SCROLL DIRECTRIX CURVES*

BY

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In his theory of surfaces based on the discussion of a non-involutory completely integrable system of two partial differential equations of the second order, Professor Wilczynski has shown that the organic curves which he calls the *directrix curves* play an important rôle.† They are defined in terms of the osculating linear complexes of the asymptotic curves. When the surface is a *scroll*, the straight line generators constitute one of the families of its asymptotic lines and any one of these asymptotic lines is contained in a quadruply infinite family of linear complexes. Thus the osculating linear complex becomes indeterminate and, therefore, the *directrix curves* cease to be defined. The purpose of the present paper is to define and discuss a set of curves on a ruled surface which shall correspond as closely as possible to the *directrix curves* of a non-ruled surface. We shall call them *scroll directrix curves*, and we shall study certain configurations involving these curves and their associated *line congruences*. The results obtained in this paper will constitute the counterpart for *scrolls* of certain theorems previously established for non-ruled surfaces.

Definition of the linear complex $C''(u)$

Since the integrating surface S is to be a *scroll*, we may assume that the system of partial differential equations has been reduced to the following canonical form:‡

$$y_{uu} + 2by_v + fy = 0, \qquad y_{vv} + gy = 0, \tag{1}$$

where

$$y_u = \frac{\partial y}{\partial u}, \qquad y_{uu} = \frac{\partial^2 y}{\partial u^2}, \qquad \text{etc.}$$

The coefficients of these equations are the fundamental seminvariants of the system. The curves $u =$ const. will be the straight line generators, and the

* Presented to the Society, Chicago, December, 1913.

† Wilczynski, *Differential Geometry of Curved Surfaces*, these Transactions, vols. 8, 9, and 10. These memoirs will be cited as M_1, M_2, M_3.

‡ M_3, pp. 293, 294.

curves $v=$ const. will be the twisted asymptotic curves on the integrating surface S.*

The integrability conditions of (1) are

$$(2) \qquad g_u = 0, \qquad b_{vv} + f_v = 0, \qquad -f_{vv} + 4gb_v + 2bg_v = 0.$$

The fundamental semi-covariants

$$y, \qquad z = y_u, \qquad \rho = y_v, \qquad \sigma = y_{uv}$$

determine the tetrahedron $P_y P_z P_\rho P_\sigma$ and a corresponding system of coördinates, in which the coördinates of the point $\alpha y + \beta z + \gamma\rho + \delta\sigma$ are defined to be $(\alpha, \beta, \gamma, \delta)$.

The linear complex $C'(v)$ osculating the asymptotic curve Γ' at the point P_y, when referred to the tetrahedron $P_y P_z P_\rho P_\sigma$, has the equation†

$$(3) \qquad -b_v\,\omega_{34} - b\omega_{14} + b\omega_{23} = 0.$$

We shall now define a linear complex $C''(u)$ to take the place of the osculating linear complex of the asymptotic curve Γ'', which is indeterminate since Γ'' is a straight line. The linear complex C'' shall contain the asymptotic line Γ''. It shall be in involution with the complex C' which osculates the asymptotic curve Γ' at the point P_y. The null-plane of the point P_y in the complex C'' shall be the tangent plane of the surface S at this point. The *unode* and *singular tangent plane* of the *Cayley cubic scroll* osculating the integrating surface S at the point P_y shall be incident elements in the null-system of C''.

Let $\sum a_{ij}\,\omega_{ij} = 0$ be the equation of a linear complex referred to $P_y P_z P_\rho P_\sigma$. The first two of the above conditions will be satisfied at the point P_y if

$$a_{12} = a_{13} = 0 \quad \text{and} \quad a_{14} = a_{23}.$$

Hence the complex C'' is contained in the family of complexes

$$\omega_{14} + \omega_{23} + \alpha_{34}\,\omega_{34} + \alpha_{42}\,\omega_{42} = 0.$$

Now consider the point P_y displaced to the point $P_{y+y_v\delta v}$ along Γ''. The semi-covariants y, z, ρ, σ will become $\bar{y}, \bar{z}, \bar{\rho}, \bar{\sigma}$, respectively, where

$$\begin{aligned} \bar{y} &= y + \rho dv + \cdots \\ \bar{z} &= z + \sigma dv + \cdots \\ \bar{\rho} &= \rho - gydv + \cdots \\ \bar{\sigma} &= \sigma - gzdv + \cdots \end{aligned}$$

* We shall hereafter refer to the curves $v=$ const. as Γ', and to the curves $u=$ const. as Γ''.

† M_2, p. 92.

Let the coördinates of a point Q be $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$ when referred to $P_{\bar{y}}\, P_{\bar{z}}\, P_{\bar{\rho}}\, P_{\bar{\sigma}}$ and (x_1, x_2, x_3, x_4) when referred to $P_y\, P_z\, P_\rho\, P_\sigma$. Then

$$(\bar{x}_1 \bar{y} + \bar{x}_2 \bar{z} + \bar{x}_3 \bar{\rho} + \bar{x}_4 \bar{\sigma})$$

must be identically equal to $(x_1 y + x_2 z + x_3 \rho + x_4 \sigma)$, apart from a factor of proportionality. Hence, if we neglect infinitesimals of the second and higher orders, we find the following equations of transformation between the coördinate systems $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$ and (x_1, x_2, x_3, x_4):

$$\omega x_1 = \bar{x}_1 - g\bar{x}_3\, dv, \qquad \omega x_2 = \bar{x}_2 - g\bar{x}_4\, dv,$$
$$\omega x_3 = \bar{x}_3 + \bar{x}_1\, dv, \qquad \omega x_4 = \bar{x}_4 + \bar{x}_2\, dv,$$

