THE GENERAL THEORY OF CONGRUENCES*

BY

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1. Introduction

The general theory of congruences may be based upon the following considerations. Let the system of partial differential equations

$$(D)\qquad \begin{aligned} y_v &= mz, \qquad z_u = ny, \\ y_{uu} &= a\ y + b\ z + c\ y_u + d\ z_v, \\ z_{vv} &= a'\, y + b'\, z + c'\, y_u + d'\, z_v, \end{aligned}$$

where

$$y_u = \frac{\partial y}{\partial u}, \qquad y_v = \frac{\partial y}{\partial v}, \qquad y_{uu} = \frac{\partial^2 y}{\partial u^2}, \qquad \text{etc.,}$$

be completely integrable. It will then have precisely four pairs of linearly independent solutions $(y^{(k)}, z^{(k)})$, $(k = 1, 2, 3, 4)$, such that the general solution will be of the form

$$y = \sum_{k=1}^{4} c^{(k)}\, y^{(k)}, \qquad z = \sum_{k=1}^{4} c^{(k)}\, z^{(k)}.$$

Let $y^{(1)}, \cdots y^{(4)}$ and $z^{(1)}, \cdots z^{(4)}$ be interpreted as the homogeneous coördinates of two points P_y and P_z. As u and v vary, these points will describe two surfaces, S_y and S_z (either or both of which may be degenerate), and the line $P_y P_z$ will generate a congruence whose focal surface consists of the two surfaces S_y and S_z. Moreover the ruled surfaces of the congruence obtained by equating either u or v to a constant will be its developables.†

All congruences whose focal surfaces have two distinct sheets may be studied by this method.

The *invariants* and *covariants* of a system of form (D) are those functions of the coefficients and variables which are left unchanged (absolutely or except for a factor), when system (D) is subjected to any transformation of the form

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† E. J. Wilczynski, *Sur la théorie générale des congruences*. Mémoire couronné par la classe des sciences. Mémoires publiés par la Classe des Sciences de l'Académie Royale de Belgique. Collection en 4°. Deuxième série. Tome III (1911). This paper will hereafter be cited as the Brussels Paper.

$$\bar{y} = \lambda(u)y, \qquad \bar{z} = \mu(v)z, \qquad \bar{u} = \phi(u), \qquad \bar{v} = \psi(v),$$

where λ, μ, ϕ, and ψ are arbitrary functions of the variables indicated. It is easy to see that the invariants and covariants of (D) are intrinsically connected with those properties of the congruence which are invariant under projective transformation. Therefore the theory of congruences based upon a consideration of the invariants and covariants of a system of form (D) is a projective theory.*

It is the purpose of the present paper to enrich the theory of congruences, making use of the analytical basis just indicated, by the introduction of a number of new geometrical concepts. The results, aside from their intrinsic interest, throw a great deal of light upon a number of questions in the theory of surfaces. To mention only one instance: we shall find a geometrical interpretation for the condition which Bianchi expresses by saying that a conjugate system is isothermally conjugate.

2. The axis of a surface point with respect to a given conjugate system. The axis congruence and the axis curves

The fundamental covariants of system (D) are y, z, ρ, and σ, where

$$(1) \qquad \rho = y_u - \frac{m_u}{m} y, \qquad \sigma = z_v - \frac{n_v}{n} z.$$

The point P_ρ whose coördinates are given by (1) is on the cuspidal edge of the developable which is formed by the tangents of the curves $v =$ const. on S_y constructed at the various points of the same curve $u =$ const. The locus of P_ρ, the surface S_ρ, is therefore the second sheet of the focal surface of the congruence which is composed of the tangents to the curves $v =$ const. on S_y. Moreover, the curves $u =$ const. and $v =$ const. on S_ρ form again a conjugate system, the first Laplacian transform of the given conjugate system on S_y. The point P_σ is related in similar fashion to the congruence composed of the tangents to the curves $u =$ const. on S_z and to the minus first Laplacian transform.

The four points $P_y\, P_z\, P_\rho\, P_\sigma$ are, in general, not coplanar and we shall make use of them systematically as the vertices of a local tetrahedron of reference, for the purpose of studying the properties of a congruence in the vicinity of one of its lines. To completely define this local coördinate system, we shall say that the coördinates of any point given by an expression of the form

$$x_1 y + x_2 z + x_3 \rho + x_4 \sigma,$$

when referred to the system $P_y\, P_z\, P_\rho\, P_\sigma$ shall be proportional to

$$(x_1, x_2, x_3, x_4).$$

* Brussels Paper, §§ 1–3.

