ON THE FOUNDATIONS OF PLANE ANALYSIS SITUS*

BY

ROBERT L. MOORE

1. Introduction

The present paper contains three systems of axioms, Σ_1, Σ_2, and Σ_3. Each of these systems is a sufficient basis for a considerable body of theorems in the domain of plane analysis situs or what may be roughly termed the non-metrical part of plane point-set theory, including the theory of plane curves. The axioms of each system are stated in terms of a class of elements called *points* and a class of point-sets called *regions*.

On the basis of Σ_1 *the existence of simple continuous curves is proved as a theorem* and it is shown that every region is the interior of a simple closed curve.

Σ_2 is equivalent† to Σ_1 as far as statements in terms of *point* and *limit point* are concerned. But Σ_2 is satisfied if in an ordinary euclidean space of two dimensions the term region is interpreted so as to apply to every bounded, connected set of points R of connected exterior such that every point of R lies in the interior of some triangle that is contained in R.

Both Σ_1 and Σ_2 contain an axiom (Axiom 1) which postulates the existence of a countable sequence of regions containing a set of subsequences that close down in a specified way on the points of space. Among other things this axiom implies that the set of all points is separable.‡

The set Σ_3 is obtained from Σ_2 by replacing Axioms 1 and 2 by two other axioms, Axioms 1′ and 2′. Here Axiom 1′ postulates the existence *for each point* P of a countable sequence of regions that closes down on P, while Axiom 2′ postulates that every two points of a region are the extremities of at least one arc lying in that region. Σ_2 implies Σ_3 but not conversely.

Though Theorems 1–52 of the present paper are all consequences of Σ_3 nevertheless there exists a space that satisfies Σ_3 but is neither metrical, descriptive§ nor separable. It is interesting that no space that satisfies Σ_3

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† See § 9.

‡ For a definition of the term separable see M. Fréchet, *Sur quelques points du calcul fonctionnel*, Rendiconti del Circolo Matematico di Palermo, vol. 22 (1906), p. 23.

§ A space S is said to be *descriptive* or *potentially descriptive* if it contains a system of open curves (as defined in § 8) such that through every two points of S there is one and only one curve of this system. I have not determined whether every space satisfying Σ_1 is potentially descriptive.

can be potentially descriptive without being separable and, indeed, *metrical* in the sense that it is in a one to one continuous correspondence with an ordinary euclidean space of two dimensions.

2. Axioms and definitions

I consider a class, S, of elements called *points* and a class of sub-classes of S called *regions*, subject to a set of postulates (axioms) as described below. Before stating these axioms I will define certain terms that will be used.

Definitions. A point P is said to be a *limit point* of a point-set M if, and only if, every region that contains P contains at least one point of M distinct from P. The *boundary* of a point-set M is the set of all points $[X]$ such that every region that contains X contains at least one point of M and at least one point that does not belong to M. If M is a point-set, M' denotes the set of points composed of M plus its boundary. If R is a region, the point-set $S - R'$ is called the *exterior* of R. A point in the exterior of R is said to be *without* R.

A set of points is said to be *connected* if, however it be divided into two mutually exclusive subsets, one of them contains a limit point of the other one.

A set of regions K is said to *cover* a point-set M if every point of M belongs to at least one region of the set K. If for every infinite set of regions K covering the point-set M there exists a finite subset of K that also covers M then M is said to *possess the Heine-Borel property*.

Axiom 1.* *There exists an infinite sequence of regions,* $K_1, K_2, K_3, \cdots$ *such that* (1) *if* m *is an integer and* P *is a point, there exists an integer* n, *greater than* m, *such that* K_n *contains* P, (2) *if* P *and* $\overline{P}$ *are distinct points of a region* R *then there exists an integer* δ *such that if* $n > \delta$ *and* K_n *contains* P *then* K'_n *is a subset of* $R - \overline{P}$.

Axiom 2. *Every region is a connected set of points.*

Axiom 3. *If* R *is a region,* $S - R'$ *is a connected set of points.*

Axiom 4. *If* R *is a region,* R' *possesses the Heine-Borel property.*

Axiom 5. *There exists an infinite set of points that has no limit point.*

Axiom 6. *If* R *and* $\overline{R}$ *are regions and* P *is a point in* $\overline{R}$ *and on the boundary of* R, *then there exist in* $\overline{R}$ *two regions* K *and* $\overline{K}$ *such that* $\overline{K}$ *contains* P, K *lies in* R *and all those points of the boundary of* R *that lie in* $\overline{K}$ *are points also of the boundary of* K.

Axiom 7. *If* R *and* $\overline{R}$ *are regions and* P *is a point in* $\overline{R}$ *and on the boundary of* R, *then there exist in* $\overline{R}$ *two regions* L *and* $\overline{L}$ *such that* $\overline{L}$ *contains* P, L *lies*

* There is a certain amount of resemblance between Axiom 1 and Veblen's Postulate of Uniformity. Cf. O. Veblen, *Definition in terms of order alone in the linear continuum and in well-ordered sets*, these Transactions, vol. 6 (1905), p. 169.

in $S - R'$ and all those points of the boundary of R that lie in $\bar{L}$ are points also of the boundary of L.

