partial u_2'}{\partial n'} ds' = \int_{AB} u_1' (-k)\, ds' + \int_{CD} u_1' k\, ds' = k\left(-\int_{AB} u_1'\, ds' + \int_{CD} u_1'\, ds'\right).$$

Hence

$$\int_{AB} \frac{\partial u_1'}{\partial n'} ds' = -\int_{AB} u_1'\, ds' + \int_{CD} u_1'\, ds' > -\int_{AB} u_1'\, ds' > -kl,$$

and this is precisely (13).

§ 6. *The Fundamental Theorem and the Method of Proof.*

Definition: A boundary point A of a region S shall be said to be *accessible* if it can be approached along a Jordan curve lying wholly within the region except for its extremity A.* For a discussion of multiply counting boundary points cf. § 7, under Theorem III.

THEOREM I. *Let S be a simply connected plane region whose boundary consists of more than one point, and let the interior of S be mapped conformally on the interior of a circle S'. Let A be an accessible boundary point of S, and let C be a curve lying within S (except for one extremity) and leading to A. Then the image of C in S' is a curve C' with a single limiting point A' on the circumference of S'; so that, if a point P approach A along C, its image P' will approach A' as its limit.*

The region S can be transformed on a finite region lying in the upper half of the $z = x + yi$-plane, the point A going into the origin.† It is this latter region which we shall henceforth consider and denote by S. Let S' be the unit circle of the $w = u + vi$-plane, the point O of S going over into $w = 0$, and let g be the Green's function of S with its pole at O; h, the conjugate function. Then the interior of S is mapped conformally on the interior of S' by the transformation

$$w = e^{-g-hi}.$$

As P approaches A along C, g approaches 0. To prove the theorem, it is necessary and sufficient to prove that, as P approaches A, a branch of h, single-valued and continuous along C, also approaches a limit.

Let ϵ be a positive quantity chosen at pleasure and then held fast. With A as center construct two circumferences

$$\sigma_1: \quad |z| = r_1; \qquad \sigma_2: \quad |z| = r_2; \qquad \frac{r_1}{r_2} = a < 1,$$

* For an example of a region having boundary points that are not accessible cf. OSGOOD, these Transactions, vol. 1 (1900), p. 311.

† OSGOOD, *Funktionentheorie*, vol. 1, p. 700.

a being a positive constant less than unity. Furthermore, r_2 shall be so chosen that O shall lie outside the circle σ_2. Let P_1 be that point of intersection of σ_1 and C such that σ_1 and the arc P_1A of C have no other point in common. It will now be shown that r_2 can be so chosen that the variation of h along C from P_1 to A is less than ϵ.

Let η be a positive quantity chosen arbitrarily small. Then it is clear that r_2 can be so restricted that the upper limit of g on the arc or arcs of σ_2 that lie in S will be less than η. Let this be done. r_2, and with it r_1, are now fixed.

Let P' be a point on the arc P_1A arbitrarily near A, and let $\mu > 0$ be so chosen that every point of the arc P_1P' lies within the curve $g = \mu$. Denote by $\bar{S}$ the portion of S composed of the curve $g = \mu$ and the points interior to

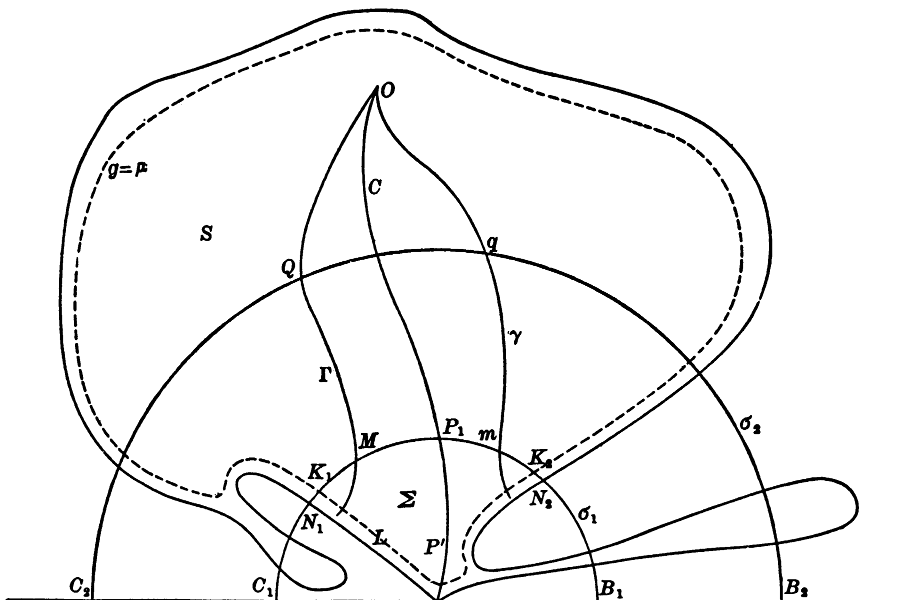

Fig. 5.