Let us consider the planes which osculate the curves u = const. and v = const. on S_y; i. e., the curves along which the developables of the congruence touch the focal sheet S_y. The osculating plane of the curve v = const. on S_y is determined by the three points whose coördinates are $(y^{(1)}, \cdots y^{(4)})$, $(y_u^{(1)}, \cdots y_u^{(4)})$, and $(y_{uu}^{(1)}, \cdots y_{uu}^{(4)})$, or as we shall henceforth say more briefly, by the three points y, y_u, and y_{uu}. Now we have

$$y_u = \frac{m_u}{m} y + \rho,$$

$$y_{uu} = \left(a + c\frac{m_u}{m}\right) y + \left(b + d\frac{n_v}{n}\right) z + c\rho + d\sigma.$$

Therefore the coördinates of these three points in the local coördinate system $P_y\, P_z\, P_\rho\, P_\sigma$ are

$$(1, 0, 0, 0), \left(\frac{m_u}{m}, 0, 1, 0\right), \quad \text{and} \quad \left(a + c\frac{m_u}{m},\ b + d\frac{n_v}{n},\ c,\ d\right)$$

respectively. Therefore we find

$$dx_2 - \left(b + d\frac{n_v}{n}\right) x_4 = 0 \tag{2}$$

as the equation of their plane. In the same way we find

$$x_3 = 0 \tag{3}$$

as the equation of the plane which osculates the curve u = const. of S_y at P_y, a result which is geometrically obvious, in so far as this is also the plane tangent to S_z at P_z. Of course the plane (2) is also tangent to S_ρ at P_ρ.

We prefer to write equation (2) in a different form obtained from it by making use of the conditions which the coefficients of (D) must satisfy in order that (D) may be completely integrable. These integrability conditions are as follows:*

$$
\text{(I)} \quad
\begin{gathered}
c = f_u, \qquad d' = f_v, \qquad b = -d_v - df_v, \qquad a' = -c'_u - c' f_u, \\
W = mn - c'd = f_{uv}, \\
m_{uu} + d_{vv} + df_{vv} + d_v f_v - f_u m_u = ma + db', \\
n_{vv} + c'_{uu} + c' f_{uu} + c'_u f_u - f_v n_v = c' a + nb', \\
2m_u n + mn_u = a_v + f_u mn + a' d, \\
m_v n + 2mn_v = b'_u + f_v mn + bc',
\end{gathered}
$$

where f may be any function of u and v.

The third of these integrability conditions enables us to replace (2) by the equivalent equation

* Brussels Paper, p. 17, eq. (12).

$$x_2 + \left(f_v + \frac{d_v}{d} - \frac{n_v}{n}\right)x_4 = 0. \tag{4}$$

The curves u = const. and v = const. form, of course, a conjugate system on S_y. *We shall call the line of intersection of the osculating planes of the two curves of this system which meet at P_y, the axis of the point P_y with respect to this conjugate system.* The equations of this axis are then

$$x_2 + \left(f_v + \frac{d_v}{d} - \frac{n_v}{n}\right)x_4 = 0, \qquad x_3 = 0. \tag{5}$$

Clearly the axis of P_y passes through P_y and all of the axes, formed for all of the points of the surface S_y, form a congruence which we shall call the *axis congruence* of the given conjugate system on S_y.

We now proceed to determine the developables of the axis congruence, or what amounts to the same thing, those curves on S_y which have the property that the axes of all of their points form a developable. These curves are the curves of intersection of S_y and the developables of the axis congruence.

Let us write

$$\tau^{(y)} = -\left(f_v + \frac{d_v}{d} - \frac{n_v}{n}\right)z + \sigma. \tag{6}$$

Then the point whose coördinates are given by $\tau^{(y)}$ is on the axis of P_y. It is, more specifically, the point in which the axis intersects $P_z P_\sigma$, and we may think of the axis as determined by the two points y and $\tau^{(y)}$. The condition that the axis shall generate a developable of the axis congruence may also be expressed in this form; two consecutive axes shall intersect. Now, the axis of the surface point, which belongs to the parameters $u + \delta u$, $v + \delta v$, is obtained by joining the point whose coördinates are given by

$$Y = y + y_u\,\delta u + y_v\,\delta v$$

to the point

$$T^{(y)} = \tau^{(y)} + \tau^{(y)}_u\,\delta u + \tau^{(y)}_v\,\delta v.$$

But we have, making use of (D) and (1),

$$y_u = \frac{m_u}{m}y + \rho, \qquad y_v = mz,$$

$$\tau^{(y)}_u = n\left(\frac{b}{d} + \frac{n_v}{n}\right)y + \left[\left(\frac{b}{d}\right)_u + mn\right]z,$$

$$\tau^{(y)}_v = \left(a' + c'\frac{m_u}{m}\right)y + \left[\left(\frac{b}{d}\right)_v + b' + \frac{n_v}{n}\left(\frac{b}{d} + d'\right)\right]z + c'\rho + \left(\frac{b}{d} + d'\right)\sigma.$$