AXIOM 8. *Every simple closed curve* is the boundary of at least one region.*

3. CONSEQUENCES OF AXIOMS 1–3, 5_1†

THEOREM 1. *No point of a region is a boundary point of that region.*

THEOREM 2. *If P is a limit point of the point-set M then every region that contains P contains infinitely many points of M.*

Proof. Suppose the region R contains P. By hypothesis R contains at least one point P_1 which belongs to M and is distinct from P. By Axiom 1 there exists in R a region R_1 containing P but not containing P_1. The region R_1 contains at least one point P_2 belonging to M and distinct from P. This process may be continued. It follows that R and M have in common an infinite sequence of distinct points, $P_1, P_2, P_3, \cdots$.

THEOREM 3. *If P is a point and M is a finite set of points not containing P then there exists a region containing P but containing no point of M.*

THEOREM 4. *Every region contains infinitely many points.*

Proof. Suppose R is a region. There exists in R at least one point O. By Axiom 5_1 there exist two points P_1 and P_2 distinct from each other and from O. By Theorem 3 there exists, about‡ P_1, a region R_1 containing neither O nor P_2. Suppose R_1' contains O or P_2. Then either O or P_2 is a limit point of R_1 and therefore, by Axiom 1 and Theorem 2, R_1 contains a point Y distinct from P_1. Hence, by Axiom 1, there exists, about P_1, a region $\bar{R}_1$ such that $\bar{R}_1'$ is a subset of R_1. It follows that $\bar{R}_1'$ contains neither O nor P_2.

Let K denote $\bar{R}_1$ or R_1 according as R_1' does or does not contain one of the points O and P_2. Then $S - K' = O + M$ where M is a point-set which does not contain O. Hence, by Axiom 3 and Theorem 3, O is a limit point of M. Hence R contains infinitely many points of M.

THEOREM 5. *If P is a point then there exists an infinite sequence of regions $R_1, R_2, R_3, \cdots$ such that* (1) *P is the only point they have in common,* (2) *for every n, R_{n+1} is a proper subset of R_n,* (3) *if R is a region about P then there exists n such that R_n' is a subset of R.*

Proof. Let R_1 denote the first fundamental region§ that contains P. Let R_2 denote the first fundamental region that follows R_1 in the fundamental

* For definition of simple closed curve see § 4.

† Here 5_1 denotes the axiom that there exist at least three points. Of course this proposition is a part of Axiom 5.

‡ In this connection "about" is synonymous with "containing."

§ Select once for all a *definite* sequence $K_1, K_2, K_3, \cdots$ satisfying the conditions stated in Axiom 1. This definite sequence will be called *the fundamental sequence* and its regions will be termed *fundamental regions*. Hereafter in this paper K_n denotes the nth region in the fundamental sequence.

sequence, contains P and is a proper subset of R_1. In general let R_{n+1} denote the first fundamental region that follows R_n, contains P and is a proper subset of R_n. It is clear that the sequence $R_1, R_2, R_3, \cdots$, thus defined, satisfies conditions (1), (2), and (3).

THEOREM 6. *If two regions have a point in common, then they have in common at least one region containing that point.*

Proof. Suppose the regions K and L have in common the point P. Let $R_1, R_2, R_3, \cdots$ denote a sequence of regions satisfying, with respect to the point P, conditions (1), (2), and (3) of Theorem 5. There exist positive integers m and n such that R_m is a subset of K and R_n is a subset of L. The region R_{m+n} is a common subset of K and L.

THEOREM 7. *If P is a limit point of $M + N$ then it is a limit point either of M or of N.*

DEFINITION. The point P is said to be a *sequential limit point* of the sequence of points $P_1, P_2, P_3, \cdots$ if and only if for every region R containing P there exists an integer δ such that if $n > \delta$ then P_n lies in R.

THEOREM 8. *If P is a sequential limit point of a sequence of points, $P_1, P_2, P_3, \cdots$, then no other point is a limit point of the point-set*

$$P_1 + P_2 + P_3 + \cdots.$$

Proof. Suppose $P_1 + P_2 + P_3 + \cdots$ has a limit point X distinct from P. By Theorem 3 there exists a region R containing P but not containing X. By Theorem 5 there exists a region K, containing P, such that K' is a subset of R. There exists δ such that if $n > \delta$ then P_n lies in K. But X does not belong to K'. Hence it is not a limit point of $P_{\delta+1} + P_{\delta+2} + P_{\delta+3} + \cdots$. By Theorem 3 it is not a limit point of $P_1 + P_2 + P_3 + \cdots + P_n$. Hence, by Theorem 7, it is not a limit point of $P_1 + P_2 + P_3 + \cdots$.