it. Since $g = \mu$ is a closed analytic curve distinct from σ_1, it is cut by σ_1 in a finite number of points. Thus σ_1 yields a finite number of cross-cuts of $\bar{S}$. One of these, K_1K_2, divides $\bar{S}$ into two simply connected regions, one of which contains the point O, and the other, —Σ, let us call it,—the point P' (and hence the arc P_1P' of C).

Since a branch of h is single-valued and harmonic in Σ and continuous on the boundary, it has an upper and a lower limit in Σ, which are reached on the boundary of Σ. Now $\partial g / \partial n > 0$ at every point of the curve $g = \mu$, and so h varies monotonically along the arc K_1LK_2. Therefore the upper limit H and the lower limit h_0 of h in Σ are reached on the arc K_1K_2 of σ_1.

Construct the curves $h = H$ and $h = h_0$, and denote them respectively by Γ and γ. Let be M and Q points in which Γ meets σ_1 and σ_2 respectively, such that the arc MQ of Γ has no point except its extremities in common with σ_1 or σ_2. Let m and q have a corresponding meaning for γ. The points m, M may lie on the arc K_1K_2 produced.

The proof turns on applying Lemmas 2 and 3 of § 5 to the analytic quadrilaterals $B_1P_1C_1C_2QB_2$ and $mMQq$ (or a corresponding one, to be determined later), and noting at the outset the following two facts:

a) The value of g on the arc qQ is less than η;

b)
$$-\int_{mM}\frac{\partial g}{\partial n}\,ds = H - h_0.$$

The truth of (*b*) depends essentially on the fact that when a point P, starting at K_1, describes the boundary of Σ in the positive sense (K_1LK_2), h attains its minimum value h_0 along the arc $K_2P_1K_1$ in a point or in several points, all of which are reached *before* h attains its maximum value H on that arc. For a proof, consider the map of $\bar{S}$ on the circle $\rho \leq e^{-\mu}$, where $w = \rho e^{\theta i}$, and the image of the arc $K_2P_1K_1$ in this circle. Since $h = -\theta + \text{const.}$, it is sufficient to prove the corresponding statement for θ, and this is readily done by mapping the above circle on the W-plane by the function $W = \log w$.

Lemma 2 is applied as follows: Form a function u_1 harmonic in the ring $r_1 < |z| < r_2$ and such that

$$u_1|_{\sigma_1} = 0, \qquad u_1|_{\sigma_2} = \eta.$$

It is

$$u_1 = \frac{\eta\,(\log r - \log r_1)}{\log r_2 - \log r_1} = c\eta\,(\log r - \log r_1),$$

where $c = -1/\log a$. The conjugate of u_1 is

$$v_1 = c\eta\varphi + \text{const.},$$

and

(14)
$$-\int_{B_1C_1}\frac{\partial u_1}{\partial n}\,ds = c\pi\eta.$$

Let u_2 be determined by the following conditions:

1) u_2 is finite and harmonic within the analytic quadrilateral $mMQq$,—we assume for the moment that γ and Γ are not tangent to σ_1 or σ_2;

2)
$$u_2|_{qQ} = \eta, \qquad u_2|_{mM} = 0;$$

3)
$$\left.\frac{\partial u_2}{\partial n}\right|_{\Gamma} = 0, \qquad \left.\frac{\partial u_2}{\partial n}\right|_{\gamma} = 0.$$

The existence and uniqueness of such a function follow from Lemma 1, § 5. We now apply Lemma 2 of § 5, taking for the region S_1 so much of the ring

$r_1 \leqq |z| \leqq r_2$ as lies above the lower half-plane, C_1 lying along σ_2; and as S_2, the analytic quadrilateral $mMQq$. Hence, with the aid of (14),

$$(15) \qquad -\int_{qQ} \frac{\partial u_2}{\partial n} ds < -\int_{B_2C_2} \frac{\partial u_1}{\partial n} ds = c\pi\eta.$$

Furthermore, from § 1, (D),

$$\int_{mMQq} \frac{\partial g}{\partial n} ds = 0,$$

and hence, by using (b), we have

$$(16) \qquad -\int_{qQ} \frac{\partial g}{\partial n} ds = \int_{mM} \frac{\partial g}{\partial n} ds = H - h_0,$$

the normal in the last integral being opposite in sense to that of the integral in (b).

