ON MULTIFORM SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS HAVING ELLIPTIC FUNCTION COEFFICIENTS*

BY

WILSON LEE MISER

1. Introduction

The researches of Hermite[1] on Lamé's equation called first attention to the class of linear differential equations having elliptic function coefficients. Having written the equation in the form used by Lamé,

$$\frac{d^2 x}{dt^2} = [\, n\,(n+1)\,k^2\, sn^2\, t + h\,]\, x,$$

where $sn\, t$ denotes the ordinary sine amplitude function of modulus k, n a positive integer, and h any constant, he showed that its fundamental set of solutions consisted of uniform doubly-periodic functions of the second kind. He defined a uniform function $F(t)$ as a doubly-periodic function of the second kind with the periods 2ω and $2\omega'$ in case it satisfied the two relations

$$F\,(t+2\omega) = \mu F\,(t), \qquad F\,(t+2\omega') = \mu' F\,(t),$$

μ and μ' being constants.

The investigation of the class of linear differential equations having elliptic function coefficients, but restricted to have only uniform solutions in the vicinity of all the poles, was systematically begun by Picard[2] who showed that, in general, the solutions of such equations are linear combinations of uniform doubly-periodic functions of the second kind. In a supplementary note to Picard's paper, Mittag-Leffler[3] pointed out the theorem, which is now known by Picard's name, to the effect that, in all cases in which all the solutions are uniform, there is at least one solution which is a uniform doubly-periodic function of the second kind. In making his assumption that the solutions shall all be uniform in the neighborhood of all the poles, he supposed that the poles of the coefficients of the differential equations were of such orders that they should all be singular points of determination for the solutions. Neces-

* Presented to the Society, March 21, 1913.

[1] The numbers refer to the bibliography at the end of this paper.

sary and sufficient conditions are known that all the solutions shall be uniform in the vicinity of such a singular point.

Since Picard many writers have treated the class of linear differential equations having elliptic function coefficients, some of them making studies of the solutions of certain particular equations. A list of their treatises and memoirs is given in the bibliography at the end of this paper. Among the most important of the memoirs are those by Floquet,[6] Stenberg,[12] Plemelj,[18] and Mercer.[20]

There are two directions in which Mercer in his paper makes his problem less restricted than that considered by all earlier writers. In the first place he does not limit himself to the case in which the coefficients are mere elliptic functions, but adopts the wider condition that the coefficients shall have for their singularities a reducible set of points. In the second place he assumes that the solutions are all uniform when considered as localized in a doubly-periodic region Φ which excludes a region Θ and its congruent regions obtained by shrinking the sides of a pseudo-parallelogram* of periods so as not to pass over any singularities of the coefficients.

The present investigation has to do only with a system of n linear homogeneous differential equations of the first order whose coefficients are elliptic functions having the common periods 2ω and $2\omega'$, and having only simple poles. The hypothesis that the coefficients are elliptic functions, rather than uniform doubly-periodic functions, which if non-elliptic have essential singularities in the finite portion of the plane, is made in order that, in each common parallelogram of periods, all the singular points of the solutions shall be isolated, and, therefore, finite in number. The elliptic functions are restricted to have only simple poles in order that, in the whole finite portion of the plane of the independent variable, all the singular points of the solutions shall be singular points of determination, and in order that use can, therefore, be made of the general theory of Fuchs respecting the nature of solutions of linear differential equations in the vicinity of such singular points.

Since n simultaneous differential equations of the first order include one differential equation of the nth order, the latter being always reducible to the former, the former is chosen for consideration. All the writers except Picard and Plemelj have used solely the latter form in their investigations. Picard, however, considered both forms and Plemelj the former, but, as has already been observed, they restricted all the solutions to be everywhere uniform.

After specifying in section 2 the explicit form of the differential equations and some results that follow immediately from the general theory, a hypothesis is made in section 3 as to the character of a solution called the Jth solution which the differential equations shall be supposed to possess, and as a

* Mercer's paper, section I, 5.

consequence of this hypothesis two equivalent forms for this Jth solution are found in section 4. By two different methods a lemma is proved in section 5 which is to the effect that a necessary and sufficient condition that the Jth solution shall return exactly to itself after its analytic continuation around a closed path encircling the singular points for a common parallelogram of periods, is that a certain sum N_J shall be an integer. Conditions for the sum N_J to be an integer are discussed in section 6. On adding the hypotheses that all the solutions of the fundamental set of solutions have the character of the Jth solution, and that all the sums $N_1, \cdots, N_n$ are integers, Picard's theorem for solutions all of which are uniform is extended in section 7 to a case of multiform solutions. In section 8 a determination is made of a type of differential equations which possess a certain solution consisting of doubly-periodic functions of the second kind.

2. The differential equations and a summary of the general theory

The system of differential equations to be considered is

$$\frac{dx_i}{dt} = \sum_{h=1}^{n} \psi_{ih}(t)\, x_h \qquad (i = 1, \cdots, n), \tag{1}$$

in which the coefficients $\psi_{ih}(t)$ are elliptic functions of the independent variable t, having the common periods 2ω and $2\omega'$, and having only simple poles.

Let P denote the pseudo-parallelogram comprising the region of the common parallelogram of periods with the vertices $A = 0$, $B = 2\omega$, $C = 2\omega + 2\omega'$, $D = 2\omega'$, from the perimeter of which common parallelogram $ABCD$ are excluded the vertices B, C, D and the sides BC, CD. Since the points in this pseudo-parallelogram P of finite area which are poles of the n^2 elliptic functions $\psi_{ih}(t)$ $(i, h = 1, \cdots, n)$, are isolated, they are finite in number. Let then $t_1, \cdots, t_m$ be all the points in P which are poles of the $\psi_{ih}(t)$, and let t_f denote the fth one of them.

The expansion of each $\psi_{ih}(t)$ in the vicinity of t_f has the form

$$\psi_{ih}(t) = \frac{c_{ih}^{(f)}}{t - t_f} + c_{ih} + \sum_{k=1}^{\infty} c_{ih}^{(kf)} (t - t_f)^k, \tag{2}$$

where $c_{ih}^{(f)}$ is zero, if t_f is only an ordinary point of $\psi_{ih}(t)$. Then the relation of the residues

$$\sum_{f=1}^{m} c_{ih}^{(f)} = 0 \tag{3}$$

holds for every i and h, since the sum of the residues of an elliptic function corresponding to its poles in P is zero. It follows by use of Hermite's formula* for the general expression of an elliptic function in terms of the Weierstrassian $\zeta(t)$ that each elliptic function $\psi_{ih}(t)$ can be written in the form

$$\psi_{ih}(t) = \sum_{f=1}^{m} c_{ih}^{(f)} \zeta(t - t_f) + c_{ih}.$$

The differential equations (1) have then the explicit form

$$(1') \qquad \frac{dx_i}{dt} = \sum_{h=1}^{n} \left[\sum_{f=1}^{m} c_{ih}^{(f)} \zeta(t - t_f) + c_{ih} \right] x_h \qquad (i = 1, \cdots, n),$$

which displays the constants c_{ih} and the residues $c_{ih}^{(1)}, \cdots, c_{ih}^{(m)}$ for each $\psi_{ih}(t)$ corresponding to the points $t_1, \cdots, t_m$ in P. The expressions for the Legendre-Jacobi elliptic functions $sn\,t$, $cn\,t$, and $dn\,t$ which have the common periods $4K$ and $4iK'$, and which have as simple poles in P the four points iK, $2K + iK'$, $3iK'$, and $2K + 3iK'$, in terms of the Weierstrassian function $\zeta(t)$, where $2\omega = 4K$ and $2\omega' = 4iK'$, are given here in order that a system of differential equations whose coefficients contain linear combinations of $sn\,t$, $cn\,t$, and $dn\,t$ can be converted into a system of the explicit form (1′). These expressions are found by Hermite's formula to be

$$sn\,t = \frac{1}{k}\left[\zeta\left(t - \frac{\omega'}{2}\right) - \zeta\left(t - \omega - \frac{\omega'}{2}\right) + \zeta\left(t - \frac{3\omega'}{2}\right) \right.$$
$$\left. - \zeta\left(t - \omega - \frac{3\omega'}{2}\right) - 2\eta \right],$$

$$cn\,t = \frac{i}{k}\left[- \zeta\left(t - \frac{\omega'}{2}\right) + \zeta\left(t - \omega - \frac{\omega'}{2}\right) + \zeta\left(t - \frac{3\omega'}{2}\right) \right.$$
$$\left. - \zeta\left(t - \omega - \frac{3\omega'}{2}\right) \right],$$

$$dn\,t = i\left[- \zeta\left(t - \frac{\omega'}{2}\right) - \zeta\left(t - \omega - \frac{\omega'}{2}\right) + \zeta\left(t - \frac{3\omega'}{2}\right) \right.$$
$$\left. + \zeta\left(t - \omega - \frac{3\omega'}{2}\right) + 2\eta' \right],$$

where k is the modulus, $\eta = \zeta(\omega)$, and $\eta' = \zeta(\omega')$.