THEOREM 9. *If P is a limit point of M then there exists an infinite sequence of points belonging to M and all distinct from P such that P is the sequential limit point of this sequence.*

Proof. Let $R_1, R_2, R_3, \cdots$ denote an infinite sequence of regions satisfying, with respect to P, conditions (1), (2), and (3) of Theorem 5. For every n, R_n contains at least one point of M distinct from P. With the help of Zermelo's postulate it follows that there exists an infinite sequence of points $P_1, P_2, P_3, \cdots$ such that P_n belongs to R_n and to M and is distinct from P. It is clear that P is the sequential limit point of this sequence.

DEFINITION. If A and B are distinct points, then a *simple chain from A to B* is a finite sequence of regions $R_1, R_2, R_3, \cdots R_n$ such that (1) R_i contains A if and only if $i = 1$, (2) R_i contains B if and only if $i = n$, (3) if $1 \leqq i \leqq n$, $1 \leqq j \leqq n$, $i < j$ then R_i has a point in common with R_j if and only if

$$j = i + 1.$$

The region R_k $(1 \leqq k \leqq n)$ is said to be the kth *link* of the chain

$$R_1 R_2 R_3 \cdots R_n.$$

THEOREM 10. *If M is a connected set of points, A and B are two distinct points of M and G is a set of regions covering M then there exists a simple chain from A to B every link of which is a region of the set G.*

Proof. If there is no such chain from A to B then the points of M can be divided into two classes S_A and S_B, where S_A is the set of all points $[P]$ such that P can be joined to A by a simple chain every link of which is a region of the set G, and S_B is the set of all other points of M. Since M is connected, one of the sets S_A and S_B must contain a point X which is a limit point of the other one. The point X lies in at least one region R of the set G. The region R contains a point A_1 belonging to S_A and a point B_1 belonging to S_B, where A_1 is X or B_1 is X according as X belongs to S_A or to S_B. The point A can be joined to A_1 by a simple chain $R_1 R_2 R_3 \cdots R_n$ every link of which is a region of the set G. Let R_k be the first link of this chain that intersects* R. Then $R_1 R_2 R_3 \cdots R_k R$ is a simple chain from A to B_1, every link of which belongs to G. Thus the supposition that Theorem 10 is false leads to a contradiction.

DEFINITION. A set of points is said to be *closed* if it contains all its limit points. A set of points M is said to be *bounded* if there exists a finite set of regions, $R_1, R_2, R_3, \cdots R_n$ such that M is a subset of

$$(R_1 + R_2 + R_3 + \cdots + R_n)'.$$

It is clear that if M is a set of points then M' is closed.

4. CONSEQUENCES OF AXIOMS 1–4, 5

THEOREM 11. *If $R_1, R_2, R_3, \cdots R_n$ is a finite set of regions, the point-set $(R_1 + R_2 + R_3 + \cdots + R_n)'$ possesses the Heine-Borel property.*

THEOREM 12. *Every closed, bounded set of points possesses the Heine-Borel property.*

Proof. Suppose that $R_1, R_2, R_3, \cdots R_n$ is a finite set of regions and that M is a closed point-set lying in $(R_1 + R_2 + R_3 + \cdots + R_n)'$ and covered by a set of regions G. Let H denote the point-set $R_1 + R_2 + R_3 + \cdots + R_n$. If every point of H' belongs to M then, by Theorem 11, M has the Heine-Borel property. Suppose that not every point of H' belongs to M. Since M is closed therefore about each point of $H' - M$ there is a region containing no point of M. Thus there exists a set of regions $\bar{G}$ covering $H' - M$ but such that no point of M belongs to any region of $\bar{G}$. By Theorem 11 there

* Two point-sets are said to intersect each other if they have at least one point in common.

exists a finite set of regions G_1 covering H and such that each region of G_1 belongs either to G or to $\overline{G}$. Let G_2 denote the set of all those regions of G_1 that intersect M. No region of G_2 can belong to $\overline{G}$. Hence every region of G_2 belongs to G. Moreover G_2 covers M.

THEOREM 13. *Every infinite, bounded set of points has at least one limit point.*

For a proof of Theorem 13 see F. Hausdorff, *Grundzüge der Mengenlehre,** page 231.

THEOREM 14. *If M_1, M_2, M_3, $\cdots$ is an infinite sequence of bounded point-sets such that, for every n, M_n contains M'_{n+1}, then the point-sets M_1, M_2, M_3, $\cdots$ have at least one point in common and the set, G, of all such common points is closed.*

Proof. For each n, M_n contains at least one point. Hence, by Zermelo's postulate, there exists a sequence of points P_1, P_2, P_3, $\cdots$ such that, for each n, P_n lies in M_n. If there exists a positive integer $\bar{n}$ such that, for every n greater than $\bar{n}$, $P_n = P_{\bar{n}}$, then $P_{\bar{n}}$ is common to all the point-sets M_1, M_2, M_3, $\cdots$. If no such $\bar{n}$ exists then the sequence P_1, P_2, P_3, $\cdots$ contains infinitely many distinct points and therefore, by Theorem 13, there exists a point P which is a limit point of $P_1 + P_2 + P_3 + \cdots$. Since P is a limit point of $(P_1 + P_2 + P_3 + \cdots + P_n) + (P_{n+1} + \cdots)$, therefore, by Theorems 7 and 3, it is a limit point of $P_{n+1} + \cdots$. But $P_{n+1} + \cdots$ is a subset of M_{n+1}, and M'_{n+1} is a subset of M_n. Therefore every M_n contains P. Hence the set G exists.