The coördinates of any point on the line $YT^{(y)}$ are given by an expression of the form $\lambda Y + \mu T^{(y)}$. Therefore, the coördinates of such a point, referred to the local tetrahedron $P_y P_z P_\rho P_\sigma$, will be

$$x_1 = \lambda\left(1 + \frac{m_u}{m}\delta u\right) + \mu\left[n\left(\frac{b}{d} + \frac{n_v}{n}\right)\delta u + \left(a' + c'\frac{m_u}{m}\right)\delta v\right],$$

$$x_2 = \lambda m\delta v + \mu\left[-\left(f_v + \frac{d_v}{d} - \frac{n_v}{n}\right) + \left\{\left(\frac{b}{d}\right)_u + mn\right\}\delta u + \left\{\left(\frac{b}{d}\right)_v + b' + \frac{n_v}{n}\left(\frac{b}{d} + d'\right)\right\}\delta v\right],$$

$$x_3 = \lambda\delta u + \mu c'\,\delta v,$$

$$x_4 = \mu\left[1 + \left(\frac{b}{d} + d'\right)\delta v\right].$$

This point will be on the axis of P_y if and only if its coördinates satisfy equations (5), i. e., if and only if λ, μ, δu, and δv can be chosen so as to satisfy the conditions

$$\lambda m\delta v + \mu\left[\left\{\left(\frac{b}{d}\right)_u + mn\right\}\delta u + \left\{\left(\frac{b}{d}\right)_v + b' - \frac{b}{d}\left(\frac{b}{d} + d'\right)\right\}\delta v\right] = 0,$$

$$\lambda\delta u + \mu c'\,\delta v = 0.$$

But we have, from (I),

$$\frac{b}{d} = -f_v - \frac{d_v}{d}, \qquad d' = f_v,$$

and therefore

$$\left(\frac{b}{d}\right)_u + mn = -f_{uv} - \frac{\partial^2 \log d}{\partial u\partial v} + mn = c'\,d - \frac{\partial^2 \log d}{\partial u\partial v},$$

$$\left(\frac{b}{d}\right)_v + b' - \frac{b}{d}\left(\frac{b}{d} + d'\right) = -\frac{1}{d}(df_{vv} + d_v f_v + d_{vv} - db')$$

$$= -\frac{m}{d}\left(a + f_u\frac{m_u}{m} - \frac{m_{uu}}{m}\right),$$

the equality of the last two members being established by means of the sixth of the integrability conditions (I). Let us put

$$I = a + f_u\frac{m_u}{m} - \frac{m_{uu}}{m},$$

(7)

$$d_1 = d\left(c'\,d - \frac{\partial^2 \log d}{\partial u\partial v}\right),$$

where the notation d_1 has been used to indicate the fact that d_1 is the coefficient which corresponds to d in the system (D_1) obtained from (D) by application of the first Laplace transformation.*

With these notations, the above conditions for intersection of $y_T^{(v)}$ with $Y_T^{(v)}$ reduce to

* See Brussels Paper, equation (137).

$$
\begin{aligned}
&\lambda d m \delta v + \mu (d_1 \delta u - m I \delta v) = 0, \\
&\lambda \delta u + \mu c' \delta v = 0,
\end{aligned} \tag{8}
$$

giving

$$
d_1 \delta u^2 - m I \delta u \delta v - c' d m \delta v^2 = 0 \tag{9}
$$

as the differential equation of the developables of the axis congruence. If we call the curves in which these developables intersect S_y its *axis curves*, we may also regard (9) as the differential equation of the axis curves on S_y.

If a point P_y moves along one of the axis curves on S_y, the corresponding axis describes a developable and meets the cuspidal edge of this developable in the point $\lambda y + \mu \tau^{(v)}$, where the ratio $\lambda : \mu$ is connected with $\delta u : \delta v$ by means of (8). Let us speak of the two points, determined in this way on the axis, as its *foci*. They are given by the expression $\lambda y + \mu \tau^{(v)}$ where $\lambda : \mu$ is determined by the quadratic equation

$$
d m \lambda^2 - m I \lambda \mu - c' d_1 \mu^2 = 0.
$$

We may express this more elegantly as follows. *The factors of the quadratic covariant*

$$
c' d_1 y^2 - m I y \tau^{(v)} - d m (\tau^{(v)})^2 \tag{10}
$$

determine the foci of the axis which passes through P_y.

If the invariant I is equal to zero, the axis curve tangents and the tangents of the given conjugate system on S_y form a harmonic pencil. Moreover, the foci of the axis in that case divide harmonically the points in which the axis meets the original surface and the line which joins the corresponding points of the first and second Laplace transforms.

The asymptotic lines of S_y are given by the differential equation*

$$
d \delta u^2 + m \delta v^2 = 0. \tag{11}
$$

Therefore the axis curves can coincide with the asymptotic lines only if the simultaneous conditions

$$
I = 0, \qquad \frac{\partial^2 \log d}{\partial u \partial v} - 2c' d = 0 \tag{12}
$$

are satisfied. If the simultaneous invariant of (9) and (11) is equal to zero, the axis curves form a conjugate system. But this condition reduces to

$$
\frac{\partial^2 \log d}{\partial u \partial v} = 0, \tag{13}
$$

which gives on integration

$$
d = U \cdot V,
$$

where U is a function of u alone, V a function of v alone. Thus, *the axis curves form a conjugate system on* S_y, *if and only if* d *is a product of a function of* u *alone by a function of* v *alone.*

* Brussels Paper, eq. (111).