and

$$\omega' \bar{x}_1 = x_1 + g x_3\, dv, \qquad \omega' \bar{x}_2 = x_2 + g x_4\, dv,$$
$$\omega' \bar{x}_3 = x_3 - x_1\, dv, \qquad \omega' \bar{x}_4 = x_4 - x_2\, dv,$$

where ω and ω' are factors of proportionality. From these we find

$$
\begin{aligned}
\bar{\omega}_{14} &= \omega_{14} - (\omega_{12} - g\omega_{34})\, dv, \\
\bar{\omega}_{23} &= \omega_{23} + (\omega_{12} - g\omega_{34})\, dv, \\
\bar{\omega}_{34} &= \omega_{34} + (\omega_{23} - \omega_{14})\, dv, \\
\bar{\omega}_{42} &= \omega_{42}.
\end{aligned} \tag{4}
$$

The canonical coördinates of the *unode* of the Cayley cubic scroll osculating the integrating surface S at the point P_y are $(0, 0, 1, 0)$.* The canonical coördinates of a point are connected with (x_1, x_2, x_3, x_4) by the equations

$$
\begin{gathered}
X = \lambda(x_2 + \beta x_4), \qquad Y = \mu(x_3 + \alpha x_4), \\
Z = \lambda\mu x_4, \\
\omega = x_1 + \alpha x_2 + \beta x_3 + \alpha\beta x_4,
\end{gathered} \tag{5}
$$

where

$$\alpha = -\frac{b_u}{4b}, \qquad \beta = \frac{b_v}{2b}, \qquad \lambda = \frac{\sqrt{-\tfrac{1}{5}h}}{b}, \qquad \mu = -\frac{3h}{10b^3},$$

h being the fundamental invariant $b^2(f + b_v) - \frac{1}{4}bb_{uu} + \frac{5}{16}b_u^2$ of the surface S.†

Consequently the coördinates of the *unode* when referred to the tetrahedron $P_y\, P_z\, P_\rho\, P_\sigma$ are

$$x_1 : x_2 : x_3 : x_4 = \alpha\beta : -\beta : -\alpha : 1 = -b_u b_v : -4bb_v : 2bb_u : 8b^2.$$

* M_3, pp. 297, 303.

† M_3, p. 299.

The equation of the *singular tangent plane**

$$\omega = 0$$

becomes

$$8b^2 x_1 - 2bb_u x_2 + 4bb_v x_3 - b_u b_v x_4 = 0,$$

when referred to the tetrahedron $P_y P_z P_\rho P_\sigma$. The polar plane of the *unode* in the complex

$$(\omega_{14} + \omega_{23}) + \alpha_{34}\omega_{34} + \alpha_{42}\omega_{42} = 0$$

is

$$8b^2 x_1 + b(2b_u - 8b\alpha_{42})x_2 + 4b(b_v + 2b\alpha_{34})x_3 \\ + (b_u b_v - 2bb_u\alpha_{34} - 4bb_v\alpha_{34})x_4 = 0.$$

If we identify these equations, we find

$$\alpha_{34} = 0, \qquad \alpha_{42} = \frac{b_u}{2b}.$$

Hence the equation of C'' is

$$2b(\omega_{14} + \omega_{23}) + b_u\omega_{42} = 0. \tag{6}$$

The invariant of C' is $I' = -b^2$, and that of C'' is $I'' = 4b^2$. From equations (4), we find the equation of the complex $C''(u, v + dv)$ to be

$$(\omega_{14} + \omega_{23}) + \left(\alpha_{42} + \frac{\partial\alpha_{42}}{\partial v}\right)\omega_{42} = 0,$$

when referred to the tetrahedron $P_y P_z P_\rho P_\sigma$. This will be identical with $C''(u, v)$ if, and only if, the function b_u/b is independent of v, i. e.,

$$\Omega' \equiv \frac{\partial^2 \log b}{\partial u \partial v} = 0.$$

Now this is precisely the condition that the asymptotic curves Γ' belong to linear complexes.† Hence we have the theorem:

THEOREM. *The complexes C'' which belong to the different points of the same generator of S will all coincide if, and only if, the curved asymptotic lines of the scroll belong to linear complexes.*

THE SCROLL DIRECTRIX CURVES OF THE FIRST AND SECOND KINDS

Let $\sum a'_{ij}\omega_{ij} = 0$ and $\sum a''_{ij}\omega_{ij} = 0$ be the equations of two linear complexes. Let A' and A'' be their respective invariants, and let (A', A'') be their mutual invariant. Then the equations of the directrices of the congruence common to these two complexes may be obtained by writing down the

* M_3, p. 303.

† See a paper by the author, these Transactions, vol. 15 (1914), p. 190.

conditions that must be satisfied by the coördinates of a point in order that the same plane may correspond to it in both complexes. We thus find

$$
\begin{aligned}
* + (a'_{12} - \omega_k a''_{12})x_2 + (a'_{13} - \omega_k a''_{13})x_3 + (a'_{14} - \omega_k a''_{14})x_4 &= 0, \\
(-a'_{12} + \omega_k a''_{12})x_1 + * + (a'_{23} - \omega_k a''_{23})x_3 + (a'_{24} - \omega_k a''_{24})x_4 &= 0, \\
(-a'_{13} + \omega_k a''_{13})x_1 + (-a'_{23} + \omega_k a''_{23})x_2 + * + (a'_{34} - \omega_k a''_{34})x_4 &= 0, \\
(-a'_{14} + \omega_k a''_{14})x_1 + (-a'_{24} + \omega_k a''_{24})x_4 + (-a'_{34} + \omega_k a''_{34})x_3 + * &= 0,
\end{aligned}
$$

where $a'_{24} = -a'_{42}$, $a''_{42} = -a''_{42}$, and where ω_1, ω_2 are the two roots of the quadratic equation

$$A'\omega^2 - (A', A'')\omega + A'' = 0.$$

For the complexes C' and C'' we find

$$\omega_1 = 2, \qquad \omega_2 = -2;$$

so that the equations of the directrices are

$$
\begin{aligned}
d &: x_4 = 0, \qquad 4bx_1 - b_u x_2 + 2b_v x_3 = 0, \\
d' &: 2bx_2 + b_v x_4 = 0, \qquad 4bx_3 - b_u x_4 = 0.
\end{aligned} \tag{7}
$$