Suppose that O is a limit point of G. Then, for every n, O belongs to M'_{n+1} and therefore to M_n. Hence O belongs to G. Thus G is closed.

DEFINITION. A *domain* is a connected set of points M such that if P is a point of M then there exists a region that contains P and lies in M.

DEFINITION.† If A and B are two distinct points, a *simple continuous arc* from A to B is a bounded, closed, connected set of points containing A and B but containing no connected proper subset that contains both A and B.

THEOREM 15. *If A and B are distinct points of a domain M, there exists a simple continuous arc from A to B that lies wholly in M.*

Proof. If P is a point of M, P lies in a region R which is contained in M.

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* Veit & Co., Leipzig, 1914. See also E. W. Chittenden, *The converse of the Heine-Borel theorem in a Riesz domain*, Bulletin of the American Mathematical Society, vol. 21 (1915), pp. 179–183, and vol. 20 (1914), p. 461. For a proof that in the presence of certain linear order postulates the Heine-Borel Theorem is equivalent to the Dedekind-cut Postulate see O. Veblen, *The Heine-Borel Theorem*, Bulletin of the American Mathematical Society, vol. 10 (1903–04) pp. 436–439.

† See N. J. Lennes, *Curves in non-metrical analysis situs with an application in the calculus of variations*, American Journal of Mathematics, vol. 33 (1911), page 308, and Bulletin of the American Mathematical Society, vol. 12 (1906), p. 284.

There exists n_{1P} greater than or equal to 1 such that $K_{n_{1P}}$ contains P and such that if $n \geqq n_{1P}$ and K_n contains P then K'_n is contained in R and therefore in M. Let G_1 be the set of all such $K_{n_{1P}}$'s for all P's in M. By Theorem 10 there exists a simple chain $R_{11} R_{12} \cdots R_{1m_1}$, from A to B, every link of which belongs to G_1. Call this chain C_1. For each $i (1 \leqq i < m_1)$ select a point P_{1i} common to R_{1i} and R_{1i+1}. Let $P_{10} = A$ and $P_{1m_1} = B$. If $0 \leqq i < m_1$ then P_{1i} can be joined to P_{1i+1} by a simple chain C_{1i+1} each link of which is a region of the sequence $K_2, K_3, \cdots$ and lies, with all its limit points, in R_{1i+1}. If for any $i (1 \leqq i \leqq m_1)$ any link except the last one of the chain C_{1i} intersects any link of C_{1i+1} then omit from C_{1i} every link that follows $\bar{R}_{1i}$ where $\bar{R}_{1i}$ is the first link of C_{1i} that intersects a link of C_{1i+1}; also omit from C_{1i+1} every link (if there be any such) that precedes the last link that intersects $\bar{R}_{1i}$. These omissions having been made for each i concerned, the remaining links of the chains $C_{11}, C_{12}, \cdots C_{1m_1}$ form a simple chain C_2 from A to B. The chain C_2 has the important property that each one of its links lies wholly in some single link of the chain C_1 and if a link x of C_2 lies in a link y of C_1 then every link that follows x in C_2 lies either in y or in some link that follows y in C_1. Similarly there exists a chain C_3 having a relation to C_2 analogous to the above indicated relation of C_2 to C_1 and such that every link of C_3 is a region of the sequence $K_3, K_4, \cdots$. This process may be continued. Thus there exists an infinite sequence of chains $C_1, C_2, C_3 \cdots$ such that (1) each link of the chain C_{n+1} lies, together with all its limit points, wholly in some single link of C_n; (2) if a link x of C_{n+1} lies in a link y of C_n then each link that follows x in C_{n+1} lies either in y or in some link that follows y in C_n; (3) every link of C_n is a region of the sequence $K_n, K_{n+1}, K_{n+2}, \cdots$.

Let $\bar{C}_n$ denote the point-set which is the sum of all the links of the chain C_n. Let C denote the set of all those points that the sets $\bar{C}_1, \bar{C}_2, \bar{C}_3, \cdots$ have in common. It will be proved that C satisfies Lennes' definition of a Jordan arc from A to B.

I. To prove that C is closed. Every limit point of a single link of C_{n+1} lies in some link of C_n. Hence, by Theorem 7, $\bar{C}'_{n+1}$ is a subset of $\bar{C}_n$. It follows by Theorem 14 that C is closed.