The points $t_1, \cdots, t_m$ constitute the singular points in P for the solutions of the differential equations (1), and the points congruent to these points $t_1, \cdots, t_m$, modulo 2ω and $2\omega'$, constitute the singular points outside of P. Let any of the points in the finite portion of the t-plane which is congruent to

* É. Goursat, *Cours d'Analyse Mathématique*, vol. 2 (1911), p. 191.

t_f, modulo 2ω and $2\omega'$, be denoted by T_f; then T_f is any one of the points $t_{f+2\nu\omega+2\nu'\omega'}$ $(\nu, \nu' = 0, 1, 2, \cdots)$ in the finite portion of the t-plane, and $T_f = t_f$ when $\nu = \nu' = 0$. All the singular points T_f are singular points of determination. Infinity is an essential singularity.

By general theory there exists in the vicinity of every singular point of determination t_f in P at least one solution of the form

$$x_{ij}^{(f)} = (t - t_f)^{r_j^{(f)}} \sum_{k=0}^{\infty} a_{ij}^{(kf)} (t - t_f)^k \qquad (i = 1, \cdots, n), \tag{4}$$

where $r_j^{(f)}$ is a root of the indicial equation

$$d_f = \begin{vmatrix} c_{ih}^{(f)} - r^{(f)}\delta_{ih} \\ i, h = 1, \cdots, n \end{vmatrix} = 0 \quad (\delta_{hh} = 1;\ \delta_{ih} = 0,\ i \neq h),$$

associated with the singular point t_f. The roots $r_1^{(f)}, \cdots, r_n^{(f)}$ of this indicial equation $d_f = 0$ are all finite, since the coefficient of $r^{(f)n}$ is $(-1)^n$.

On account of the double periodicity of the $\psi_{ih}(t)$ the indicial equation for every singular point T_f congruent, modulo 2ω and $2\omega'$, to t_f in P is $d_f = 0$, the same indicial equation as for t_f. Hence it follows that there is in the vicinity of every singular point T_f a solution of the form

$$x_{ij}^{(f)} = (t - T_f)^{r_j^{(f)}} \sum_{k=0}^{\infty} a_{ij}^{(kf)} (t - T_f)^k \qquad (i = 1, \cdots, n), \tag{5}$$

where $r_j^{(f)}$ and the $a_{ij}^{(kf)}$ have the same values as in the form (4) for the solution in the vicinity of t_f in P, associated with the root $r_j^{(f)}$.

The expression for $x_{ij}^{(f)}$ can also be written

$$x_{ij}^{(f)} = (t - T_f)^{r_j^{(f)}} P_{ij}^{(f)} \qquad (i = 1, \cdots, n), \tag{6}$$

where

$$P_{ij}^{(f)} = \sum_{k=0}^{\infty} g_i^{(k)}(r_j^{(f)}) (t - T_f)^k,$$

the symbols $g_i^{(k)}(r_j^{(f)})$ denoting that the coefficients $g_i^{(k)}$ of the power series $P_{ij}^{(f)}$ are functions of $r_j^{(f)}$.

Two theorems from the general theory which are fundamental in this investigation are

THEOREM I: If the indicial determinant d_f has as elementary divisors $r^{(f)} - r_1^{(f)}, \cdots, r^{(f)} - r_n^{(f)}$, and if between $r_1^{(f)}, \cdots, r_n^{(f)}$ there is no difference which is an integer (different from zero), then the differential equations (1) possess a fundamental set of solutions of the form

$$x_{1j}^{(f)} = (t - T_f)^{r_1^{(f)}} P_{1j}^{(f)}, \cdots, x_{nj} = (t - T_f)^{r_n^{(f)}} P_{nj}^{(f)} \quad (j = 1, \cdots, n),$$

where the $P_{ij}^{(f)}$ are power series in $t - T_f$.

THEOREM II: If $r_1^{(f)}, r_2^{(f)}, r_3^{(f)}, \cdots$ are simple roots of the indicial equation $d_f = 0$, if the differences $r_1^{(f)} - r_2^{(f)}, r_2^{(f)} - r_3^{(f)}, \cdots$ are positive integers, and if there are no other roots that differ from them by integers, then there belong to the exponents $r_1^{(f)}, r_2^{(f)}, r_3^{(f)}, \cdots$ linearly independent solutions of the form

$$
\begin{aligned}
x_{i1}^{(f)} &= (t - T_f)^{r_1^{(f)}} P_{i1}^{(f)}, \\
x_{i2}^{(f)} &= (t - T_f)^{r_2^{(f)}} P_{i2}^{(f)} + (t - T_f)^{r_1^{(f)}} P_{i2}^{(1f)} \log (t - T_f) \quad (i = 1, \cdots, n), \\
x_{i3}^{(f)} &= (t - T_f)^{r_3^{(f)}} P_{i3}^{(f)} + (t - T_f)^{r_2^{(f)}} P_{i3}^{(1f)} \log (t - T_f) \\
&\qquad + (t - T)^{r_1^{(f)}} P_{i3}^{(2f)} \log^2 (t - T_f), \\
&\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

where all the P_{ij} are power series in $t - T_f$. In order that the solutions $x_{i1}^{(f)}, x_{i2}^{(f)}, x_{i3}^{(f)}, \cdots$ shall be free from logarithms the coefficients of the different powers of $\log (t - T_f)$,

$$
\begin{aligned}
P_{i2}^{(1f)} &= \sum_{k=0}^{\infty} g_i^{(k)} (r_2^{(f)}) (t - T_f)^k \qquad\qquad (i = 1, \cdots, n), \\
P_{i3}^{(1f)} &= \sum_{k=0}^{\infty} g_i^{(k)} (r_3^{(f)}) (t - T_f)^k, \qquad P_{i3}^{(2f)} = \sum_{k=0}^{\infty} g_i^{\prime(k)} (r_3^{(f)}) (t - T_f)^k, \\
&\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

where

$$
g_i^{\prime(k)} (r_3^{(f)}) = \left[\frac{\partial}{\partial r^{(f)}} g_i^{(k)} (r^f) \right]_{r^{(f)} = r_3^{(f)}},
$$

must be identically zero in $t - T_f$. These conditions simplify into the following necessary and sufficient conditions that the solutions $x_{i1}^{(f)}, x_{i2}^{(f)}, x_{i3}^{(f)}, \cdots$ shall be wholly free from logarithms:

$$
\begin{aligned}
&g_i^{(r_1 - r_2)} (r_2) = 0; \qquad\qquad (i = 1, \cdots, n), \\
&g_i^{(r_1 - r_2)} (r_3) = 0; \quad g_i^{\prime(r_2 - r_3)} (r_3) = 0, \quad g_i^{\prime(r_1 - r_3)} (r_3) = 0; \\
&\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

where, for simplicity in writing, $r_1 = r_1^{(f)}$, $r_2 = r_2^{(f)}$, $r_3 = r_3^{(f)}, \cdots$.