Finally, we apply Lemma 3 to the functions g and u_2 considered in $mMQq$, g corresponding to u_1. Hence

$$-\int_{qQ} \frac{\partial g}{\partial n} ds < -\int_{qQ} \frac{\partial u_2}{\partial n} ds,$$

or, from (15) and (16),

$$H - h_0 < c\pi\eta.$$

If, then, we choose η so that $2c\pi\eta < \epsilon$,—the reason for the factor 2 will appear presently,—our proof is complete, except for the assumption that Γ and γ are not tangent to σ_1 or σ_2.

In case of tangency, Γ and γ can be replaced by curves near by, cutting σ_1 and σ_2, but not tangent: $h = H'$ and $h = h_0'$, H' and h_0' differing but slightly from H and h_0 respectively; so that, in particular, $H' - h_0' > \frac{1}{2}(H - h_0)$. The point m', as above in the case of m and M, will thus be reached before M', so that (b) will hold. Using the analytic quadrilateral bounded in part by these curves instead of Γ and γ, we infer that

$$H' - h_0' < c\pi\eta, \qquad \text{or} \qquad H - h_0 < 2c\pi\eta < \epsilon.$$

This completes the proof.

Let the limit approached by h as P approaches A along C be denoted by $\bar{h}$.

Extension of Theorem I.

The analysis by which the foregoing theorem has been established yields more than the result stated in that theorem. Extend the arc mM of σ_1 in both directions until it meets the boundary of S in the points N_1 and N_2. The cross-cut $N_1P_1N_2$ divides S into two simply connected regions. Let Σ^0 be the one of these regions that contains the arc P_1A of C.

From the theorem just proved it follows that the function h approaches a limit when P approaches N_1, and also when P approaches N_2, along the arc mM produced. h has, therefore, a finite upper limit $\bar{H}$ and a finite lower limit $\bar{h}_0$ along the cross-cut. Moreover, it is readily seen from the foregoing analysis that

$$\bar{H} - \bar{h}_0 \leqq \epsilon;$$

and since, at any interior point of Σ^0, h lies between $\bar{H}$ and $\bar{h}_0$, we have, for any such point

$$(17) \qquad | h - \bar{h} | \leqq \epsilon.$$

Consider, now, a succession of circles with A as center and with radii

$$r^0 = r_1 > r' > r'' > \cdots, \qquad \lim_{n=\infty} r^{(n)} = 0,$$

and the corresponding regions cut from S:

$$\Sigma^0, \ \Sigma', \ \Sigma'', \ \cdots,$$

the general region being so taken as to include the whole of a certain arc of C abutting on A. Let $z_1, z_2, \cdots$ be any set of points interior to S with A as a cluster-point and such that, for a given m arbitrarily large, z_n lies in $\Sigma^{(m)}$ provided $n > n_m$. If h_n denotes the value of h in z_n, then h_n will approach a limit, and

$$(18) \qquad \lim_{n=\infty} h_n = \bar{h}.$$

We have, therefore, obtained the following

Extension of Theorem I. *If $w_1, w_2, \cdots$ be the images in S' of z_1, $z_2, \cdots$, and if $\bar{w}$ correspond to A', then w_n will approach a limit and*

$$(19) \qquad \lim_{n=\infty} w_n = \bar{w}.$$

It is to be noted that the maximum diameter of $\Sigma^{(n)}$ does not necessarily converge toward zero when $n = \infty$, and also that the points z_n may have other cluster-points than A,—which points will, of course, lie on the boundary of S.

Corollary. *A point A' of the circumference of S' which is the image of an accessible boundary point A of S has other such points in every neighborhood of itself and on either side; i. e., the points A' form a set that is "in sich dicht".*

In fact, the extremities of the cross-cuts $|z| = r^{(n)}$ just described are accessible points of the boundary of S, which go over into points of the circumference of S' approaching A' from both sides.