The directrix of the first kind d lies in the tangent plane to the surface S at the point P_y; while the directrix of the second kind d' passes through the point P_y. The directrix of the second kind is the line joining the point P_y to the point P_τ, where

$$\tau = -2b_v z + b_u \rho + 4b\sigma.$$

As the point P_y moves over the surface S, the directrix d' generates the *directrix congruence of the second kind G_2*, one line of which passes through each point of S. The two one-parameter families of curves on S which correspond to the two ways of assembling the lines of G_2 into developables, we shall call *the directrix curves of the second kind (C_1, C_2)*.

We shall now determine the directrix curves (C_1, C_2) on S. Let P_y and P_τ be displaced to $P_{y+zdu+\rho dv}$ and $P_{\tau+\tau_u du+\tau_v dv}$ respectively; the directrix d' will be displaced from the position determined by the former pair of points to that determined by the latter pair. By direct calculation, we find

$$
\begin{aligned}
\tau_u &= P\, y + Q\, z + R\, \rho + S\, \sigma, \\
\tau_v &= P' y + Q' z + R' \rho + S' \sigma,
\end{aligned}
$$

where

$$P = 8b^2 g + 2b_v f - 4bf_v, \quad Q = -2b_{uv}, \quad R = b_{uu} - 4b(f + b_v), \quad S = 5b_v,$$

$$P' = -b_u g, \quad Q' = -2(2bg + b_{vv}), \quad R' = b_{uv}, \quad S' = 2b_v.$$

An arbitrary point on the displaced directrix will be given by

$$\lambda(y + z\delta u + \rho\delta v) + \mu(\tau + \tau_u \delta u + \tau_v \delta v).$$

The coördinates of such a point are therefore

$$\begin{aligned} x_1 &= \lambda + \mu(P\delta u + P'\delta v), \\ x_2 &= \lambda\delta u + \mu(-2b_v + Q\delta u + Q'\delta v), \\ x_3 &= \lambda\delta v + \mu(b_u + R\delta u + R'\delta v), \\ x_4 &= \mu(4b + S\delta u + S'\delta v). \end{aligned}$$

In order that this point may be on the directrix d' determined by the point P_y, its coördinates must satisfy equations (7); i. e.,

$$\begin{aligned} 2b\delta u\lambda + [2b(Q\delta u + Q'\delta v) + b_v(S\delta u + S'\delta v)]\mu &= 0, \\ 4b\delta v\lambda + [4b(R\delta u + R'\delta v) - b_u(S\delta u + S'\delta v)]\mu &= 0. \end{aligned}$$

Whence

$$\begin{vmatrix} \delta u & 2b(Q\delta u + Q'\delta v) + b_v(S\delta u + S'\delta v) \\ 2\delta v & 4b(R\delta u + R'\delta v) - b_u(S\delta u + S'\delta v) \end{vmatrix} = 0. \tag{8}$$

By means of the integrability conditions, we find

$$\begin{aligned} 4bR - b_uS &= -16h, \\ (4bR' - buS') - (4bQ + 2b_vS) &= 3/2\, bC', \\ 4bQ' + 2b_vS' &= \frac{\theta'}{16}. \end{aligned}$$

Equation (8) therefore reduces to

$$2^8 h\delta u^2 - 3 \cdot 2^7 bC' \delta u\delta v + \theta' \delta v^2 = 0. \tag{8'}$$

This is the differential equation of the directrix curves (C_1, C_2) on S. Its coefficients are fundamental invariants of the surface.*

Let us consider *the congruence of the first kind* (G_1) generated by the directrix d of the first kind as the point P_y moves over S. The directrix d joins the points

$$r = b_u y + 4bz, \qquad s = -b_v y + 2b\rho.$$

Let P_y be displaced to $P_{y+z\delta u+\rho\delta v}$. The directrix d will be displaced to the position determined by the points

$$r + r_u \delta u + r_v \delta v, \qquad s + s_u \delta u + s_v \delta v.$$

By direct calculation, we find

* M_3, p. 299.