If this condition is satisfied, the conjugate system formed by the axis curves will coincide with the original conjugate system of S_y, if and only if the further conditions

$$c'\, d = c'\, m = 0 \tag{14}$$

are fulfilled. If m were to vanish, the whole theory of conjugate systems would be inapplicable since the surface S_y would, in that case, degenerate into a curve. Excluding that case, (14) gives $c' = 0$. Therefore S_z must be developable.* Equations (7) show that d_1 will also vanish on account of (13) and (14), i. e., S_1 is also developable. Thus, if the axis curves of a conjugate net on a non-degenerate surface coincide with the given conjugate net, the first and minus first Laplace transformations both give rise to nets on developable surfaces.

If

$$d_1 = mI = c'\, dm = 0 \tag{15}$$

the axis curves of S_y are indeterminate and the axis congruence reduces to the system of lines through a fixed point. If S_y does not degenerate into a curve, $m \neq 0$, and we must have

$$c'\, d = 0, \qquad d\frac{\partial^2 \log d}{\partial u \partial v} = 0, \qquad I = 0. \tag{16}$$

It remains to state briefly the corresponding formulas for the second sheet S_z of the focal surface. The axis of P_z joins this point to the point whose coördinates are given by

$$\tau^{(z)} = \left(\frac{a'}{c'} + \frac{m_u}{m}\right) y + \rho = -\left(f_u + \frac{c'_u}{c'} - \frac{m_u}{m}\right) y + \rho. \tag{17}$$

The differential equation of the axis curves of the surface S_z is

$$c'\, dn\delta u^2 + nJ\delta u \delta v - c'_{-1}\, \delta v^2 = 0, \tag{18}$$

where

$$J = b' + f_v \frac{n_v}{n} - \frac{n_{vv}}{n}, \tag{19}$$

$$c'_{-1} = c'\left(c'\, d - \frac{\partial^2 \log c'}{\partial u \partial v}\right).$$

3. The ray of a surface point with respect to a given conjugate system. The ray congruence and the ray curves

Let us apply the principle of duality to the notions of § 2. The osculating plane of the curve v = const. is replaced by the point of intersection of three consecutive tangent planes of S_y along such a curve v = const. But this is the point P_ρ. Similarly three consecutive tangent planes of S_y along a curve

* Brussels Paper, end of § 4.

$u =$ const. intersect in P_z. Thus, the line $P_\rho P_z$ corresponds dualistically to the axis of the point P_ρ. We shall call the line $P_\rho P_z$, which joins corresponding points of the first and minus first Laplace transformed nets, and which corresponds dualistically to the axis of P_y, *the ray of* P_y. We shall moreover speak of the totality of these rays, as the *ray congruence,* and call the curves on S_y which correspond to the developables of this congruence, *the ray curves of* S_y. Further, we shall speak of the points in which any ray touches the cuspidal edges of the two developables of the ray congruence to which it belongs, as the *foci of the ray.*

By means of a method closely analogous to that of § 2, we find the following results. The foci of any ray are given by an expression of the form $\lambda z + \mu\rho$ where the ratio $\lambda : \mu$ is either root of the quadratic

$$mn\lambda^2 + mI\lambda\mu - dm_1\mu^2 = 0, \tag{20}$$

where

$$m_1 = m\left(mn - \frac{\partial^2 \log m}{\partial u \partial v}\right). \tag{21}$$

This may also be expressed as follows. *The foci of the ray are given by the factors of the quadratic covariant*

$$dm_1 z^2 + mIz\rho - mn\rho^2. \tag{22}$$

The ray curves on S_y *are determined by the differential equation*

$$dmn\delta u^2 - mI\delta u\delta v - m_1\delta v^2 = 0. \tag{23}$$

The simultaneous invariant of (23) and (11) is

$$d\frac{\partial^2 \log m}{\partial u \partial v},$$

and d is different from zero if S_y is not developable. Therefore we obtain the following theorem. *The ray curves form a conjugate net on a non-developable surface, if and only if*

$$\frac{\partial^2 \log m}{\partial u \partial v} = 0. \tag{24}$$

Now, in accordance with the theory of Darboux,* the conjugate system composed of the curves $u =$ const. and $v =$ const. on S_y may be studied by means of the equation

$$y_{uv} = \frac{m_u}{m} y_v + mny, \tag{25}$$

which follows from (D). The Laplace-Darboux invariants of this equation are

$$h = mn, \qquad k = -\frac{\partial^2 \log m}{\partial u \partial v} + mn, \tag{26}$$

* *Théorie des surfaces,* vol. 2.

so that (24) is equivalent to the condition $h = k$. We have found the following new interpretation for this condition.