II. To prove that C is connected. Suppose that C can be divided into two mutually exclusive subsets S_1 and S_2 neither of which contains a limit point of the other one. Since C is closed every limit point of S_1 belongs to $S_1 + S_2$ and therefore must belong to S_1. Hence S_1 is closed. Likewise S_2 is closed. About each point P of S_1 there is a region R containing no point of S_2. There exists $\bar{n}$ such that $K_{\bar{n}}$ contains P while $K'_{\bar{n}}$ lies in R. Hence, by Theorem 10, there exists a finite set G_1 of regions such that every point of S_1 belongs to some region of G_1 but no point of S_2 belongs to, or is a limit point of, any region of G_1. Since the set G_1 is finite it follows, by Theorem 7, that no point of S_2 is a limit point of $\bar{G}_1$ (the point-set which is the sum of all the regions of the set G_1). About each point of S_2 there is a region such that no point of $\bar{G}_1'$ is a point or a limit point of that region. Hence there exists a finite set G_2 of regions, covering S_2 and such that no point of $\bar{G}_1'$ is a point or a limit point of any region of G_2. Let $\bar{G}_2$ denote the point-set which is the sum of all the regions of the set G_2. Then the two closed point-sets $\bar{G}_1'$ and $\bar{G}_2'$ have no point in common. But $\bar{C}_n$ contains at least one point of $\bar{G}_1'$ and at least one point of $\bar{G}_2'$. Hence, since $\bar{C}_n$ is connected it must contain at least one point that does not belong to $\bar{G}_1' + \bar{G}_2'$. Let T_n denote the set of all such points. For every n, T_{n+1}' is a subset of T_n. Hence, by Theorem 14, the point-sets $T_1, T_2, \cdots$ have in common at least one point O. The point O belongs to C and therefore to $\bar{G}_1 + \bar{G}_2$ as well as to T_1. Thus the supposition that C is not connected leads to a contradiction.

III. To prove that no connected proper subset of C contains both A and B. Let us first order the points of C. If X_1 and X_2 are two distinct points of C then there exists n such that X_1 and X_2 do not both lie in the same region of the set $K_n, K_{n+1}, K_{n+2}, \cdots$. But every link of C_n is a region of this set. Thus for every two distinct points X_1 and X_2 belonging to C there exists n such that X_1 and X_2 do not belong to the same link of C_n. Furthermore it is clear that if X_1 and X_2 do not lie in the same link of C_n but X_1 lies in a link of C_n that precedes one in which X_2 lies, then if $m > n$ every link of C_m that contains X_1 precedes every link of C_m that contains X_2. The point X_1 is said to precede the point X_2 ($X_1 < X_2$) if there exists n such that every link of C_n that contains X_1 precedes every link of C_n that contains X_2. From facts observed above it follows that if X_1 and X_2 are distinct points of C then either $X_1 < X_2$ or $X_2 < X_1$, while if $X_1 < X_2$ then it is not true that $X_2 < X_1$. Furthermore if $X_1 < X_2$ and $X_2 < X_3$ then $X_1 < X_3$. For there exist n_1 and n_2 such that X_1 and X_2 do not lie in the same link of C_{n_1} and X_2 and X_3 do not lie in the same link of C_{n_2}. Hence every link of $C_{n_1+n_2}$ that contains X_1 precedes every link of $C_{n_1+n_2}$ that contains X_2 and every link of $C_{n_1+n_2}$ that contains X_2 precedes every one that contains X_3. Hence every one that contains X_1 precedes every one that contains X_3. Hence $X_1 < X_3$.

Suppose now that H is a proper subset of C that contains both A and B. Then there exists a point P belonging to C but different from A and from B such that H is a subset of $C - P$. Now $C - P = S_A + S_B$ where S_A is the set of all points of C that precede P and S_B is the set of all points of C that follow P. It is clear that S_A contains A and S_B contains B. Suppose that P_A is a point of S_A. Then there exists n such that every link of C_n that contains P_A precedes every one that contains P. Suppose that some link y of the chain C_n contains P_A and also a point P_B of the set S_B. Since y precedes every link of C_n that contains P it follows that P_B precedes P, which is contrary to hypothesis. Hence no region of C_n that contains P_A contains any point of S_B. But some region of C_n does contain P_A. Hence P_A is not a limit point of S_B. Similarly no point of S_B is a limit point of S_A. But H contains a point A that belongs to S_A and a point B that belongs to S_B. Moreover H is a subset of $S_A + S_B$. It follows that H is not connected.

It is thus established that C is a simple continuous arc from A to B.

THEOREM 16. *Every two points of a region R can be joined by an arc* lying entirely in R while every two points without R can be joined by an arc lying entirely without R.*

5. CONSEQUENCES OF AXIOMS 2–5 AND THEOREM 5

On the basis of Axioms 2–5 and Theorem 5 it is possible to prove Theorems 2–9 of § 4 of Lennes' paper.† In particular if X is any point of the arc AB then AB is the sum of two arcs AX and XB that have no common point other than X. If X and Y are points of the arc AB distinct from A and from B then AB contains as a subset only one arc that has X and Y as its endpoints. This arc XY is called the *interval* XY of the arc AB. If Z is a point (distinct from X and from Y) of the interval XY of the arc AB then Z is said to be between X and Y on the arc AB. If X is between A and Y on the arc AB then X is said to *precede* Y *on* AB and Y is said to *follow* X *on* AB. If X precedes Y on AB then Y precedes X on BA.