3. THE HYPOTHESIS ON THE JTH SOLUTION

It has been seen from the general theory that there exists in the vicinity of every singular point T_f at least one solution of the form

$$
x_{iJ}^{(f)} = (t - T_f)^{r_J^{(f)}} P_{iJ}^{(f)} \qquad (i = 1, \cdots, n), \tag{6}
$$

where $r_J^{(f)}$ is a root of the indicial equation $d_f = 0$, and where the $P_{iJ}^{(f)}$ are power series in $t - T_f$. The question naturally arises whether the expansions $x_{iJ}^{(f)}$ are elements, or branches, of a solution, which will be called the Jth solution, consisting of multiform monogenic functions $x_{iJ}(t)$ $(i = 1, \cdots, n)$. Some examples can be given where such is the case; but they are simple and seem to indicate that there exists a more general class of differential equations than the types of these examples, which possess one or more solutions having the character of the Jth solution. It has not so far been found possible to determine what are necessary and sufficient conditions on the coefficients of the system of differential equations (1), in order that they shall be such a class of differential equations. On account of this lack of knowledge as to the nature of necessary and sufficient conditions on the $\psi_{ih}(t)$ of (1), the hypothesis is made that the coefficients $\psi_{ih}(t)$ are such that the differential equations (1) shall possess a Jth solution consisting of multiform monogenic functions $x_{1J}(t), \cdots, x_{nJ}(t)$ whose expansions in the vicinity of every singular point T_f are of the form (6).

4. Two equivalent forms for the Jth solution

By the hypothesis of section 3 the Jth solution has in the vicinity of the singular point t_f the form (6) which has already in (5) been written more fully in the form

$$x_{iJ}^{(f)} = (t - t_f)^{r_J^{(f)}} \sum_{k=0}^{\infty} a_{iJ}^{(kf)} (t - t_f)^k \qquad (i = 1, \cdots, n), \tag{7}$$

where $r_J^{(f)}$ is a root of the indicial equation $d_f = 0$. It will now be shown that $x_{iJ}^{(f)}$ of (7) is uniquely expressible in the form

$$x_{iJ}^{(f)} = e^{\int \phi_J(t) dt} \sum_{k=0}^{\infty} b_{iJ}^{(kf)} (t - t_f)^k \qquad (i = 1, \cdots, n), \tag{8}$$

in which

$$\phi_J(t) = \sum_{f=1}^{m} r_J^{(f)} \zeta(t - t_f),$$

the $r_J^{(1)}, \cdots, r_J^{(m)}$ being the Jth roots of the indicial equations $d_1 = 0, \cdots, d_m = 0$, respectively, and the $\zeta(t - t_f)$ being the Weierstrassian ζ-function. In order to show this it is only necessary to show that, when the two forms are placed equal to each other, the a_{iJ}-coefficients of (7) are uniquely expressible in the b_{iJ}-coefficients of (8).

On placing the two forms (7) and (8) equal, there are obtained the identities in $t - t_f$,

$$(t - t_f)^{r_J^{(f)}} \sum_{k=0}^{\infty} a_{iJ}^{(kf)} (t - t_f)^k \equiv e^{\int \phi_J(t) dt} \sum_{k=0}^{\infty} b_{iJ}^{(kf)} (t - t_f)^k \qquad (i = 1, \cdots, n). \tag{9}$$

In the vicinity of the point t_f,

$$\phi_J(t) = \frac{r_J^{(f)}}{t - t_f} + c_{1J}^{(f)} + c_{2J}^{(f)}(t - t_f) + \cdots,$$

whence it follows that

$$(10) \qquad \int \phi_J(t)\,dt = r_J^{(f)} \log (t - t_f) + \log c_J^{(f)} + c_{1J}^{(f)}(t - t_f) + \cdots,$$

where $\log c_J^{(f)}$ is the constant of integration. On substituting (10) in (9) and simplifying, it is found that

$$(11) \qquad \sum_{k=0}^{\infty} a_{iJ}^{(kf)}(t - t_f)^k \equiv c_J^{(f)} e^{[c_{1J}^{(f)}(t-t_f)+\cdots]} \sum_{k=0}^{\infty} b_{iJ}^{(kf)}(t - t_f)^k \qquad (i = 1, \cdots, n),$$

which can be written

$$(12) \qquad \sum_{k=0}^{\infty} a_{iJ}^{(kf)}(t - t_f)^k \equiv c_J^{(f)}\left[1 + \sum_{k=1}^{\infty} g_{kJ}^{(f)}(t - t_f)^k\right] \sum_{k=0}^{\infty} b_{iJ}^{(kf)}(t - t_f)^k \qquad (i = 1, \cdots, n).$$

Since $c_J^{(f)}$ can be chosen to be unity, one obtains, by equating the coefficients of the terms in (12) independent of $t - t_f$,

$$(13) \qquad a_{1J}^{(0f)} = b_{1J}^{(0f)}, \qquad \cdots, \qquad a_{nJ}^{(0f)} = b_{nJ}^{(0f)},$$

and, by equating the coefficients of $(t - t_f)^k$,

$$(14) \qquad a_{iJ}^{(kf)} = b_{iJ}^{(kf)} + g_{kJ}^{(f)} b_{iJ}^{(0f)} + g_{k-1J}^{(f)} b_{iJ}^{(1f)} + \cdots + g_{1J}^{(f)} b_{iJ}^{(k-1f)} \qquad (i = 1, \cdots, n).$$

In these equations (13) and (14) the a_{iJ}-coefficients are expressed uniquely in terms of the b_{iJ}-coefficients, and *conversely*. Hence the two forms (7) and (8) are equivalent.

On account of the hypothesis made upon this solution $x_{1J}(t), \cdots, x_{nJ}(t)$, the power series

$$(15) \qquad \sum_{k=0}^{\infty} b_{1J}^{(kf)}(t - t_f)^k, \qquad \cdots, \qquad \sum_{k=0}^{\infty} b_{nJ}^{(kf)}(t - t_f)^k \quad (f = 1, \cdots, m)$$

are elements, or branches, of monogenic functions, say $\xi_{1J}(t), \cdots, \xi_{nJ}(t)$. Since the power series (15) are finite everywhere in the finite portion of the t-plane, the functions $\xi_{1J}(t), \cdots, \xi_{nJ}(t)$ are finite, and hence holomorphic in the whole finite portion of the t-plane. Now since

$$\int \zeta(t - t_f)\,dt = \log \sigma(t - t_f) + \log B^{(f)},$$

$B^{(f)}$ being an arbitrary constant of integration, it follows that

$$\int \phi_J(t)\,dt = \int \sum_{f=1}^{m} r_J^{(f)} \zeta(t-t_f)\,dt = \sum_{f=1}^{m} \{\log[\sigma(t-t_f)]^{r_J^{(f)}} + \log B_J^{(f)}\},$$

and, therefore, that

$$e^{\int \phi_J(t)dt} = B_J \prod_{f=1}^{m} [\sigma(t-t_f)]^{r_J^{(f)}},$$

where $B_J = \prod_{f=1}^{m} B_J^{(f)}$. This arbitrary constant B_J can be chosen to be unity, since the solution $x_{1J}(t), \cdots, x_{nJ}(t)$ carries an arbitrary constant as a multiplier. Therefore the

LEMMA. If the differential equations (1) are such that they possess a solution consisting of multiform monogenic functions $x_{1J}(t), \cdots, x_{nJ}(t)$ whose expansions at the singular points $t_1, \cdots, t_m$ in the pseudo-parallelogram P are of the form (6),

$$x_{iJ}^{(f)} = (t-t_f)^{r_J^{(f)}} P_{iJ}^{(f)} \qquad (i = 1, \cdots, n),$$

the $P_{iJ}^{(f)}$ being power series in $t - t_f$, this Jth solution has the form, namely,

$$x_{iJ}(t) = e^{\int \phi_J(t)dt} \xi_{iJ}(t) \qquad (i = 1, \cdots, n), \tag{16}$$

where

$$\phi_J(t) = \sum_{f=1}^{m} r_J^{(f)} \zeta(t-t_f),$$

and the equivalent form,

$$x_{iJ}(t) = \prod_{f=1}^{m} [\sigma(t-t_f)]^{r_J^{(f)}} \xi_{iJ}(t) \qquad (i = 1, \cdots, n), \tag{17}$$

the functions $\xi_{iJ}(t)$ being holomorphic in the whole finite part of the t-plane.