$$r + r_u\,\delta u + r_v\,\delta v = P_1\,y + Q_1\,z + R_1\,\rho + S_1\,\sigma,$$

$$s + s_u\,\delta u + s_v\,\delta v = P_1'\,y + Q_1'\,z + R_1'\,\rho + S_1'\,\sigma,$$

where

$$P_1 = b_u + (b_{uu} - 4bf)\,\delta u + b_{uv}\,\delta v,$$

$$Q_1 = 4b + 5b_u\,\delta u + 4b_v\,\delta v,$$

$$R_1 = -\,8b^2\,\delta u + bu\delta v, \qquad S_1 = 4b\delta v,$$

$$P_1' = -\,(b_v + b_{uv}\,\delta u + \overline{b_{vv} + 2bg}\delta v), \qquad Q_1' = -\,b_v\delta u,$$

$$R_1' = 2b + 2b_u\,\delta u + b_v\,\delta v, \qquad S_1' = 2b\delta u.$$

The coördinates of an arbitrary point

$$\lambda'(r + r_u\,\delta u + r_v\,\delta v) + \mu'(s + s_u\,\delta u + s_v\,\delta v)$$

on this line are

$$x_1 = \lambda'\,P_1 + \mu'\,P_1',$$

$$x_2 = \lambda'\,Q_1 + \mu'\,Q_1',$$

$$x_3 = \lambda'\,R_1 + \mu'\,R_1',$$

$$x_4 = \lambda'\,S_1 + \mu'\,S_1'.$$

This point will be on the directrix d if its coördinates satisfy the equations (7), i. e., if

$$2\lambda'\,\delta v + \mu'\,\delta u = 0,$$

(9)
$$\begin{aligned}\lambda'\,[4(b + b_{uu} - 4b^2 f - 4b^2\,b_v - 5/4b_u^2)\,\delta u + 2\,(2bb_{uv} - b_u\,b_v)\,\delta v]\\ + \mu'\,[(5b_u\,b_v - 4bb_{uv})\,\delta u + 2\,(b_v^2 - 2bb_{vv} - 4b^2\,g)\,\delta v] = 0.\end{aligned}$$

Hence, after some simplifications and reductions, we find the equation of the directrix curves (C_1', C_2') to be

(8′)
$$2^8\,h\delta u^2 - 3\cdot 2^7\,bC'\,\delta u\delta v + \theta'\,\delta v^2 = 0,$$

i. e., the same as equation (8). Hence *the directrix curves of the second kind coincide with those of the first kind.*

The first equation of (9) shows that

$$-\frac{1}{2}\,\frac{\delta u}{\delta v} = \frac{\lambda'}{\mu'};$$

so that $\lambda' : \mu'$ will be the two roots of the quadratic equation

$$2^{10}\,h\lambda'^2 + 3\cdot 2^8\,bC'\,\lambda'\,\mu' + \theta'\,\mu'^2 = 0.$$

Thus every directrix of the first kind belongs to two developables of the congruence formed by their totality. The points P_i $(i = 1, 2)$ where the directrix d intersects the cuspidal edges of these two developables are given by

$$\lambda_i' r + \mu_i' s = \lambda_i' (b_u y + 4bz) + \mu_i' (-b_v y + 2b\rho),$$

where $\lambda_i' : \mu_i'$ are the two roots of the above quadratic. The tangent t_k to one of the directrix curves through P_y joins this point to the point $P_{y+zdu_k+\rho dv_k}$, where $du_k : dv_k$ is a root of equation (8). The point of intersection of t_k and d is

$$ly + m\delta u_k z + m\delta v_k \rho,$$

where

$$\frac{l}{m} = \frac{b_u \delta u_k - 2b_v \delta v_k}{4b}.$$

Thus the point (t_k, d) is given by

$$(b_u \delta u_k - 2b_v \delta v_k) y + 4(z\delta u_k + \rho\delta v_k) = r\delta u_k + 2s\delta v_k = 2(-r\lambda_k' + s'\mu_k'),$$

which is the harmonic conjugate of the intersection of d with the cuspidal edge of the corresponding developable.

The cuspidal edges of the two developables to which d' belongs intersect this line in the points

$$P_i = \lambda_i y + \mu_i \tau,$$

where $\lambda_i : \mu_i$ $(i = 1, 2)$ are the roots of the equation

$$\lambda^2 - \left[\frac{C'}{8} - \frac{b_u b_v}{b}\right]\lambda\mu + \left[\frac{\theta' h}{2^4 b^2} - \frac{C'^2}{2^5} - \frac{b_u b_v}{2^2 b^2}(2bb_{uv} - 3b_u b_v)\right]\mu^2 = 0. \tag{10}$$

Referred to the tetrahedron $P_y P_z P_\rho P_\sigma$, the coördinates of P_1 and P_2 are

$$x_1 = \lambda_i, \qquad x_2 = -2b_v \mu_i, \qquad x_3 = b_u \mu_i, \qquad x_4 = 4b\mu_i.$$

The canonical coördinates of these points will be

$$X = 0, \qquad Y = 0, \qquad Z = -\lambda\mu \cdot 4b\mu_i,$$

$$\omega = \lambda_i + \frac{b_u b_v}{2b}\mu_i;$$

or, in non-homogeneous form,

$$x = 0, \qquad y = 0, \qquad z = \frac{-\lambda\mu \cdot 4b}{\dfrac{\lambda_i}{\mu_i} + \dfrac{b_u b_v}{2b}} = -\frac{4b\lambda\mu}{K} = \frac{6h}{5b^3 K}\sqrt{\frac{-h}{5}},$$

where

$$K = \frac{b_u b_v}{2b} + \frac{\lambda_i}{\mu_i}.$$

Hence

$$\frac{\lambda_i}{\mu_i} = K - \frac{b_u b_v}{2b},$$

where the two values of K are the two roots of the quadratic equation

$$K^2 - \frac{C'}{8} K + \frac{2h\theta' - b^2 C'}{32b^2} = 0.^* \tag{11}$$

The coefficients of this equation are invariants, as was to be expected. We may state these results in the form of the following theorem:

THEOREM. *As a point P_y describes a directrix curve (C_1 or C_2) on a scroll, its directrices (d and d') of both kinds describe developables. Let the tangent to one of the directrix curves be constructed at the point, as well as the asymptotic tangents at this point. The harmonic conjugate of the first line with respect to the other two intersects the directrix of the first kind in a point on the cuspidal edge of the developable which it describes. The non-homogeneous canonical coördinates of the points in which the directrix of the second kind intersects the edges of regression of the respective developables to which it belongs are*

$$x = 0, \qquad y = 0, \qquad z = \frac{6h}{5b^3 K}\sqrt{\frac{-h}{5}},$$

where K is a root of the invariant equation (11).