The set of points A' is not, however, in general perfect, as is shown by the following example. Consider the region S_1 of the $z_1 = x_1 + iy_1$-plane:

$$S_1: \qquad 1 < x_1 \leqq 2, \qquad \frac{1}{x_1 - 1} \leqq y_1 \leqq \frac{1}{x_1 - 1} + 1,$$

and let a definite interior point O_1 be chosen as the pole of the Green's function for S_1. If S_1 be mapped on S_2 by the transformation

$$z_2 = \frac{1}{z_1},$$

then all the boundary points of S_2, inclusive of the cusp at the origin, are accessible. Let $h = h'$ be the line of flow to this point.

On the other hand, let S_1 be wrapped round a circle by the transformation

$$z_3 = e^{z_1}.$$

Then the curve $h = h'$ winds round this circle indefinitely, and its only points of condensation not pertaining to itself are the points of the circle. Let A be an arbitrary point of this circle, and let $z_1, z_2, \cdots$ be a set of points in S with A as their limiting point. Then their images w_n will converge toward a point A' on the boundary of S', and A' will be the same point for all of the above choices of A.

§ 7. *Further Theorems.*

THEOREM II. *Let C_1 be a second curve of S also leading from O to A and meeting C only in those points. Let C_1' be the image of C_1 in S', A_1' its limiting point on the boundary of S'. The necessary and sufficient condition that A_1' coincide with A' is that the simple closed curve $\bar{C}$ consisting of C and C_1 may be drawn together continuously to the point A without passing out of S; or, in other words, that $\bar{C}$ shall contain in its interior only interior points of S.*

That the condition is sufficient follows at once from the extension of Theorem I.

To prove the condition necessary, consider the simple closed curves $\bar{C}$ and $\bar{C}'$. Suppose $\bar{C}$ contains boundary points of S in its interior. Denote the part of the latter region lying within S by R, and consider the function w of z, defined by the map, in R. When z, remaining in R, approaches a boundary point B of S, passing through any set of points $z_1, z_2, \cdots$ which have B as their sole limiting point, w approaches a limit, namely, the value $\bar{w}$ of w in A'. By an extension of a theorem due to Painlevé* w must, therefore, reduce to a constant, $\bar{w}$, and this is not true.

Closely related to the foregoing theorem is the following.

* PAINLEVÉ, Toulouse Annales, vol. 2 (1888), p. B. 29, stated the following theorem: *If $f(z)$ is analytic in a region T and approaches one and the same value along an arc of the boundary, then $f(z)$ is a constant.* He proved the theorem for ordinary arcs. But it is true generally, when the words: "an arc of the boundary" are replaced by the words: "a connected piece of the boundary containing two distinct points, each of which can be approached along a curve lying within the region." Cf. TAYLOR, Bulletin of the American Mathematical Society, vol. 19, June, 1913.

THEOREM III. *If A and A_1 are two distinct accessible points of the boundary of S, then the corresponding points A' and A_1' on the circumference of S' will also be distinct.*

Conversely, let A' and A_1' be two distinct points of the circumference of S', corresponding to points A and A_1 (distinct or coincident) which can be reached along curves C and C_1; the latter moreover shall not intersect. Then each of the regions into which S is divided by the cross-cut (C, C_1) will contain among its boundary points other boundary points of S than A and A_1.

For, suppose A' and A_1' coincide. Then the reasoning in the proof of the last theorem is applicable, and it follows that A and A_1 coincide. Thus the first part of the proposition is established.

The second part of the theorem follows at once from the extension of Theorem I.

Theorem II may be regarded as coming directly under Theorem III if we agree to consider as distinct two coincident points of the boundary of S which can be approached along curves C and C_1 emanating from O and lying in S, when each of the regions into which S is divided by the cross-cut (C, C_1) has among its boundary points other boundary points of S than the given points. Such points of the boundary, if they exist, shall be called *multiple points.*

Again, a boundary point may be accessible for a certain avenue of approach, and still be inaccessible for a set of points lying within S and condensing on it as their sole limiting point. As an example of such a region, take the interior points of the circle $|z| < 5$ with the exception of those points (a) which lie on the line $x = 0, -1 \leqq y \leqq 1$, and ($b$) which lie on the curve

$$y = \sin\frac{1}{x}, \qquad 0 < x < x_0,$$

where x_0 is the smallest positive x for which this curve cuts the circumference of the circle.

It is, then, in general impossible to divide the boundary points into accessible and inaccessible points except as they are multiply counted. We will not attempt such a classification, but refer to Condition A below as giving a working definition for Theorem V.