5. On the analytic continuation of the Jth solution

Consider the analytic continuation of the Jth solution (16) around a closed path L enclosing within its interior the singular points $t_1, \cdots, t_m$ of the pseudo-parallelogram P. Let an ordinary point Q on L be a starting point for the analytic continuation. Since the points $t_1, \cdots, t_m$ are isolated, the path L is reconcileable into successive loops, L_1 from Q around t_1 back to Q, and so on, L_m from Q around t_m back to Q. Each loop L_f is reconcileable into a circle C_f whose center is t_f, and whose radius is greater than zero and less than the absolute value of $t_h - t_f$ $(h = 1, \cdots, f-1, f+1, \cdots, m)$, and a path l_f, joining Q and C_f and containing no one of the points $t_1, \cdots, t_{f-1}, t_{f+1}, \cdots, t_m$. Since the Jth solution is left unchanged by analytic continuation from Q along each path l_f and back to Q, it is necessary only to consider the effect of taking the Jth solution around each circle C_f successively. From its expansion (6) in the vicinity of t_f,

$$x_{iJ}^{(f)} = (t - t_f)^{r_J^{(f)}} P_{iJ}^{(f)} \qquad (i = 1, \cdots, n),$$

it follows that, after its circuit around the circle C_f, the Jth solution is multiplied by the constant

$$e^{2\pi\sqrt{-1}r_J^{(f)}}.$$

This being true for the circuit around each circle C_f, the Jth solution, when continued analytically around all the circles $C_1, \cdots, C_m$, is multiplied by the constant $e^{2\pi\sqrt{-1}N_J}$, where

$$N_J = \sum_{f=1}^{m} r_J^{(f)}. \tag{18}$$

Since $e^{2\pi\sqrt{-1}N_J} = 1$ only when N_J is an integer, there results the

LEMMA. A necessary and sufficient condition that the Jth solution shall return exactly to itself after its analytic continuation around a path L enclosing wholly within its interior all the singular points $t_1, \cdots, t_m$ of the pseudo-parallelogram P is that the sum N_J shall be an integer.

Another proof of this lemma is as follows. Since

$$\sigma(t + 2\omega) = -e^{2\eta(t+\omega)}\sigma(t),$$
$$\sigma(t + 2\omega') = -e^{2\eta'(t+\omega')}\sigma(t),$$

it follows that, by substitution of $t + 2\omega$ and $t + 2\omega'$ for t in $x_{1J}(t), \cdots, x_{nJ}(t)$,

$$x_{iJ}(t+2\omega) = [-e^{2\eta(t+\omega)}]^{N_J}\prod_{f=1}^{m}[\sigma(t - t_f)]^{r_J^{(f)}}\xi_{iJ}(t + 2\omega) \qquad (i = 1, \cdots, n), \tag{19}$$

$$x_{iJ}(t + 2\omega') = [-e^{2\eta'(t+\omega')}]^{N_J}\prod_{f=1}^{m}[\sigma(t - t_f)]^{r_J^{(f)}}\xi_{iJ}(t + 2\omega'). \tag{20}$$

On substituting $t + 2\omega'$ for t in (19), and $t + 2\omega$ for t in (20), it follows also that

$$x_{iJ}(t + 2\omega' + 2\omega) = [-e^{2\eta(t+2\omega'+\omega)+2\eta'(t+\omega')}]^{N_J} \times\prod_{f=1}^{m}\sigma(t - t_f)]^{r_J^{(f)}}\xi_{iJ}(t + 2\omega' + 2\omega) \quad (i = 1, \cdots, n), \tag{21}$$

$$x_{iJ}(t + 2\omega + 2\omega') = [-e^{2\eta'(t+2\omega+\omega')+2\eta(t+\omega)}]^{N_J} \times\prod_{f=1}^{m}[\sigma(t - t_f)]^{r_J^{(f)}}\xi_{iJ}(t+2\omega+2\omega'). \tag{22}$$

Since the functions $\xi_{iJ}(t)$ are holomorphic, it is evident that

$$\xi_{iJ}(t + 2\omega' + 2\omega) = \xi_{iJ}(t + 2\omega + 2\omega') \quad (i = 1, \cdots, n).$$

In order that

$$x_{iJ}(t + 2\omega' + 2\omega) = x_{iJ}(t + 2\omega + 2\omega') \quad (i = 1, \cdots, n),$$

it is necessary in (21) and (22) to have

$$e^{4\eta\omega' N_J} = e^{4\eta'\omega N_J},$$

which can be written

$$e^{4(\eta\omega' - \eta'\omega) N_J} = 1.$$

By use of Legendre's formula* the relation $e^{2\pi\sqrt{-1}N_J} = 1$ is again obtained which holds only if the sum N_J is an integer.

6. On the sum N_J

Since it is not known what are necessary and sufficient conditions on the $\psi_{ih}(t)$ in order that the differential equations (1) shall possess a Jth solution of the form (16), viz.,

$$x_{iJ}(t) = e^{\int \phi_J(t)dt}\, \xi_{iJ}(t) \qquad (i = 1, \cdots, n),$$

where

$$\phi_J(t) = \sum_{f=1}^{m} r_J^{(f)} \zeta(t - t_f),$$

and where the $\xi_{iJ}(t)$ are holomorphic everywhere in the whole finite part of the t-plane, it is therefore not known what are necessary and sufficient conditions that the sum N_J shall be an integer. The following examples and discussion give, however, some information as to what are not sufficient conditions, and as to what are sufficient but not necessary conditions, in order that a sum N_J shall be an integer. Finally, an example 4 makes more explicit a difficulty which besets the investigation, viz., the difficulty of determining whether or not solutions in the vicinities of the singular points $t_1, \cdots, t_m$ in P of the form (5)

$$x_{iJ}^{(f)} = (t - t_f)^{r_J^{(f)}} \sum_{k=0}^{\infty} a_{iJ}^{(kf)} (t - t_f)^k \quad (i = 1, \cdots, n; f = 1, \cdots, m),$$

are elements, or branches, of a set of multiform monogenic functions $x_{1J}(t), \cdots, x_{nJ}(t)$ which have the expansions (5) in the vicinities of the points $t_1, \cdots, t_m$ in P.

The first example will show that the hypothesis of Theorem I, section 2 on a

* É. Goursat, *Cours d'Analyse Mathématique*, vol. 2 (1911), p. 187.

set of indicial determinants $d_1, \cdots, d_m$ are not sufficient to insure that a sum N_J shall be an integer.

Example 1. Take the indicial equations to be

$$d_1 = \begin{vmatrix} -1 - r^{(1)}, & 0 \\ -2, & \sqrt{-1} - r^{(1)} \end{vmatrix} = 0, \quad d_2 = \begin{vmatrix} 1 - r^{(2)}, & -\sqrt{-1} \\ 1, & -\sqrt{-1} - r^{(2)} \end{vmatrix} = 0,$$

$$d_3 = \begin{vmatrix} -r^{(3)}, & \sqrt{-1} \\ 1, & -r^{(3)} \end{vmatrix} = 0.$$

Their coefficients satisfy the relations (3), viz.,

$$\sum_{f=1}^{3} c_{ih}^{(f)} = 0 \qquad (i, h = 1, 2),$$

and their roots are -1, $\sqrt{-1}$; 0, $1 - \sqrt{-1}$; $\frac{1}{2}\sqrt{2}(1 + \sqrt{-1})$, $-\frac{1}{2}\sqrt{2}(1 + \sqrt{-1})$, respectively. The roots being simple, the elementary divisors of the determinants d_1, d_2, d_3 are simple. There is for each pair of roots no difference which is an integer. So the hypothesis of Theorem I, section 2 is fulfilled, but there is evidently no way to add the roots such that either sum N_1 or N_2 shall be an integer.