A conjugate system on a non-developable surface has equal Laplace-Darboux invariants, if and only if its ray curves also form a conjugate system.

The condition $h = k$, or in our notation

$$\frac{\partial^2 \log m}{\partial u \partial v} = 0,$$

has been interpreted, by Darboux, in another quite different fashion. We shall now explain and at the same time generalize Darboux's interpretation.

If we let v remain constant, a line of our congruence will generate a developable as the variable u changes. The point in which this line meets the cuspidal edge of the developable which it describes, will be the point P_z, and the osculating plane of the cuspidal edge at P_z will be the plane of the three points

$$z, \qquad z_u = ny, \qquad z_{uu} = \left(n_u + n\frac{m_u}{m}\right)y + n\rho. \tag{27}$$

This plane may be regarded as determined by three consecutive points of the cuspidal edge, and we may determine a two-parameter family of conics each of which passes through these three consecutive points, or in other words, has second order contact with the cuspidal edge at P_z.

Let Z_k be the coördinates of any point of the cuspidal edge in the immediate neighborhood of P_z. Then

$$Z_k = z_k + \frac{\partial z_k}{\partial u}\delta u + \frac{1}{2}\frac{\partial^2 z_k}{\partial u^2}\delta u^2 + \cdots \qquad (k = 1, 2, 3, 4),$$

or, on account of (27),

$$Z_k = \left[n\delta u + \tfrac{1}{2}\left(n_u + n\frac{m_u}{m}\right)\delta u^2 + \cdots\right]y_k + z_k + (\tfrac{1}{2}n\delta u^2)\rho_k.$$

Thus, the parametric equations of the cuspidal edge, referred to the tetrahedron $P_y P_z P_\rho P_\sigma$, in the neighborhood of P_z, may be expanded in the form

$$\begin{aligned} x_1 &= n\delta u + \tfrac{1}{2}\left(n_u + n\frac{m_u}{m}\right)\delta u^2 + \cdots, \\ x_2 &= 1 + \cdots, \qquad x_3 = \tfrac{1}{2}n\delta u^2 + \cdots, \end{aligned} \tag{28}$$

where the omitted terms are of higher than the second order in δu.

In order to find the most general conic of the plane $x_4 = 0$ which has second order contact with the cuspidal edge at P_z, it suffices to determine the coefficients of the quadratic equation

$$A_{11}x_1^2 + \cdots + 2A_{12}x_1x_2 + \cdots = 0$$

in such a way that this equation shall be satisfied by the expressions (28) up to and including second order terms. We find in this way

(29) $$A_{11}(x_1^2 - 2nx_2 x_3) + A_{33} x_3^2 + 2A_{13} x_1 x_3 = 0$$

as the equation of the most general conic which has second order contact with the cuspidal edge of the developable v = const. at the point P_z.

The point P_ρ stands in the same relation to the curves u = const. of S_y which connects the point P_z with the curves v = const. If we give a fixed value to u and allow v to vary, the line $P_y P_\rho$ generates a developable of the first Laplacian transformed congruence whose cuspidal edge is the locus of P_ρ. Since we have

$$\rho_v = \frac{m_1}{m}\rho, \qquad \rho_{vv} = \left(\frac{m_1}{m}\right)_v y + m_1 z,$$

this cuspidal edge is represented, up to and including terms of the second order, by the expansions

(30) $$x_1 = \frac{m_1}{m}\delta v + \tfrac{1}{2}\left(\frac{m_1}{m}\right)_v \delta v^2 + \cdots, \qquad x_2 = \tfrac{1}{2}m_1 \delta v^2 + \cdots, \qquad x_3 = 1 + \cdots,$$

again referred to the coördinate system of $P_y P_z P_\rho P_\sigma$. Equations (30) show that the coefficients of (29) may always be determined so that (29) shall touch the cuspidal edge (30) at P_ρ, but that the contact will be of the second order only if

$$\frac{\partial^2 \log m}{\partial u \partial v} = 0,$$

if we exclude the case $m_1 = 0$ from consideration since, in that case, the surface described by P_ρ would degenerate into a curve.

We have therefore proved Darboux's theorem which states that there exists a conic having second order contact with both of these cuspidal edges if and only if the given conjugate system of curves on S_y has equal invariants. We may combine the two results in the following statement.

Given two one-parameter families of curves (u = const. and v = const.) forming a conjugate system on a non-degenerate surface S_y. Consider the developables composed of the tangents of curves of one of these families and circumscribed about S_y along a fixed curve of the other family. Let P_z and P_ρ be the points in which the tangents of the curves u = const. and v = const., which cross at a point P_y of S_y, meet the cuspidal edges of these circumscribing developables. Moreover, let the loci of P_z and P_ρ for all possible values of u and v be non-degenerate surfaces. If there exists a conic in the tangent plane of S_y at P_y, which has second order contact, at P_z and P_ρ respectively, with the cuspidal edges of both of these circumscribing developables, then the ray curves of S_y with respect to the given conjugate system themselves form a conjugate system and the original conjugate system has equal Laplace-Darboux invariants. Moreover, each of these three properties implies the other two.