COROLLARY 1. *If the boundary of S consist of a simple closed Jordan curve, then the conformal map of the interior of S on the interior of the circle S' will be one-to-one and continuous on the boundary.*

From Theorems II and III it follows that a one-to-one relation exists between the totality of points of the boundary of S and a set of points (not as yet known to be everywhere dense) on the circumference of S'. It follows further, however, from the extension of Theorem I that the points of the latter set depend continuously on the points of the former set. And so Hurwitz's condition that the points of the latter set form a Jordan curve is seen to be fulfilled.*

COROLLARY 2: *If the boundary has no inaccessible and no multiple points, it is a simple closed Jordan curve.*

First, there will be a one-to-one relation between the points of the boundary of S and those of a set situated on the circumference of S'.

Secondly, the latter points must comprise all the points of that circumference. For otherwise consider a radius leading to an omitted point. The image of this radius in S would be a curve having more than one point of condensation on the boundary, and hence entering every neighborhood of two accessible points. But this is absurd.

Thirdly, if A' is an arbitrary point of the circumference of S' and A its image on the boundary of S; and if a circle of arbitrarily small radius be described about A, then there exists an arc extending equal distances to either side of A', all of whose points have their images in the circle in question. For otherwise there would exist a set of boundary, and therefore also a set of interior points of S, each of whose distances from A is greater than a certain positive constant, and such that their images in S' cluster about A'. But these points would have a cluster point A_1 on the boundary of S distinct from A, and hence go over into points of S' condensing on A_1'.

Thus the boundary points of S fulfil Hurwitz's definition of a Jordan curve, and the theorem is proved.

Remark: It may however happen, even when the boundary has inaccessible points, that the accessible points are related in a one-to-one manner to the points of the circumference of S'. An example of such a region is the domain considered by Osgood, *Funktionentheorie*, vol. 1, p. 154.

This example shows, moreover, that Painlevé's theorem cannot be extended as follows: Let $f(z)$ be analytic in a region T and vanish along a portion of the boundary consisting of an arc of an analytic curve; then $f(z) \equiv 0$. For the foregoing function w approaches one and the same value, w_0, along the right line segment $x = 0$, $0 \leqq y \leqq 1$. But $w - w_0 \not\equiv 0$.

THEOREM IV. *The points A' are everywhere dense on the circumference of S'.*

Suppose the proposition were not true. Let $h = -\theta$, and let the arc $-L \leqq \theta \leqq -l$ lie within an arc free from points A'. The region S will be divided by the curves $h = l$ and $h = L$ into two regions, one of which, S_1, —namely, that in which $l < h < L$,—has no accessible boundary points except those of the curves $h = l$ and $h = L$.

* A. HURWITZ, *Verhandlungen des ersten Internationalen Mathematikerkongresses*, 1898, p. 102; OSGOOD, *Funktionentheorie*, vol. 1, p. 147.

Consider the curve $h = l_1 = \frac{1}{2}(l + L)$, and let A be a point of condensation of this curve not pertaining to the curve. Then A will be a boundary point of S. Let the plane be transformed as in the proof of Theorem I so that A will come to the origin and S lie in the upper half-plane; S and A now referring to the transformed region and point.

Choose the positive quantity η arbitrarily small and draw the circles σ_1 and σ_2 as before, the Green's function g of S being less than η wherever it is defined on and inside of σ_2, and $r_1 / r_2 = a < 1$.

The curve $h = l_1$ must enter the inner circle. Let P_1 be a point in which it meets that circle. Then an arc of the circle, emanating from P_1 and remaining in S_1, will approach the boundary of S_1 in a point P_2 lying on one of the curves $h = l$, $h = L$; and the arc P_1P_2 will lie within S. The variation in h along this arc is $\frac{1}{2}(L - l)$.

Choose $\mu > 0$ so that the curve $g = \mu$ includes the arc P_1P_2 in its interior, and denote the points of intersection of this arc produced with $g = \mu$ by K_1 and K_2. We have now a region Σ cut off from $\bar{S}$ (the interior and boundary of $g = \mu$) by the arc K_1K_2 precisely like the region Σ of the proof of Theorem I. Let H and h_0 be the maximum and minimum values of h in Σ. They are attained on the arc K_1K_2 of the boundary. And now the earlier reasoning leads here, as in that case, to the conclusion that

$$H - h_0 < 2\pi c\eta .$$

But $H - h_0$ is at least as great as $\frac{1}{2}(L - l)$, and η may be so chosen that $2\pi c\eta < \frac{1}{2}(L - l)$. In this contradiction lies the proof of the theorem.