In case the hypothesis of Theorem II, section 2, is fulfilled for each one of the indicial equations $d_1, \cdots, d_m$, where all the roots $r_1^{(1)}, \cdots, r_n^{(1)}; \cdots; r_1^{(m)}, \cdots, r_n^{(m)}$ are integers, all the sums $N_1, \cdots, N_n$ are integers. Only in this case and under the further conditions that all the solutions are free from logarithms are all the solutions uniform. If not all the roots of the indicial equations $d_1, \cdots, d_m$ are integers, then a sum N_J may or may not be an integer, as is shown by the two following examples.

Example 2. Let the indicial equations be

$$d_1 = \begin{vmatrix} 1 - r^{(1)}, & 0 \\ -2, & -1 - r^{(1)} \end{vmatrix} = 0, \quad d_2 = \begin{vmatrix} 2 - r^{(2)}, & 2 \\ 1, & 3 - r^{(2)} \end{vmatrix} = 0,$$

$$d_3 = \begin{vmatrix} -3 - r^{(3)}, & -2 \\ 1, & -2 - r^{(3)} \end{vmatrix} = 0.$$

Their coefficients satisfy the relations

$$\sum_{f=1}^{3} c_{ih}^{(f)} = 0 \qquad (i, h = 1, 2),$$

and their roots are 1, -1; 1, 4; $\frac{1}{2}(-5 + \sqrt{-7})$, $\frac{1}{2}(-5 - \sqrt{-7})$. The roots are simple and the first two pairs differ by integers, but there is no way to add them such that either sum N_1 or N_2 shall be an integer.

Example 3. Consider the indicial equations

$$d_1 = \begin{vmatrix} -2-r^{(1)}, & 0 \\ -1+\sqrt{-1}, & -\sqrt{-1}-r^{(1)} \end{vmatrix} = 0, \quad d_2 = \begin{vmatrix} 1+\sqrt{-1}-r^{(2)}, & -1-\sqrt{-1} \\ 0, & -1-r^{(2)} \end{vmatrix} = 0,$$

$$d_3 = \begin{vmatrix} 1-r^{(3)}, & \sqrt{-1} \\ 1, & \sqrt{-1}-r^{(3)} \end{vmatrix} = 0, \qquad d_4 = \begin{vmatrix} -\sqrt{-1}-r^{(4)}, & 1 \\ -\sqrt{-1}, & 1-r^{(4)} \end{vmatrix} = 0.$$

Their coefficients satisfy the relations $\sum_{f=1}^{4} c_{ih}^{(f)} = 0 \ (i, h = 1, 2)$. Their roots are

$$r_1^{(1)} = -2, \quad r_2^{(1)} = -\sqrt{-1}; \qquad r_1^{(2)} = 1+\sqrt{-1}, \quad r_2^{(2)} = -1;$$

$$r_1^{(3)} = 0, \quad r_2^{(3)} = 1+\sqrt{-1}; \qquad r_1^{(4)} = 1-\sqrt{-1}, \quad r_2^{(4)} = 0,$$

respectively, each pair of which satisfies the hypothesis of Theorem I, section 2, and both of whose sums $N_1 = \sum_{f=1}^{4} r_1^{(f)}$ and $N_2 = \sum_{f=1}^{4} r_2^{(f)}$ are zero.

If the differential equations (1) possess a Jth solution of the form (16), a sufficient condition for the sum N_J to be zero is found as follows. On substituting the solution (16) in the differential equations (1) and using the expansions (4) of the $x_{tJ}(t)$ in the vicinity of t_f, there are obtained, by equating the coefficients of $(t-t_f)^{-1}$, the m sets of equations

$$\sum_{h=1}^{n} [c_{ih}^{(f)} - r_J^{(f)} \delta_{ih}] \, a_{hJ}^{(0f)} = 0 \qquad (i = 1, \cdots, n; f = 1; \cdots, m), \tag{23}$$

where $\delta_{hh} = 1$ and $\delta_{ih} = 0 \ (i \neq h)$. Suppose now that the $a_{hJ}^{(0f)}$ of (23) satisfy the conditions

$$a_{hJ}^{(0f)} = q_f \, a_{hJ}^{(01)} \quad (h = 1, \cdots, n; f = 1, \cdots, m), \tag{24}$$

where $q_1, \cdots, q_m$ are constants not zero. Then, after dividing out $q_1, \cdots, q_m$, the equations (23) are

$$\sum_{h=1}^{n} [c_{ih}^{(f)} - r_J^{(f)} \delta_{ih}] \, a_{hJ}^{(01)} = 0 \quad (i = 1, \cdots, n; f = 1, \cdots, m). \tag{25}$$

On summing the equations (25) with respect to f, they become, after interchanging the order of the sums,

$$\sum_{h=1}^{n} \left[\sum_{f=1}^{m} (c_{ih}^{(f)} - r_J^{(f)} \delta_{ih}) \right] a_{hJ}^{(01)} = 0 \qquad (i = 1, \cdots, n). \tag{26}$$

Every non-diagonal coefficient of the $a_{hJ}^{(01)}$ in (26) is zero, since $\sum_{f=1}^{m} c_{ih}^{(f)} = 0$

$(i, h = 1, \cdots, n)$, and since $\delta_{ih} = 0\ (i \neq h)$. Then, since $\delta_{hh} = 1$, the equations (26) become

$$\sum_{f=1}^{m} (c_{hh}^{(f)} - r_J^{(f)})\, a_{hJ}^{(01)} = 0 \qquad (h = 1, \cdots, n),$$

whence it follows that

$$\sum_{f=1}^{m} (c_{hh}^{(f)} - r_J^{(f)}) = 0 \qquad (h = 1, \cdots, n),$$

because at least one of the $a_{hJ}^{(01)}$ is distinct from zero. Since

$$\sum_{f=1}^{m} c_{hh}^{(f)} = 0 \qquad (h = 1, \cdots, n),$$

$$N_J = \sum_{f=1}^{m} r_{hJ}^{(f)} = 0.$$

Therefore the conditions (24) on the $a_{hJ}^{(0f)}$ imply that the sum N_J is zero.

That these conditions (24) are not necessary conditions for a sum N_j to be zero is shown by example 3, because the N_1 and N_2 are both zero, while the $a_{hj}^{(0f)}$, defined by the equations

$$\sum_{h=1}^{2} [\, c_{ih}^{(f)} - r_j^{(f)}\, \delta_{ih}\,]\, a_{hj}^{(0f)} = 0 \quad (i = 1, 2; f = 1, 2, 3, 4),$$

do not satisfy the conditions (24) for either $j = 1$ or $j = 2$. In example 4 it is found that all the sums N_j are zero, and that the $a_{hj}^{(0f)}$ satisfy the conditions (24) for $j = 1, \cdots, n$.