The asymptotic tangent to the surface S at the point $(\alpha y + \beta\rho)$ of the generator Γ'' joins this point to the point $(\alpha z + \beta\sigma)$.† Hence the coördinates of an arbitrary point on the *osculating hyperboloid* H are

$$x_1 = \lambda\alpha, \qquad x_2 = \mu\alpha, \qquad x_3 = \lambda\beta, \qquad x_4 = \mu\beta.$$

Therefore the equation of H referred to $(P_y\, P_z\, P_\rho\, P_\sigma)$ is

$$x_1 x_4 - x_2 x_3 = 0. \tag{12}$$

The directrix d' intersects H in the points $(-b_u b_v,\ -4bb_v,\ 2bb_u,\ 8b^2)$ and P_y. But the former is the *pinch point* of the Cayley cubic scroll osculating S at the point P_y (see equation (5)).

If $Y = y + l_\rho$, where l is an arbitrary constant, the canonical equations of S considered as the locus of P_Y are‡

$$\eta_{uu} + 2B\eta_v + F\eta = 0, \qquad \eta_{vv} + G\eta = 0, \tag{13}$$

where

$$\begin{aligned} B &= b + \frac{ll_v - \frac{1}{2}l^2 f_v}{1 + l^2 g}, \\ F &= \frac{f + lf_v + 2l^2 gb_v + l^2 bg_v}{1 + l^2 g} + \frac{l^2 g_v (b_v - \frac{1}{2}lf_v)}{(1 + l^2 g)^2}, \\ G &= g + \frac{lg_v + \frac{1}{2}l^2 g_{vv}}{1 + l^2 g} + \frac{l^4 g_v}{(1 + l^2 g)^2}. \end{aligned} \tag{14}$$

* Cf. M_2, pp. 117, 118, 119.

† E. J. Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, p. 146.

‡ M_3, pp. 311, 312.

The fundamental semi-covariants of this system of equations are

$$\eta = \frac{1}{\lambda}(y + l\rho), \qquad Z = \frac{1}{\lambda}(z + l\sigma),$$

$$\Sigma = \frac{1}{\lambda}\left[-\left(lg + \frac{\frac{1}{2}l^2 g_v}{1 + l^2 g} \right) z + \left(1 - \frac{\frac{1}{2}l^3 g_v}{1 + l^2 g} \right) \sigma \right],$$

$$\Sigma = \frac{1}{\lambda}\left[-\left(lg + \frac{\frac{1}{2}l^2 g_v}{1 + l^2 g} \right) z + \left(1 - \frac{\frac{1}{2}l^3 g_v}{1 + l^2 g} \right) \sigma \right].$$

The equations of the directrix d' of the point P_Y are, according to equations (7),

$$\begin{aligned} 2BX_2 + B_v X_4 &= 0, \\ 4BX_3 - B_u X_4 &= 0, \end{aligned} \tag{15}$$

the tetrahedron of reference being that determined by (η, Z, P, Σ). From the definition of the coördinates of the point $Q(x_1 y + x_2 z + x_3 \rho + x_4 \sigma)$ and the above expressions for (η, Z, P, Σ), we find the following equations of transformation between the coördinate systems determined by the semi-covariants (η, Z, P, Σ) and (y, z, ρ, σ) respectively:

$$\begin{aligned} \omega X_1 &= Q x_3 - Q_1 x_1, & \omega X_2 &= Q x_4 - Q_1 x_2, \\ \omega X_3 &= l x_1 - x_3, & \omega X_4 &= l x_2 - x_4, \end{aligned}$$

where

$$Q = -\left(lg + \tfrac{1}{2}\frac{l^2 g_v}{1 + l^2 g} \right), \qquad Q_1 = \left(1 - \tfrac{1}{2}\frac{l^3 g_v}{1 + l^2 g} \right),$$

and where ω is a factor of proportionality.

If we apply this transformation to equations (15), we shall find the equations of the directrix d' of the point P_Y to be

$$l(b_v x_2 + \beta_2 x_4) + (2bx_2 + b_v x_4) = 0,$$

$$l\left(4x_1 - \frac{\partial \log B}{\partial u} x_2 \right) - \left(4x_3 - \frac{\partial \log B}{\partial u} x_4 \right) = 0,$$

where

$$\beta_2 = 2bg - f_v,$$

when referred to the tetrahedron $(P_y P_z P_\rho P_\sigma)$. Now the equation of the surface generated by d' as P_y moves along Γ'' is found by eliminating the parameter l between these equations. This surface turns out to be a quartic which we shall denote, for brevity, by Q.