If the arc AB has at least one point in common with the closed set of points K then there exists (1) a point P_1 common to AB and K such that if X_1 is common to AB and K and $X_1 \neq P_1$ then X_1 is between P_1 and B on the arc AB, (2) a point P_2 common to AB and K such that if X_2 is common to AB and K and $X_2 \neq P_2$ then X_2 is between P_2 and A on the arc AB. The point P_1 is said to be *the first point that* AB *has in common with* K while P_2 is said to be *the last point that* AB *has in common with* K. It is to be observed that the first point that AB has in common with K is the last point that BA has in common with K.

If P is a point on an arc AB and M is a point-set on AB then P is a limit point of M if and only if every interval of AB that contains P (but does not have P as an endpoint) contains also a point of M distinct from P.

DEFINITION.‡ A *simple closed curve* is a set of points composed of two arcs AXB and AYB that have no point in common except A and B.

* Hereafter in this paper, "arc" and "simple continuous arc" will be considered synonymous terms.

† Loc. cit.

‡ See N. J. Lennes, loc. cit., p. 314.

If A and B are two distinct points of the simple closed curve J then there are two and only two distinct arcs of J from A to B. These two arcs, AXB and AYB, have no point in common except their endpoints, A and B. Every point of J belongs either to AXB or to AYB and A and B are said to separate J into the two arcs AXB and AYB. If A, B, C, and D are four distinct points of J such that C lies on one and D lies on the other of the two arcs into which A and B separate J then A and B are said to separate C and D on J. If A and B separate C and D on J then C and D separate A and B on J.

THEOREM 17. *If R is a region, the exterior of R contains an infinite set of points that has no limit point.*

Theorem 17 is a consequence of Axiom 5 and Theorem 13.

THEOREM 18. *The exterior of a region is not a bounded point-set.*

THEOREM 19. *Every region has at least one boundary point.*

Proof. Suppose R is a region. By Theorems 4 and 5 there exist in R two points P and X and a region R_n such that R_n contains P but R_n' is a subset of $R - X$. It follows by Theorem 17 that $S - R_n' = M_1 + M_2$ where M_1 is a subset of R and M_2 is a subset of $S - R$. No point of M_1 is a limit point of M_2. Hence, by Axiom 3, M_2 contains a limit point of M_1. Every such point is a boundary point of R

6. CONSEQUENCES OF AXIOMS 2–8 AND THEOREMS 5 AND 15

It is clear that Theorems 1–19 are consequences of Axioms 2–5 and Therems 5 and 15.

THEOREM 20. *Every point of the boundary of a region is a limit point of the exterior of that region.*

Theorem 20 can easily be proved with the help of Axiom 7.

THEOREM 21. *If K and R are regions and the boundary of R is a subset of K' then R is a subset of K.*

Proof. By Theorem 18, $S - K'$ contains at least one point that does not belong to R. If it contains also a point of R then $S - K' = S_1 + S_2$ where S_1 is a subset of R but no point of S_2 belongs to R. The point-set S_1 cannot contain a limit point of S_2. Hence, by Axiom 3, S_2 must contain a limit point of S_1. Hence S_2 contains a point of the boundary of R. But this is contrary to hypothesis. It follows that R is a subset of K'. If R contained a point on the boundary of K then, by Theorem 20, it would contain a point of $S - K'$. It follows that R is a subset of K.

THEOREM 22. *The set of all points is connected.*

Theorem 22 can be easily proved with the help of Theorem 19, Axioms 2 and 3 and Theorem 20.

THEOREM 23. *No region is a subset of an arc.*

Proof. Suppose that the region R is a subset of the arc AB. Then R

must contain at least one point O belonging to AB but distinct from A and from B. By Theorems 3 and 5 there exists about O a region K such that A and B are both without K. By Theorem 16 there exists an arc AXB lying without K. There exist on AB two points A_1 and B_1 such that A_1 is the last point that AXB has in common with the interval AO of the arc AB and B_1 is the first point following A_1 on AXB that AXB has in common with the interval OB of the arc AB. The arc $A_1 OB_1$ on AB and the arc from B_1 to A_1 on AXB constitute together a closed curve $\bar{J}$. There exists about O a region R_1 containing no point of the closed point-set $AA_1 + B_1 B$, where AA_1 and BB_1 are intervals of AB. By Theorem 6 the regions R and R_1 contain in common a region R_2 that contains O. Since R_2 is a subset of R that contains no point of $AA_1 + B_1 B$ it must be a subset of $A_1 OB_1$ and therefore of $\bar{J}$. But O is on the boundary of $\bar{R}$, the interior of $\bar{J}$. Therefore, since R_2 contains O, R_2 must contain a point of $\bar{R}$. Thus the supposition that R is a subset of AB leads to a contradiction.