Example 4. Let the coefficients of the differential equations (1) have only two points, t_1 and t_2, in P as poles. The differential equations can then be written

$$\text{(27)} \qquad \frac{dx_i}{dt} = \sum_{h=1}^{n} \{c'_{ih}\, [\, \zeta\, (t - t_1) - \zeta\, (t - t_2)\,] + c_{ih}\} x_h \quad (i = 1, \cdots, n),$$

The indicial equations are

$$d_1 = \begin{vmatrix} c'_{ih} - r^{(1)}\, \delta_{ih} \\ i, h = 1, \cdots, n \end{vmatrix} = 0, \qquad d_2 = \begin{vmatrix} -\, c'_{ih} - r^{(2)}\, \delta_{ih} \\ i, h = 1, \cdots, n \end{vmatrix} = 0,$$

and their roots satisfy the relations $r_j^{(2)} = -\, r_j^{(1)}\ (j = 1, \cdots, n)$, whence

$$N_j = \sum_{f=1}^{2} r_j^{(f)} = 0 \qquad (j = 1, \cdots, n).$$

Suppose there exist in the vicinities of t_1 and t_2 fundamental sets of solutions of the form (4). Then the sets of equations (23), belonging to t_1 and t_2, are, respectively,

$$\sum_{h=1}^{n} [c'_{ih} - r_j^{(1)} \delta_{ih}] a_{hj}^{(01)} = 0 \qquad (i, j = 1, \cdots, n), \tag{28}$$

$$\sum_{h=1}^{n} [- c'_{ih} - r_j^{(2)} \delta_{ih}] a_{hj}^{(02)} = 0. \tag{29}$$

On replacing $r_j^{(2)}$ by $- r_j^{(1)}$ and changing signs throughout, the set of equations (29) becomes

$$\sum_{h=1}^{n} [c'_{ih} - r_j^{(1)} \delta_{ih}] a_{hj}^{(02)} = 0 \qquad (i, j = 1, \cdots, n). \tag{30}$$

From comparison of (28) and (30), it is evident that the $a_{hj}^{(01)}$ and the $a_{hj}^{(02)}$ satisfy the conditions

$$a_{hj}^{(0f)} = q_f \, a_{hj}^{(01)} \qquad (h, j = 1, \cdots n; f = 1, 2), \tag{31}$$

which are precisely the conditions (24) for $j = 1, \cdots, n$.

Even in this ideal case where the differential equations (1) have only two poles in P, where in the vicinity of the poles the fundamental sets of solutions $x_{ij}^{(1)}$ and $x_{ij}^{(2)}$ $(i, j = 1, \cdots, n)$ are wholly free from logarithms, and where the $a_{hj}^{(01)}$ and the $a_{hj}^{(02)}$ satisfy the conditions (24) for $j = 1, \cdots, n$, it is not known whether or not one of the solutions of the fundamental set $x_{ij}^{(1)}$ and one of the solutions of the fundamental set $x_{ij}^{(2)}$ are elements, or branches, of a set of multiform functions of the character supposed for the Jth solution in section 3.

7. Extension of Picard's theorem to a case of multiform solutions

Picard's theorem for the case in which all the solutions are everywhere uniform has been stated in the introduction, section 1. A corresponding theorem is true for the case of n multiform solutions each of which has the character of the Jth solution (16), and which constitute a fundamental set of solutions. The Theorems I and II, section 2 give necessary and sufficient conditions in order that there shall exist in the vicinity of every singular point T_f a fundamental set of solutions which are wholly free from logarithms. Just as in example 4, section 6, it is not at present known what are further necessary and sufficient conditions on the coefficients $\psi_{ih}(t)$ in order that the differential equations (1) shall possess a fundamental set of solutions having the character of the Jth solution (16). The theorem as it is proved is as follows:

Theorem: If the differential equations (1) possess a fundamental set of solutions which have the form

$$x_{ij}(t) = e^{\int \phi_j(t)\,dt} \, \xi_{ij}(t) \qquad (i, j = 1, \cdots, n), \tag{32}$$

where

$$\phi_j(t) = \sum_{f=1}^{m} r_j^{(f)} \zeta(t - t_f),$$

the sums $N_j = \sum_{f=1}^{m} r_j^{(f)} (j = 1, \cdots, n)$ being integers, and the functions $\xi_{ij}(t)$ being holomorphic in the whole finite portion of the t-plane, then there exists a solution

$$X_1(t), \quad \cdots, \quad X_n(t)$$

which consists of doubly-periodic multiform functions of the second kind, that is, which satisfy the relations

$$X_i(t + 2\omega) = \mu X_i(t) \qquad (i = 1, \cdots, n),$$
$$X_i(t + 2\omega') = \mu' X_i(t),$$

where μ and μ' are constants.

The method of proof by E. W. Barnes[15] of Picard's theorem in the case where all the solutions are everywhere uniform is adopted here.

The differential equations (1) being unchanged when $t + 2\omega$ is written for t, it follows from the fundamental set of solutions (32) that

(33) $$x_{ij}(t + 2\omega) \qquad (i, j = 1, \cdots, n)$$

is a fundamental set of solutions. The differential equations being unchanged when $t + 2\omega'$ is written for t, it follows that

(34) $$x_{ij}(t + 2\omega') \qquad (i, j = 1, \cdots, n)$$

is also a fundamental set of solutions. In the same way it follows that

(35) $$x_{ij}(t + 2\omega' + 2\omega) \quad \text{and} \quad x_{ij}(t + 2\omega + 2\omega') \quad (i, j = 1, \cdots, n)$$

are fundamental sets of solutions.

The fundamental sets of solutions (33) and (34) are, by general theory, expressible linearly in terms of the fundamental set (32) by relations of the form

(36) $$\begin{aligned} x_{ij}(t + 2\omega) &= \sum_{h=1}^{n} x_{ih}(t)\,\alpha_{hj} \qquad (i, j = 1, \cdots, n), \\ x_{ij}(t + 2\omega') &= \sum_{h=1}^{n} x_{ih}(t)\,\beta_{hj}, \end{aligned}$$

both determinants,

(37) $$\left| \begin{matrix} \alpha_{hj} \\ h, j = 1, \cdots, n \end{matrix} \right| \quad \text{and} \quad \left| \begin{matrix} \beta_{hj} \\ h, j = 1, \cdots, n \end{matrix} \right|,$$

being distinct from zero. When all the sums $N_1, \cdots, N_n$ are integers, it follows, by the Lemma of section 5, that

$$x_{ij}(t + 2\omega' + 2\omega) = x_{ij}(t + 2\omega + 2\omega') \qquad (i, j = 1, \cdots n),$$

and, therefore, that the two substitutions of (35) are commutative, whence

$$\sum_{h=1}^{n} \alpha_{ih}\, \beta_{hj} = \sum_{h=1}^{n} \beta_{ih}\, \alpha_{hj} \qquad (i, j = 1, \cdots, n), \tag{38}$$

Suppose now that

$$X_1(t), \quad \cdots, \quad X_n(t)$$

is a solution so chosen that

$$X_i(t + 2\omega) = \mu X_i(t) \qquad (i = 1, \cdots, n), \tag{39}$$

μ being a constant. Since

$$X_i(t) = \sum_{h=1}^{n} x_{ih}(t)\, \lambda_h \qquad (i = 1, \cdots, n), \tag{40}$$

it follows that

$$\sum_{j=1}^{n} x_{ij}(t + 2\omega)\, \lambda_j = \sum_{j=1}^{n} \sum_{h=1}^{n} x_{ih}(t)\, \alpha_{hj}\, \lambda_j = \mu \sum_{j=1}^{n} x_{ij}(t)\, \lambda_j \qquad (i = 1, \cdots, n), \tag{41}$$

whence the ratios of the constants $\lambda_1, \cdots, \lambda_n$ are determined by the equations

$$\sum_{j=1}^{n} \alpha_{ij}\, \lambda_j = \mu \lambda_i \qquad (i = 1, \cdots, n) \tag{42}$$

which are linear and homogeneous in the λ_i.