Equations (5) and (7) show that the locus of the *unodes* of the Cayley cubic scrolls osculating S along Γ'' is situated on the surface Q. This locus is the twisted cubic whose parametric equations are*

* M_3, p. 312.

$$
\begin{aligned}
x_1 &= -\left[b_u + b_{uv}\, l + \tfrac{1}{2}(b_{uvv} + 2gb_u)\, l^2 \right]\left[b_v + (b_{vv} + 2bg)\, l \right], \\
x_2 &= -\left[4b + 4b_v\, l + 2(b_{vv} + 2bg)\, l^2 \right]\left[b_v + (b_{vv} + 2bg)\, l \right], \\
x_3 &= \left[b_u + b_{uv}\, l + \tfrac{1}{2}(b_{uvv} + 2gb_u)\, l^2 \right]\left[2b + b_v\, l \right], \\
x_4 &= \left[4b + 4b_v\, l + 2(b_{vv} + 2bg)\, l^2 \right]\left[2b + b_v\, l \right].
\end{aligned}
$$

This twisted cubic will be an asymptotic curve on Q, if its osculating plane at the point

$$
\pi = -\, b_u\, b_v\, y - 4bb_v\, z + 2bb_u\, \rho + 8b^2\, \sigma \quad \text{(see (5), (7))}
$$

contains the corresponding generator d' of Q. Now if we determine the plane osculating this curve at π, we shall find, after a rather long calculation, that it can pass through d' only when the invariant

$$
\Delta \equiv \frac{3b\theta'}{2^4}\,(2b_u\, \theta' - b\theta'_u)
$$

vanishes. Hence we have the theorem:

Theorem. *The lines of the congruence (G_2) can be assembled into a one-parameter family of quartic scrolls having the generators of S as directrices. The twisted cubic which is the locus of the unodes of the Cayley cubic scrolls osculating S along Γ'' will be an asymptotic curve on Q only when the invariant Δ vanishes.*

Lines common to the complexes $C'(u, v)$, $C''(u, v)$, $C'(u, v + dv)$, $C''(u + du, v)$

We shall first determine the complexes $C'(u, v + dv)$, $C''(u + du, v)$ which belong to the points $P_{y+\rho dv}$ and P_{y+zdu} respectively. Let P_y receive displacements to P_{y+zdu} and $P_{y+\rho dv}$, and let $(\bar{y}, \bar{z}, \bar{\rho}, \bar{\sigma})$ and $(\bar{\bar{y}}, \bar{\bar{z}}, \bar{\bar{\rho}}, \bar{\bar{\sigma}})$ be the semi-covariants corresponding to these new positions. The values of these semi-covariants in terms of those of the position P_y are given by the following equations:

$$
(17)\qquad
\begin{aligned}
\bar{y} &= y + zdu + \cdots, & \bar{z} &= z - (fy + 2b\rho)\,du + \cdots, \\
\bar{\rho} &= \rho + \sigma du + \cdots, & \bar{\sigma} &= \sigma + (\beta_2\, y + \beta_4\, \rho)\,du + \cdots, \\
\bar{\bar{y}} &= y + \rho dv + \cdots, & \bar{\bar{z}} &= z + \sigma dv + \cdots, \\
\bar{\bar{\rho}} &= \rho - gydv + \cdots, & \bar{\bar{\sigma}} &= \sigma - gzdv + \cdots
\end{aligned}
$$

where

$$
\beta_2 = 2bg - f_v, \qquad \beta_4 = -(f + 2b_v).
$$

From these, by the method used in determining equations (4), we find the following equations of transformation between the coördinate systems $(\bar{y}, \bar{z}, \bar{\rho}, \bar{\sigma})$, $(\bar{\bar{y}}, \bar{\bar{\rho}}, \bar{\bar{z}}, \bar{\bar{\sigma}})$ and, (y, z, ρ, σ):

$$
(18)\quad
\begin{aligned}
x_1 &= \bar{x}_1 + (-f\bar{x}_2 + \beta_2 \bar{x}_4)\,du = \bar{\bar{x}}_1 - g\bar{\bar{x}}_3\,dv,\\
x_2 &= \bar{x}_2 + \bar{x}_1\,du \qquad\qquad\quad\;\; = \bar{\bar{x}}_2 - g\bar{\bar{x}}_4\,dv,\\
x_3 &= \bar{x}_3 + (-2b\bar{x}_2 + \beta_4 \bar{x}_4)\,du = \bar{\bar{x}}_3 + \bar{\bar{x}}_1\,dv,\\
x_4 &= \bar{x}_4 + \bar{x}_3\,du \qquad\qquad\quad\;\; = \bar{\bar{x}}_4 + \bar{\bar{x}}_2\,dv.
\end{aligned}
$$

whence

$$
(18')\quad
\begin{aligned}
\bar{\bar{x}}_1 &= x_1 + gx_3\,dv, & \bar{\bar{x}}_3 &= x_2 + gx_4\,dv,\\
\bar{\bar{x}}_3 &= x_3 - x_1\,dv, & \bar{\bar{x}}_4 &= x_4 - x_2\,dv,\\
\bar{x}_1 &= x_1 + (fx_2 - \beta_2 x_4)\,du, & \bar{x}_2 &= x_2 - x_1\,du,\\
\bar{x}_3 &= x_3 + (2bx_2 - \beta_4 x_4)\,du, & \bar{x}_4 &= x_4 - x_3\,du,
\end{aligned}
$$

and

$$
(18'')\quad
\begin{aligned}
\bar{\omega}_{14} &= \omega_{14} - (\omega_{13} + f\omega_{42})\,du, \qquad \bar{\omega}_{23} = \omega_{23} + (\beta_4 \omega_{42} - \omega_{13})\,du,\\
&\qquad \bar{\omega}_{42} = \omega_{42} + (\omega_{14} + \omega_{23})\,du,\\
\bar{\bar{\omega}}_{14} &= \omega_{14} + (+g\omega_{34} - \omega_{12})\,dv, \qquad \bar{\bar{\omega}}_{23} = \omega_{23} + (\omega_{12} - g\omega_{34})\,dv,\\
&\qquad \bar{\bar{\omega}}_{34} = \omega_{34} + (\omega_{23} - \omega_{14})\,dv.
\end{aligned}
$$