We are now in a position to state a necessary and sufficient condition that a curve C of S, any complete arc of which is a Jordan curve and which proceeds out toward the boundary of S, have as its image in S' a curve C' abutting on a single boundary point of S'.

Condition A. It may happen that there are two distinct accessible boundary points A and A_1 of S and neighborhoods

$$\Sigma^0,\ \Sigma',\ \Sigma'',\ \cdots,\qquad \Sigma_1^0,\ \Sigma_1',\ \Sigma_1'',\ \cdots$$

as described under the extension of Theorem I such that each $\Sigma^{(n)}$ and each $\Sigma_1^{(n)}$ is traversed by C. If this is not the case, we say that C fulfills Condition A.

THEOREM V. *A necessary and sufficient condition that the image C' of the above curve C abut on a single boundary point of S' is that C fulfil Condition A.*

That the condition is necessary, is obvious. To show that it is sufficient, suppose B and B_1 were two distinct boundary points of S' which are points of condensation of C'. Since C', as it advances, ultimately has its later points all outside of any circle concentric with S' and lying within it, at least one of the arcs BB_1 must have all of its points points of condensation of C'. But the points A' which are the images of accessible boundary points A are by Theorem IV everywhere dense along the circumference of S'. Hence there must be two distinct accessible points of the boundary of S precluded by the hypothesis, and the proposition is established.

COROLLARY. *The image of a curve C' of S' having but a single point of condensation on the circumference of S' is a curve C satisfying Condition A.*

§ 8. *Multiply Connected and Multiple-Leaved Regions.*

The foregoing results apply directly to the case of an arbitrary region S, single or multiple-leaved, simply or multiply connected, and bounded in part by a set of points $\mathfrak{Q}$ more than one in number and such that two of these points, A and A_1, can be connected by a cross-cut C cutting off from the region a simply connected piece S_1 bounded completely by C and $\mathfrak{Q}$. If S be so mapped that S_1 goes over into a region S_1' bounded in part by an analytic curve $\mathfrak{Q}'$ corresponding to $\mathfrak{Q}$, then the same conditions for continuity on the boundary of the map exist as in the case hitherto discussed.

For, the interior of S_1 can be mapped conformally on the interior of a circle $\bar{S}_1'$, the part of S_1 abutting on $\mathfrak{Q}$ going over into a part of $\bar{S}_1'$ abutting on an arc $\bar{\mathfrak{Q}}_1'$ of the circumference. To the map in the neighborhood of $\bar{\mathfrak{Q}}_1'$ the earlier results apply. On the other hand, the complete neighborhood of any point of $\bar{\mathfrak{Q}}_1'$ (exclusive of the extremities of the arc) is mapped conformally on the complete neighborhood of the corresponding point of $\mathfrak{Q}'$.

As an example, consider a plane region S with a boundary of n pieces, each piece consisting of more than a single point. This region can first be transformed into a finite region by considering the part of the whole plane bounded by a single piece and having an interior point lying within S. This latter region can then be transformed on the interior of a circle, and thus the given region goes over into a region lying within a circle and having the circle as a piece of the boundary.

Proceed now in a similar manner with a piece of the boundary of the new region lying inside the circle. Thus again a region is obtained lying within a circle and having the circle as one piece of its boundary, a second piece of its boundary being now a simple closed analytic curve.

By repeating the process we finally obtain a region $\mathfrak{S}$ whose boundary consists of n simple closed analytic curves not cutting one another. The map on the boundary obeys the same laws as in the case of the simply connected plane region discussed in the foregoing paragraphs. Thus a curve of S abutting on an accessible point of the boundary goes over into a curve of $\mathfrak{S}$ having a single point of condensation on the boundary.*

Or, again, since $\mathfrak{S}$ can be mapped on an n-sheeted Riemann's surface spread out over the upper half-plane, its leaves being connected by $2n - 2$ branch points, it follows in particular that S can be mapped on such a surface, and the map on the boundary (the axis of reals) will obey the laws obtained in the preceding paragraphs.

* It is only the part of this theorem that relates to the map on the boundary which is new. The part relating to the map in the interior was obtained by the authors prior to June, 1906, but was not published by them. It has since appeared in the literature.

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CAMBRIDGE, MASSACHUSETTS,
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