THEOREM 24. *If the points A and B separate the points C and D on the closed curve* J and AXB is an arc such that* $\underline{AXB}$† *is a subset of R, the interior of J, then* (1) R_1, *the interior of the closed curve AXBCA, is a subset of R,* (2) $\underline{ADB}$ *is entirely without* R_1, (3) R_1 *has no point in common with* R_2, *the interior of AXBDA.*

Proof. That R_1 is a subset of R is a consequence of Theorem 21. Hence, by Theorem 1, R_1 contains no point of ADB.

Suppose that R_1 and R_2 have a point in common. Since the boundary of R_2 contains points that are neither in R_1 nor on the boundary of R_1, R_2 is not a subset of R_1. Hence $R_2 = S_1 + S_2$ where S_1 is a subset of R_1 but no point of S_2 belongs to R_1. The point-set S_1 cannot contain a limit point of S_2. Hence, by Axiom 2, S_2 must contain a point P which is a limit point of S_1. Clearly P must be on the boundary of R_1. Thus R_2 would contain a point of AXB or of ACB. But this is contrary to hypothesis and (2). It follows that R_1 and R_2 have no point in common.

THEOREM 25. *Under the same hypothesis as in Theorem* 24,

$$R = \underline{AXB} + R_1 + R_2.$$

Proof. Suppose it is not true that $R = \underline{AXB} + R_1 + R_2$. Then, by hypothesis and Theorem 24, $R = \underline{AXB} + R_1 + R_2 + Y$ where the point-sets $\underline{AXB}$, R_1, R_2, and Y are mutually exclusive. It is clear that no one of the three sets R_1, R_2, and Y contains a limit point of either of the other

* Hereafter in this paper, "closed curve" will be considered synonymous with "simple closed curve."

† If AxB is an arc, $\underline{AXB}$ denotes the point-set $AXB - A - B$. Likewise if AB is an arc, $\underline{AB}$ denotes the point-set $AB - A - B$.

two. Let E denote a point of Y. By Theorem 16 there exists an arc EX lying entirely in R. There exists a point O (Fig. 1) which is the first point that EX has in common with AXB. Let OE denote the interval of EX whose endpoints are O and E. Now, (1) $OE - O$ is connected, (2) every point of $OE - O$ belongs to R_1 or to R_2 or to Y, (3) no one of the sets R_1, R_2, and Y contains a limit point of either of the other two. It follows that since E be-

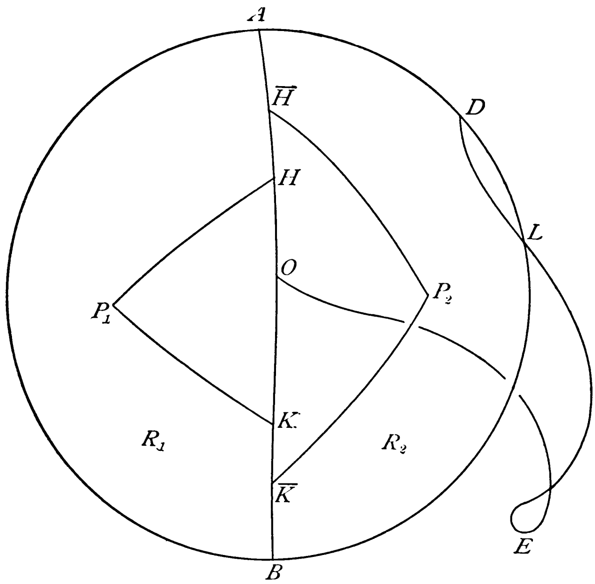

Fig. 1

longs to Y therefore $OE - O$ is a subset of Y. Let F denote a point of R_1. The point O divides the arc AXB into two arcs $A\bar{A}O$ and $B\bar{B}O$. There exists about $\bar{A}$ a region T which contains no point of the closed point-set $F + O\bar{B}B + J$. The region T contains at least one point G in common with R_1. By Theorem 16 there is an arc FG lying entirely in R_1 and an arc $G\bar{A}$ lying entirely in T. It can easily be shown that the point-set $FG + G\bar{A}$ contains as a subset an arc FH such that H is in T and on the boundary of R_1 (and therefore on $A\bar{A}O$) while $FH - H$ is a subset of R_1. Similarly there exists an arc FK such that K is a point of $O\bar{B}B$ and $FK - K$ is a subset of R_1. The point-set $FH + FK$ contains as a subset an arc HP_1K. Clearly $\overline{HP_1K}$ is a subset of R_1. Likewise there exist a point $\bar{H}$ between A and H, a point $\bar{K}$ between B and K and an arc $\bar{H}P_2\bar{K}$ such that $\overline{\bar{H}P_2\bar{K}}$ is a subset of R_2. Let $\bar{J}$ denote the closed curve $\bar{H}P_2\bar{K} + \bar{K}K + KP_1H + H\bar{H}$. By Theorem 21, $\bar{R}$ (the interior of $\bar{J}$) is a subset of R. But P_1 is on $\bar{J}$ and is therefore a limit point of $\bar{R}$. Moreover P_1 belongs to R_1 and no point of R_1 is a limit point of AXB or of R_2 or of Y. It follows that $\bar{R}$ must contain at least one point of R_1. Similarly P_2 is a limit point of $\bar{R}$ and $\bar{R}$ must contain at least one point of R_2. Since D and E are without the region R_1 therefore there exists an arc DE lying entirely without R_1. There exists on ADB a point L which is the first point that ED has in common with ADB. The point-set $EL - L$ lies in R and contains no point of AXB. It easily follows that $EL - L$ is a subset of Y. But $EO - O$ is a subset of Y. It follows that there exists an arc OL such that $\underline{OL}$ is a subset of Y. Hence OL contains no point of $\bar{J}$. But L is without $\bar{J}$. Therefore O, and consequently $\underline{HOK}$, is without $\bar{J}$. Hence every point of $\bar{R}$ belongs to R_1, R_2, or Y. But no one of these three point-sets contains a limit point of either of the other two and it has been shown that $\bar{R}$ contains at least one point of R_1 and at least one point of R_2. Hence $\bar{R}$ is not connected. But this is contrary to Axiom 2. Thus the supposition that Y exists leads to a contradiction. It follows that