In general, i. e., whether $\partial^2 \log m/\partial u \partial v$ is zero or not, we can always find a unique curve of the family (29) which shall have simple contact with the cuspidal edge (30) at P_ρ, viz.;

$$C \equiv x_1^2 - 2nx_2 x_3 = 0. \tag{31}$$

Thus the conic C has second order contact with (28) at P_z and simple contact with (30) at P_ρ. Similarly the conic

$$C' \equiv x_1^2 - 2\frac{m_1}{m^2} x_2 x_3 = 0 \tag{32}$$

has second order contact with (30) at P_ρ and simple contact with (28) at P_z. We shall speak of the two conics in the tangent plane of P_y, which are defined in this way, as the *Darboux conics* of the point P_y, although they have been considered by Darboux only in the special case ($h = k$) in which they coincide. We may then state the previous theorem more briefly by saying that a conjugate system with equal invariants is one for which the Darboux conics of every surface point coincide.

Let us consider any line $x_3 = kx_2$ through the point P_y, and in the tangent plane. Such a line will intersect each of the Darboux conics in a pair of points. If we denote by λ the double ratio of these four points, without breaking up the pairs determined by each of the conics, we shall find

$$\left(\frac{1+\lambda}{1-\lambda}\right)^2 = \frac{\left(\dfrac{m_1}{m^2 n} + 1\right)^2}{4\dfrac{m_1}{m^2 n}}, \tag{33}$$

so that λ is independent of k. Clearly λ is an absolute invariant of the conjugate system formed by the curves $u =$ const. and $v =$ const. on S_y. If we prefer a different form of statement, we may say that λ is an absolute invariant of the congruences formed by the tangents of these curves, or of a single one of these congruences since each of them determines the other.

The quantity

$$\delta = \frac{c' d}{mn}$$

is also an absolute invariant of the given conjugate system. If we consider the congruence formed by the tangents of the curves $u =$ const. on S_y, the invariant δ may be defined as follows. To every line of the congruence, there belong two linear complexes (the Wälsch associated complexes of the line); δ is the double ratio of these complexes with respect to the two special linear complexes of their pencil.* We shall speak of δ as the *Wälsch invariant of*

* Brussels Paper, eq. (69).

the curves u = const. The Wälsch invariant of the curves v = const., is

$$\delta_1 = \frac{c_1' d_1}{m_1 n_1} = \frac{m d_1}{m_1 d}.$$

Let us suppose that the curves u = const. and v = const. form a conjugate system on S_y of the particular kind which Bianchi calls an isothermally conjugate system. We can see without much difficulty that this will be the case if and only if

$$\frac{\partial^2 \log d/m}{\partial u \partial v} = 0. \tag{34}$$

In fact, the surface S_y is determined except for projective transformations, by the simultaneous system of partial differential equations

$$\begin{aligned} m y_{uu} - d y_{vv} &= a m y + c m y_u + \left(b - d \frac{m_v}{m} \right) y_v, \\ y_{uv} &= m n y + \frac{m_u}{m} y_v, \end{aligned} \tag{35}$$

obtained from system (D) by differentiation and elimination of z and the partial derivatives of z. But from (35) it is evident that the conjugate system formed by the curves u = const. and v = const. will be isothermally conjugate, if and only if (34) is satisfied.

The relative invariant

$$W = mn - c' d$$

vanishes, if and only if the congruence composed of the tangents of the curves u = const. on S_y is a W congruence.* The invariant

$$W_1 = m_1 n_1 - c_1' d_1$$

has the same significance for the congruence composed of the tangents of the curves v = const. on S_y. We easily find

$$W_1 - W = \frac{\partial^2 \log d/m}{\partial u \partial v}, \tag{36}$$

a relation which gives rise to a theorem discovered recently by Demoulin and Tzitzéica. *If both of these congruences are W-congruences, the curves u = const. and v = const. form an isothermally conjugate system on S_y and these same properties are possessed by all of the conjugate systems and congruences obtained from the original ones by any number of Laplace transformations.* This result may also be stated as follows. *If the first Laplacian transform of a W-congruence is again a W-congruence, the same is true of all of its Laplacian transforms, and each of these congruences determines, by means of its developables, an isothermally conjugate system on each of its focal surfaces.* I may be per-

* Brussels Paper, eq. (59) and (84).

mitted to mention that I published a similar theorem about congruences belonging to linear complexes in my Brussels paper* several years before the announcement of the above theorem by Demoulin and Tzitzéica. Moreover, although all congruences which belong to linear complexes are W congruences, my theorem is not a special case of Demoulin's. It is of interest however to remark that there exist a number of similar theorems concerning properties of conjugate systems which are preserved by Laplacian transformations.