Take now

$$\rho_h = \sum_{j=1}^{n} \beta_{hj}\, \lambda_j \qquad (h = 1, \cdots, n); \tag{43}$$

then

$$\sum_{h=1}^{n} \alpha_{ih}\, \rho_h = \sum_{h=1}^{n} \sum_{j=1}^{n} \alpha_{ih}\, \beta_{hj}\, \lambda_j \qquad (i = 1, \cdots, n). \tag{44}$$

On applying in succession the equations (38), (42), and (43) in the right members of the equations (44), it results that

$$\begin{aligned} \sum_{h=1}^{n} \alpha_{ih}\, \rho_h &= \sum_{h=1}^{n} \sum_{j=1}^{n} \beta_{ih}\, \alpha_{hj}\, \lambda_j = \sum_{h=1}^{n} \beta_{ih}\, \mu \lambda_h \\ &= \mu \sum_{h=1}^{n} \beta_{ih}\, \lambda_h, \text{ for } \mu \text{ is a constant,} \\ &= \mu \rho_i \qquad (i = 1, \cdots, n). \end{aligned} \tag{45}$$

The comparison of the equations (42) in the λ_i,

$$\sum_{j=1}^{n} \alpha_{ij}\, \lambda_j = \mu \lambda_i \qquad (i = 1, \cdots, n)$$

with the equations (45) in the ρ_i,

$$\sum_{j=1}^{n} \alpha_{ij}\,\rho_j = \mu\rho_i \qquad (i = 1, \cdots, n),$$

shows that there exists some constant μ' such that

$$\rho_i = \mu'\,\lambda_i \qquad (i = 1, \cdots, n).$$

Hence, from equations (43),

$$\sum_{j=1}^{n} \beta_{ij}\,\lambda_j = \mu'\,\lambda_i \qquad (i = 1, \cdots, n).$$

Wherefore it follows that

$$X_i(t + 2\omega') = \mu'\,X_i(t) \qquad (i = 1, \cdots, n)$$

The proof of Picard's theorem is thus completed for a case of multiform solutions.

8. On the determination of a type of differential equations which possess a certain solution consisting of doubly-periodic functions of the second kind

In the problem of this section there is added the hypothesis that, in the Jth solution (16), viz.,

$$x_{iJ}(t) = e^{\int \phi_J(t)dt}\,\xi_{iJ}(t) \qquad (i = 1, \cdots, n),$$

where $\phi_J(t) = \sum_{f=1}^{m} r_J^{(f)}\,\zeta(t - t_f)$, the $\xi_{1J}(t), \cdots, \xi_{nJ}(t)$ are doubly-periodic functions of the second kind, that is, that they satisfy the relations

$$\xi_{iJ}(t + 2\omega) = \mu_J\,\xi_{iJ}(t) \qquad (i = 1, \cdots, n)$$

$$\xi_{iJ}(t + 2\omega') = \mu'_J\,\xi_{iJ}(t),$$

where μ_J and μ'_J are constants.

A general expression of a uniform doubly-periodic function* $F(t)$ of the second kind having the periods 2ω and $2\omega'$ and having the constant multipliers μ and μ', is

$$F(t) = ae^{st}\,\frac{H(t - t_0)}{H(t)}\,\psi(t),$$

a being an arbitrary constant, $H(t)$ the H-function of the Jacobi Θ-functions, $\psi(t)$ an elliptic function with the periods 2ω and $2\omega'$, and s and t_0 defined by the relations

$$s = \frac{1}{2\omega}\log\mu,$$

$$t_0 = \frac{1}{\pi\sqrt{-1}}\,(\omega\log\mu' - \omega'\log\mu).$$

* Appell et Lacour, *Théorie des Fonctions Elliptiques*, p. 328.

It has just as many zeros as poles. Since the $\xi_{iJ}(t)$ have no poles, they have then no zeros. The $F(t)$ which has no poles nor zeros reduces to

$$F(t) = ae^{st}.$$

Therefore, by the hypothesis on the $\xi_{iJ}(t)$, it follows that

$$\xi_{iJ}(t) = a_{iJ}\, e^{s_J t} \qquad (i = 1, \cdots, n),$$

where the a_{iJ} are constants, and where s_J satisfies the relation

$$s_J = \frac{1}{2\omega} \log \mu_J.$$

It will now be determined what the conditions on the coefficients of the differential equations (1′) are in order that they shall possess a Jth solution of the form

$$(46) \qquad x_{iJ}(t) = a_{iJ}\, e^{\int \phi_J(t)dt}\, e^{s_J t} \qquad (i = 1, \cdots, n).$$

On substituting this solution (46) in the differential equations (1′), expanding $\zeta(t - t_f)$ in powers of $t - t_f$, and then equating the coefficients of $(t - t_f)^{-1}$, all this being done for every f, the $m + 1$ sets of n linear homogeneous equations in the $a_{1J}, \cdots, a_{nJ}$,

$$(47) \qquad \begin{aligned} &\sum_{h=1}^{n} [c_{ih}^{(f)} - r_J^{(f)}\, \delta_{ih}]\, a_{hJ} = 0 \quad (i = 1, \cdots, n; f = 1, \cdots, m), \\ &\sum_{h=1}^{n} [c_{ih} - s_J\, \delta_{ih}]\, a_{hJ} = 0, \end{aligned}$$

are obtained.

Since, in each set of (47), the number of equations is equal to the number of a_{iJ}, a necessary and sufficient condition for solutions other than $a_{1J} = a_{2J} = \cdots = a_{nJ} = 0$ is that the determinants of the coefficients be zero. Therefore the following equations are obtained,

$$(48) \qquad \begin{aligned} d_{fJ} &= \begin{vmatrix} c_{ih}^{(f)} - r_J^{(f)}\, \delta_{ih} \\ i, h = 1, \cdots, n \end{vmatrix} = 0 \quad (f = 1, \cdots, m; f = 1, \cdots, n). \\ d_{0J} &= \begin{vmatrix} c_{ih} - s_J\, \delta_{ih} \\ i, h = 1, \cdots, n \end{vmatrix} = 0. \end{aligned}$$

In order that these $m + 1$ sets of equations (47) in $a_{1J}, \cdots, a_{nJ}$ shall have their solutions unique other than $a_{1J} = a_{2J} = \cdots = a_{nJ} = 0$, it is necessary that the rank of the determinants (48) be $n - 1$, and that the $m + 1$ sets of ratios of their corresponding n^2 first minors be equal. A necessary and sufficient condition in order that the rank of the $m + 1$ determinants (48) be $n-1$ is that for every f the indicial determinant d_f have as a simple elementary divisor $r^{(f)}-r_J^{(f)}$, and that the determinant d_0 have as a simple elementary divisor $s-s_J$.

By (17), another form of this Jth solution (46) is

$$x_{iJ}(t)=a_{iJ}\prod_{f=1}^{m}[\sigma(t-t_f)]^{r_J^{(f)}}e^{s_Jt}\qquad(i=1,\cdots,n). \tag{49}$$

On substituting $t+2\omega$ and $t+2\omega'$ for t in (49), it follows that

$$\begin{aligned} x_{iJ}(t+2\omega)&=[-e^{2\eta(t+\omega)}]^{N_J}e^{s_J\omega}x_{iJ}(t)\qquad(i=1,\cdots,n),\\ x_{iJ}(t+2\omega')&=[-e^{2\eta'(t+\omega)}]^{N_J}e^{2s_J\omega'}x_{iJ}(t). \end{aligned} \tag{50}$$

In order that $x_{1J}(t),\cdots,x_{nJ}(t)$ shall be doubly-periodic functions of the second kind, the sum $N_J=\sum_{f=1}^{m}r_J^{(f)}$ must be zero. That the N_J is zero follows directly from the equations (47) in which the a_{hJ} satisfy the conditions (24), which are sufficient for N_J to be zero. Hence the following

THEOREM: If the determinants $d_1,\cdots,d_m,d_0$ have as simple elementary divisors $r^{(1)}-r_J^{(1)},\cdots,r^{(m)}-r_J^{(m)},s-s_J$, respectively, and if the $m+1$ sets of ratios of the corresponding n^2 first minors of the determinants $d_{1J},\cdots,d_{mJ},d_{0J}$ are equal, then the system of differential equations (1′) possesses the solution

$$x_{iJ}(t)=e^{\int\phi_J(t)dt}e^{s_Jt}\qquad(i=1,\cdots,n),$$

where $\phi_J(t)=\sum_{f=1}^{m}r_J^{(f)}\zeta(t-t_f)$, which consists of multiform doubly-periodic functions of the second kind.