From these we find that the equations of the complexes $C'(u, v + dv)$ and $C''(u + du, v)$ become

$$(-b_v\,\omega_{34} - b\omega_{14} + b\omega_{23}) + dv\,(2b\omega_{12} - \beta_2\,\omega_{34}) = 0,$$

$$[2b(\omega_{14} + \omega_{23}) + b_u\,\omega_{42}] + du\,[3b_u(\omega_{14} + \omega_{23}) - 4b\omega_{13} - (b_{uv} + \alpha_3)\,\omega_{42}] = 0,$$

where

$$\alpha_3 = 2(2bf + 2bb_v - b_{uu}).$$

Then clearly the lines required are those common to the four complexes

$$
(19)\quad
\begin{aligned}
&1.\quad C' : -b_v\,\omega_{34} - b\omega_{14} + b\omega_{23} = 0,\\
&2.\quad C'' : 2b(\omega_{14} + \omega_{23}) + b_u\,\omega_{42} = 0,\\
&3.\quad 2b\omega_{12} - \beta_2\,\omega_{34} = 0,\\
&4.\quad 3b_u(\omega_{14} + \omega_{23}) - 4b\omega_{13} - (b_{uu} + \alpha_3)\,\omega_{42} = 0.
\end{aligned}
$$

The lines common to these complexes must intersect d and d'. The points

$$p = b_u\,y + 4bz, \qquad q = -b_v\,y + 2b\rho$$

are on d, and the points y, $r = -2b_v\,z + b_u\,\rho + 4b\sigma$ are on d'. Hence the coördinates of an arbitrary point on d will be

$$
\begin{aligned}
x_1 &= \lambda b_u - \mu b_v, & x_2 &= 4b\lambda,\\
x_3 &= 2b\mu, & x_4 &= 0;
\end{aligned}
$$

and those of an arbitrary point on d' will be

$$x_1' = \lambda', \qquad x_2' = -2b_v \mu',$$
$$x_3' = b_u \mu', \qquad x_4 = 4b\mu'.$$

Therefore the coördinates of the line joining these points will be

$$\begin{aligned}
\omega_{12} &= -4b\lambda\lambda' - 2b_u b_v \lambda\mu' + 2b_v^2 \mu\mu', \\
\omega_{13} &= -2b\mu\lambda' + b_u^2 \lambda\mu' - b_u b_v \mu\mu', \\
\omega_{14} &= 4bb_u \lambda\mu' - b_u b_v \mu\mu', \\
\omega_{23} &= 4bb_u \lambda\mu' + b_u b_v \mu\mu', \\
\omega_{34} &= 8b^2 \mu\mu', \\
\omega_{42} &= -16b^2 \lambda\mu'.
\end{aligned}$$

In order that this line may belong to the second pair of complexes (19), the equations

$$\begin{aligned}
-2^7 b\lambda\lambda' - 2^6 b_u b_v \lambda\mu' + \theta' \mu\mu' &= 0, \\
ab\mu\lambda' + 2^4 h\lambda\mu' + b_u b_v \mu\mu' &= 0.
\end{aligned} \tag{20}$$

must be satisfied. Thus we find

$$\lambda^2 : \mu^2 = -\frac{\theta'}{64} : h,$$

$$\lambda' : \mu' = -\left(b_u b_v + 2^4 h \frac{\lambda}{\mu}\right) : 2b.$$

Clearly, when θ' and h are both equal to zero, the four complexes have a pencil of lines in common; this pencil is determined by d and the *pinch-point* of the Cayley cubic scroll osculating S at P_y. If θ' and h are different from zero, then the four complexes have two lines in common which intersect d in two points which form a harmonic group with the points p and q; and their intersections with d' are harmonic conjugates with respect to H (i. e., with respect to the pinch-point of the Cayley cubic scroll osculating S at P_y, and P_y). These results may be combined in the following theorem:*

THEOREM. *Consider a scroll whose flecnode curve consists of two distinct branches, and a point on S for which the invariant h is different from zero. The four complexes $C'(u, v)$, $C''(u, v)$, $C'(u, v+dv)$, $C''(u+du, v)$ have two lines in common. Let there be marked on the directrix of the first kind its intersections p and q with the asymptotic tangents of P_y. The two lines common to the four complexes intersect the directrix of the first kind in two points which are harmonic conjugates with respect to p and q. These two lines intersect the directrix of the second kind in two points which are harmonic conjugates with respect to the pair of points in which this directrix intersects the osculating hyper-*

* Cf. M_2, pp. 97, 98.

boloid. If the two branches of the flecnode curve on S coincide and the invariant h is different from zero, the four complexes have one line in common; if, however, the invariant h is equal to zero, the four complexes have a pencil of lines in common. This pencil is determined by the pinch-point of the Cayley cubic scroll osculating S at the point and the directrix of the first kind.