Let us return to the more general case of an isothermally conjugate system. We now see that the condition (34) for such a system may be written in the form $W_1 = W$, or

$$\frac{m_1}{m^2 n} - \frac{d_1}{dmn} = 1 - \delta,$$

or since $\delta_1 = md_1/m_1 d$, in the form

$$\frac{m_1}{m^2 n} = \frac{1 - \delta}{1 - \delta_1},$$

or finally, on account of (33), in the form

$$\left(\frac{1+\lambda}{1-\lambda}\right)^2 = \frac{\left(1 - \frac{\delta + \delta_1}{2}\right)^2}{(1-\delta)(1-\delta_1)}. \tag{37}$$

This equation contains an important result. *The condition that a system of curves be isothermally conjugate is equivalent to the simple algebraic relation* (37) *between the absolute invariants* λ, δ, *and* δ_1 *of the given conjugate system where, it should be recalled, each of these invariants has been completely interpreted from the point of view of projective geometry.*

The dual considerations may be carried out as follows. Let $s^{(1)}$, $s^{(2)}$, $s^{(3)}$, $s^{(4)}$ be the co-factors of x_1, x_2, x_3, x_4 in the determinant

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ y^{(1)} & y^{(2)} & y^{(3)} & y^{(4)} \\ y_u^{(1)} & y_u^{(2)} & y_u^{(3)} & y_u^{(4)} \\ y_v^{(1)} & y_v^{(2)} & y_v^{(3)} & y_v^{(4)} \end{vmatrix}$$

Then $s^{(1)}$, $s^{(2)}$, $s^{(3)}$, $s^{(4)}$ are the coördinates of the plane tangent to S_y at P_y, and any one of these four quantities may be represented by a symbol of the form

$$s = D(y, y_u, y_v) = mD(y, y_u, z),$$

where D stands for a determinant of the third order. Similarly

$$r = D(z, z_u, z_v) = nD(z, y, z_v)$$

* Brussels Paper, § 11.

represents the coördinates of the plane tangent to S_z at P_z. If we put

$$Y = \frac{r}{n} e^{-f}, \qquad Z = \frac{s}{m} e^{-f},$$

we find that Y and Z are solutions of the adjoint system of (D), viz.:*

$$(\overline{D}) \qquad \begin{aligned} Y_v &= MZ, \qquad Z_u = NY, \\ Y_{uu} &= A\ Y + B\ Z + C\ Y_u + D\ Z_v, \\ Z_{vv} &= A'\ Y + B'\ Z + C'\ Y_u + D'\ Z_v, \end{aligned}$$

where

$$M = c', \qquad N = d,$$

$$(38) \qquad \begin{aligned} A &= a - c_u, & B &= nd' - n_v, & C &= -c, & D &= n, \\ A' &= mc - m_u, & B' &= b' - d'_v, & C' &= m, & D' &= -d. \end{aligned}$$

The fundamental covariants of ($\overline{D}$) are Y, Z, and

$$(39) \qquad R = Y_u - \frac{M_u}{M} Y, \qquad \Sigma = Z_v - \frac{N_v}{N} Z,$$

and it is easy to see that R and Σ are the coördinates of the planes tangent to S_σ and S_ρ at P_σ and P_ρ respectively.

Any expression of the form

$$(40) \qquad u_1 Y + u_2 Z + u_3 R + u_4 \Sigma$$

will represent a plane. We choose a local system of reference, whose tetrahedron is composed of the four planes Y, Z, R, and Σ, in such a way that, with respect to it, the coördinates of the plane (40) shall be proportional to u_1, u_2, u_3, u_4. It may be noted that the tetrahedron Y, Z, R, Σ does not, in general, coincide with y, z, ρ, σ.

If now we let u remain constant while v changes, the plane Y, which is tangent to S_z at P_z, will generate a one-parameter family of planes which envelop a developable whose cuspidal edge is a curve $u = \text{const.}$ on S_y. Since we have

$$Y_v = MZ, \qquad Y_{vv} = \left(M_v + M \frac{N_v}{N} \right) Z + M\Sigma,$$

the parametric equations of this one-parameter family of planes will be

$$(41) \qquad \begin{aligned} u_1 &= 1 + \cdots, & u_2 &= M\delta v + \tfrac{1}{2} \left(M_v + M \frac{N_v}{N} \right) \delta v^2 + \cdots, \\ u_3 &= 0 + \cdots, & u_4 &= \tfrac{1}{2} M \delta v^2 + \cdots, \end{aligned}$$

where the expansion is complete up to and including terms of the second

* Brussels Paper, § 4.

order. The equation

$$(42) \qquad A_{22}(u_2^2 - 2Mu_1 u_4) + A_{44} u_4^2 + 2A_{24} u_2 u_4 = 0,$$

with arbitrary values for A_{22}, A_{44}, A_{24}, represents the most general quadric cone which has second order contact, along the generator through P_z, with the developable circumscribed about S_z along a curve u = const.