An illustration of the foregoing theorem is furnished by the differential equations

$$\begin{aligned} \frac{dx_1}{dt}&=[\psi_1(t)+7]x_1+[\psi_2(t)-9]x_2,\\ \frac{dx_2}{dt}&=[\psi_2(t)-4]x_1+[\psi_1(t)+7]x_2, \end{aligned}$$

where

$$\begin{aligned} \psi_1(t)&=\tfrac{1}{5}[\zeta(t-t_1)-\zeta(t-t_2)],\\ \psi_2(t)&=\tfrac{1}{9}[\zeta(t-t_1)-2\zeta(t-t_2)+\zeta(t-t_3)]. \end{aligned}$$

The indicial equations d_1,d_2,d_3 and the equation d_0 are

$$d_1=\begin{vmatrix}\frac{1}{5}-r^{(1)}, & \frac{1}{4}\\ \frac{1}{9}, & \frac{1}{5}-r^{(1)}\end{vmatrix}=0,\qquad d_2=\begin{vmatrix}-\frac{1}{5}-r^{(2)}, & -\frac{1}{2}\\ -\frac{2}{9}, & -\frac{1}{5}-r^{(2)}\end{vmatrix}=0,$$

$$d_3=\begin{vmatrix}-r^{(3)}, & \frac{1}{4}\\ \frac{1}{9}, & -r^{(3)}\end{vmatrix}=0,\qquad d_4=\begin{vmatrix}7-s, & -9\\ -4, & 7-s\end{vmatrix}=0,$$

whose roots are $r_1^{(1)} = \frac{1}{5} + \frac{1}{6}$, $r_2^{(1)} = \frac{1}{5} - \frac{1}{6}$; $r_1^{(2)} = -\frac{1}{5} - \frac{1}{8}$, $r_2^{(2)} = -\frac{1}{5} + \frac{1}{8}$; $r_1^{(3)} = \frac{1}{6}$, $r_2^{(3)} = -\frac{1}{6}$; $s_1 = 1$, $s_2 = 13$, respectively. Two solutions are

$$x_{11}(t) = 3a_1 e^{\int \phi_1(t)dt} e^t, \qquad x_{21}(t) = 2a_1 e^{\int \phi_1(t)dt} e^t,$$

$$x_{12}(t) = 3a_2 e^{\int \phi_2(t)dt} e^{13t}, \qquad x_{22}(t) = -2a_2 e^{\int \phi_2(t)dt} e^{13t},$$

where

$$\phi_1(t) = \psi_1(t) + \tfrac{3}{2}\psi_2(t),$$

$$\phi_2(t) = \psi_1(t) - \psi_{22}(t),$$

and they constitute a fundamental set. The $x_{ij}(t)$ are multiform doubly-periodic functions of the second kind, for they satisfy the relations

$$x_{ij}(t + 2\omega) = \mu_j x_{ij}(t) \qquad (i, j = 1, 2),$$

$$x_{ij}(t + 2\omega') = \mu_j' x_{ij}(t),$$

where $\mu_1 = e^{2\omega}$, $\mu_1' = e^{2\omega'}$, $\mu_2 = e^{26\omega}$, $\mu_2' = e^{26\omega'}$.

Bibliography

[1] Ch. Hermite: *Sur quelques applications des fonctions elliptiques*, Oeuvres de Ch. Hermite par É. Picard, vol. 3 (1912), pp. 267–421; Comptes Rendus, vols. 85 (1877), 86 (1878), 89 (1879), 90 (1881), 94 (1884).

[2] É. Picard: *Sur les équations différentielles linéaires à coefficients doublement périodiques*, Comptes Rendus, vol. 90 (1880), pp. 293–296, 1065–1067, 1118–1119; Journal für Mathematik, vol. 90 (1881), pp. 281–303.

[3] G. Mittag-Leffler: *Sur les équations différentielles linéaires à coefficients doublement périodiques*, Comptes Rendus, vol. 90 (1880), pp. 299–300.

[4] P. Appell: *Sur une classe d'équations différentielles à coefficients doublement périodiques*, Comptes Rendus, vol. 92 (1881), pp. 1005–1008.

[5] M. Elliot: *Sur une équation linéaire du second ordre à coefficients doublement périodiques*, Acta Mathematica, vol. 2 (1883), pp. 233–260.

[6] G. Floquet: *Sur les équations différentielles linéaires à coefficients doublement périodiques*, Comptes Rendus, vol. 98 (1884), pp. 82–85; Annales de l'École Normale, 3d series, vol. 1 (1884), pp. 181–238.

[7] E. Sjoblom: *Om de entydiga integralerna till en lineer homogen differential eqvation med dubbelperiodiska koefficienter*, Stockholm Öfversigt, vol. 41, no. 5 (1884), pp. 155–168; *Studier inom teorim for de lineera homogena differential eqvationer, hvilkaskkoefficienter aro dubbelperiodiska funktioner*, Helsingfors, (1884)*, 43 pages.

[8] Th. Craig: *A Treatise on Linear Differential Equations*, vol. 1 (1889), pp. 52–53, 496–514.

[9] C. Bigiavi: *Sopra una classe di equazioni differenziali lineari a coefficienti doppiamente periodici*, Pisa Annali, vol. 6 (1889), pp. 163–252; *Sulle equazioni differenziali lineari a coefficienti doppiamente periodici*, Accademia dei Lincei Rendiconti (4), vol. 6 (1890), pp. 339–346; *Sulla riducibilita della equazioni differenziali lineari a coefficienti doppiamente periodici*, Annali di matematica (3), vol. 5 (1901), pp. 107–139.

[10] F. Bremer: *Ueber lineare homogene Differentialgleichungen mit doppelt-periodischen Coefficienten*, Dissertation, Giessen, (1890), 30 pages.

[11] O. Venska: *Integration eines speciellen Systems linearer homogener Differentialgleichungen mit doppelt-periodischen Funktionen als Coefficienten*, Göttinger Nachrichten (1891), pp. 85–88.

[12] A. E. Stenberg: *Ueber die allgemeine Form der eindeutigen Integrale der linearen homogenen Differentialgleichungen mit doppelt-periodischen Coefficienten,* Acta Mathematica, vol. 15 (1891), pp. 259–278; *Zur Theorie der linearen homogenen Differentialgleichungen mit doppelt-periodischen Coefficienten,* Acta Societas Fennicae, vol. 19, no. 11 (1893), 7 pages.

[13] E. Naetsch: *Zur Theorie der homogenen linearen Differentialgleichungen mit doppelt-periodischen Coefficienten,* Dissertation, Leipzig, (1894), 66 pages; *Untersuchungen über die Reduction und Integration von Picard'schen Differentialgleichungen,* Leipziger Berichte, vol. 48 (1896), pp. 1–78.

[14] C. Jordan: *Cours d'Analyse,* vol. 3 (1896), pp. 276–299.

[15] E. W. Barnes: *A new proof of Picard's theorem,* Messenger of Mathematics, vol. 27 (1897), pp. 16–17.

[16] M. Malmberg: *Om integrationen af en klass af lineara differential-ekvationer med dubbelperiodiska koefficienter, analog de s. k. Hermet'ska differential-ekvationerna,* Upsala Universitets arsskrift, (1897)* and Dissertation, Upsala, 31 pages.

[17] L. Schlesinger: *Handbuch des linearen Differentialgleichungen,* vol. 2 (1898), pp. 403–424.

[18] J. Plemelj: *Über Systeme linearer Differentialgleichungen erster Ordnung mit doppelt-periodischen Koefficienten.* Monatshefte für Mathematik und Physik, vol. 12 (1901), pp. 203–218.

[19] A. R. Forsyth: Theory of Differential Equations, 3d part, vol. 4 (1904), pp. 441–477.

[20] J. Mercer: *On the solutions of ordinary linear differential equations having doubly-periodic coefficients,* Cambridge Transactions, vol. 20 (1908), pp. 383–436.

* Date of *Jahrbuch über die Fortschritte der Mathematik* in which a review appeared.