Surfaces for which Ω' vanishes and θ' is different from zero

Several of the preceding results assume a simpler form when the invariant $\Omega' \equiv (\partial^2 \log b)/(\partial u \partial v)$ vanishes. Because, if Ω' vanishes, it is possible to find a transformation of the form

$$y = \frac{1}{\sqrt{\dfrac{d\bar{u}}{du} \cdot \dfrac{d\bar{v}}{dv}}}, \qquad \bar{u} = \int \sqrt[3]{U}\, du, \qquad \bar{v} = \int \frac{dv}{\sqrt[3]{V}},$$

where U and V are functions of u alone and v alone respectively, that transforms equations (1) into the following system:*

$$y_{uu} + 2y_v + fy = 0, \qquad y_{vv} + gy = 0,$$

where f is a function of u alone and g is a constant. Thus the invariant h is a function of u alone and the invariant θ' is a constant. The equation (8) becomes

$$\overline{dv}^2 - \phi(u)\overline{du}^2 = 0,$$

or

$$\overline{U} \pm \overline{V} = \text{const.},$$

where

$$\overline{U} = \int \sqrt{\phi(u)}\, du, \qquad \overline{V} = v.$$

Hence the directrix curves (C_1) and (C_2) form a conjugate system on S. From equations (7), (14), (21), it is easily found that the equations of d and d' of the point P_Y become

$$\begin{aligned} d &: x_4 - lx_2 = 0, \qquad x_1 + lgx_3 = 0, \\ d' &: x_2 + lgx_4 = 0, \qquad lx_1 - x_3 = 0, \end{aligned}$$

when referred to the coördinate system $(P_y P_z P_\rho P_\sigma)$ of the point P_y. These equations show that d and d' belong to the same regulus of a quadric surface (H_1). The equation of (H_1) is

$$x_1 x_2 + g x_3 x_4 = 0.$$

Now (H) and (H_1) intersect in the two generators Γ'' and Γ_1'' (Γ_1'' being the generator on S_1, the derived ruled surface of S, which corresponds to Γ'' on S^*) which belong to the same regulus; they must, therefore, intersect in

* E. J. Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, p. 124.

two other generators which belong to the complementary regulus. It is easy to see that these generators pass through the points $y \pm 1/(\sqrt{-g})\rho$ on Γ''. The flecnodes on Γ'' are given by the factors of the quadratic covariant*

$$u'_{12}\rho^2 - u'_{21}y^2 + (u'_{11} - u'_{22})y\rho,$$

where

$$u'_{11} = -4f, \qquad u'_{12} = -8b, \qquad u'_{22} = -4(f + 2b_v),$$
$$u'_{21} = -4(f_v - 2bg) = 4(b_{vv} + 2bg).$$

Since the asymptotic curves Γ'' belong to linear complexes, the surface S has two straight-line directrices.† Hence the second pair of generators of intersection of (H) and (H_1) must coincide with these directrices (δ, δ') of S.

If the invariants h, θ', C' all vanish, the directrix curves (C_1) and (C_2) become indeterminate. But the vanishing of these invariants is completely characteristic of a Cayley cubic scroll.‡ In fact, from the second of equations (21), we find

$$y = U_1 v + U_2$$

where U_1 and U_2 are functions of u alone. If this value of y be substituted in the first of equations (21) and the resulting equations

$$\frac{d^2 U_1}{du^2} = 0, \qquad \frac{d^2 U_2}{du^2} + 2U_1 = 0,$$

be integrated, we find

$$y = (au + b)v + \left(-\frac{a}{3}u^3 - bu^2 + cu + d\right),$$

where a, b, c, d are constants. Hence the equation of S is

$$3(y_2 y_3 - y_1 y_4)y_4 = 2y_3^3.$$

In this case we readily find the following equations of transformation between the coördinate systems (y, z, ρ, σ) and the fixed system of reference (y_1, y_2, y_3, y_4):

$$\begin{aligned} y_1 &= (-\tfrac{1}{3}u^3 + uv)x_1 + (-u^2 + v)x_2 + ux_3 + x_4, \\ y_2 &= (-u^2 + v)x_1 - 2ux_2 + x_3, \\ y_3 &= ux_1 + x_2, \qquad y_4 = x_1, \end{aligned} \tag{22}$$

whence

* M_3, p. 313.

† See a paper by the author, these Transactions, vol. 15 (1914), p. 171. Since, apart from Γ'' and Γ''', H and H_1 intersect along two generators for which $(x_2/x_4)^2 = -1/g$, and (δ, δ') are the only generators on H for which this is true, H and H_1 have (δ, δ') in common.

‡ M_3, p. 307.

$$
\begin{aligned}
x_1 &= y_4, \qquad x_2 = y_3 - uy_4, \\
x_3 &= y_2 + 2uy_3 - (u^2 + v)\,y_4, \\
x_4 &= y_1 - uy_2 - (u^2 + v)\,y_3 + \left(\frac{u^3}{3} + uv\right) y_4.
\end{aligned}
$$

The complexes C' and C'' now become (when referred to the fixed system of coördinates)

$$
\begin{gathered}
\omega_{14} - \omega_{23} - 2v\omega_{34} = 0, \\
\omega_{14} + \omega_{23} - 2u\,(u\omega_{34} - \omega_{42}) = 0.
\end{gathered}
$$

The equations of the directrices are

$$
\begin{gathered}
d : y_4 = 0, \qquad y_1 - uy_2 - (u^2 + v)\,y_3 + \left(\frac{u^3}{3} + uv\right) y_4 = 0, \\
d' : y_3 - uy_4 = 0, \qquad y_2 + 2uy_3 - (u^2 + v)\,y_4 = 0.
\end{gathered}
$$

All of the directrices d are situated in the singular tangent plane $y_4 = 0$; and all of the directrices d' pass through the *unode* $(1, 0, 0, 0)$. These results may be stated in the form of the theorem:

THEOREM. *If a scroll S have distinct straight line directrices, the rays of the directrix congruences of both kinds which correspond to the points of a generator Γ'' of S belong to the same regulus of a quadric (H_1); and this quadric contains the directrices (δ, δ') of the surface S itself. The directrix curves (C_1) and (C_2) become indeterminate when, and only when, the surface S is a Cayley cubic scroll. In this case the directrix congruences of both kinds degenerate; that of the first kind consists of the net of lines in the singular tangent plane, while that of the second kind consists of the sheaf of lines through the unode.*

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