The only cone of this family which also has contact, at P_1, with the developable circumscribed about S_ρ along a curve v = const., is the cone

$$(43) \qquad u_2^2 - 2Mu_1 u_4 = 0,$$

whose vertex is the point P_y. In the same way we find the equation of a second cone

$$(44) \qquad u_2^2 - 2\frac{N_{-1}}{N^2} u_1 u_4 = 0,$$

whose vertex is also at P_y, and which has second order contact at P_ρ with the developable circumscribed about S_ρ along the corresponding curve v = const. and simple contact with the developable circumscribed about S_z along the corresponding curve u = const. These cones, whose equations may, on account of (38), be written in the forms

$$(45) \qquad u_2^2 - 2c' u_1 u_4 = 0, \qquad u_2^2 - 2\frac{d_1}{d^2} u_1 u_4 = 0$$

respectively, shall be called the *Darboux cones* of the point P_y. They coincide if and only if

$$(46) \qquad \frac{\partial^2 \log d}{\partial u \partial v} = 0,$$

i. e., if the axis curves form a conjugate net on S_y, or if the Laplace-Darboux invariants of the given conjugate system on S_y are equal when this system is referred to tangential coördinates. It is only this special case which Darboux has considered.

Any line through the point P_y and in the tangent plane, will determine a pair of planes tangent to each of the Darboux cones. Let μ be the cross-ratio of the two pairs of planes thus determined; we find

$$(47) \qquad \left(\frac{1+\mu}{1-\mu}\right)^2 = \frac{\left(1 + \dfrac{d_1}{c' d^2}\right)^2}{4\dfrac{d_1}{c' d^2}}.$$

In the particular case of an isothermally conjugate system we obtain the relation

$$(48)\qquad \left(\frac{1+\mu}{1-\mu}\right)^2 = \frac{\left(\delta\delta_1 - \frac{\delta+\delta_1}{2}\right)^2}{\delta\delta_1(1-\delta)(1-\delta_1)},$$

an equation which corresponds dualistically to (37).

There is, of course, no difficulty in carrying out corresponding considerations for the surface S_z. We shall confine ourselves to writing down the most important equations.

The differential equation of the ray curves of S_z is found to be

$$(49)\qquad -n_{-1}\,\delta u^2 - nJ\delta u\delta v + c'\,mn\delta v^2 = 0,$$

where

$$(50)\qquad n_{-1} = n\left(mn - \frac{\partial^2 \log n}{\partial u \partial v}\right),$$

and the foci of the ray $P_y\,P_\sigma$ of any point P_z of the surface S_z are given by the factors of the covariant

$$(51)\qquad c'\,n_{-1}\,y^2 + nJy\sigma - mn\sigma^2.$$

The axis curves and ray curves on S_y coincide if and only if

$$(52)\qquad d_1 = dmn, \qquad m_1 = c'\,dm.$$

If S_y is neither degenerate nor developable, these conditions are equivalent to

$$(53)\qquad W = mn - c'\,d = \frac{\partial^2 \log m}{\partial u \partial v} = -\frac{\partial^2 \log d}{\partial u \partial v}.$$

4. Some further loci connected with the congruence

It is easy to verify that the axes of P_y and P_z can not intersect. But the osculating planes of the curve $v =$ const. on S_y and of the corresponding curve $u =$ const. on S_z intersect in a line which we shall call the *joint axis* of the points P_y and P_z. It is the line which joins the points given by the expressions $\tau^{(y)}$ and $\tau^{(z)}$. The developables of the joint-axis congruence are obtained by integrating the differential equation

$$(54)\qquad nd_1\,J\delta u^2 + \left[c'\,dmn\left(f_u + \frac{c'_u}{c'} - \frac{m_u}{m}\right)\left(f_v + \frac{d_v}{d} - \frac{n_v}{n}\right) - c'_{-1}\,d_1 - mnIJ\right]\delta u\delta v + mc'_{-1}\,I\delta v^2 = 0,$$

an equation which may also be regarded as giving the joint-axis curves on the surface S_y.

Similarly, if we define the line joining P_ρ to P_σ as the *joint ray* of P_y, we find

$$(55)\quad mn_{-1}\,I\delta u^2 - \left[c'\,dmn\left(f_u + \frac{c'_u}{c'} - \frac{m_u}{m}\right)\left(f_v + \frac{d_v}{d} - \frac{n_v}{n}\right) - m_1\,n_{-1} - mnIJ\right]\delta u\delta v + m_1\,nJ\delta v^2 = 0$$

as the differential equation of the joint-ray curves on S_y.

The characteristics of the osculating planes of the curves v = const. for constant v and variable u, are of course the tangents of these curves, and the cuspidal edges of the corresponding developables are the curves v = const. themselves.

The one-parameter family composed of the osculating planes of the curves v = const. constructed at all of the points of a fixed curve u = const. on S_y, also fails to give rise to anything essentially new. For the characteristics of this family of planes are the tangents of the curves v = const. on S_ρ and their congruence is therefore one of those obtained from the original congruence by a Laplace transformation.

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