$$R = R_1 + R_2 + \underline{AXB}.$$

THEOREM 26. *If A and B are distinct points and ACB, ADB, and AFB are three arcs no two of which have any point in common except A and B, then the regions which are bounded by the closed curves $ACBFA$, $ACBDA$, and $ADBFA$ respectively are not mutually exclusive.*

Proof. Denote the curves $ACBFA$, $ACBDA$, and $ADBFA$ by J, J_1, J_2, their interiors by R, R_1, R_2 and their exteriors by E, E_1, E_2 respectively.

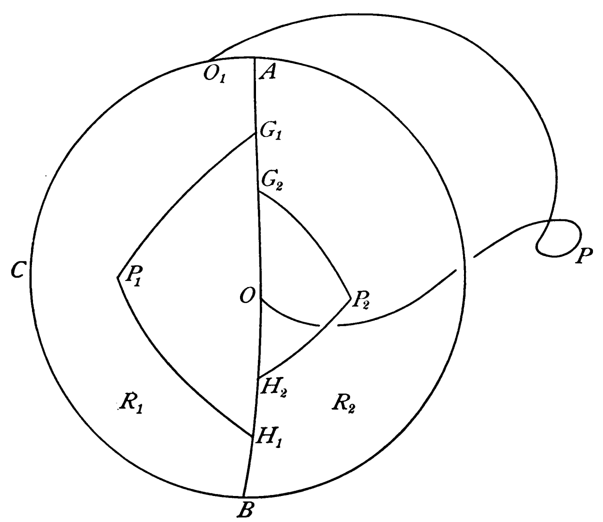

FIG. 2

Suppose that R, R_1, R_2 are mutually exclusive. Then E contains $\underline{ADB}$, R_1, and R_2. It may be easily proved with the assistance of Theorems 17 and 13 that E contains at least one point which does not belong to R_1, R_2, or ADB.

Let Y denote the set of all such points. It is clear that Y is composed of all points that are common to E, E_1, and E_2. Let P (Fig. 2) be a point of Y. There exists an arc PD which lies entirely in E. Let O be the first point that PD has in common with $\underline{ADB}$. Then $PO - O$ is clearly a subset of Y. Similarly there exists an arc PO_1 such that O_1 is a point on $\underline{ACB}$ and such that $PO_1 - O_1$ is a subset of Y. It can easily be shown* that there exist points G_1, G_2, H_1, H_2 lying on ADB in the order $AG_1\,G_2\,OH_2\,H_1\,B$ and arcs $G_1\,P_1\,H_1$, $G_2\,P_2\,H_2$ such that $\underline{G_1\,P_1\,H_1}$ and $\underline{G_2\,P_2\,H_2}$ lie in R_1 and R_2 respectively. Let $\bar{J}$, $\bar{J}_1$, $\bar{J}_2$ denote the closed curves $G_1\,P_1\,H_1\,H_2\,P_2\,G_2\,G_1$, $G_1\,P_1\,H_1\,OG_1$, $G_2\,P_2\,H_2\,OG_2$ respectively. Let $\bar{R}$, $\bar{R}_1$, $\bar{R}_2$ denote their respective interiors. It follows from Theorem 21 that $\bar{R}_1$ and $\bar{R}_2$ are subsets of R_1 and R_2 respectively.

There are three cases to be considered.

Case I. Suppose that O lies in $\bar{R}$. Then, by Theorem 25,

$$R = \bar{R}_1 + \bar{R}_2 + \underline{G_2\,OH_2}.$$

It follows that if O is a limit point of a point set M then M must contain a point of $\bar{R}_1$ or of $\bar{R}_2$ or of $\underline{G_2\,OH_2}$. But O is a limit point of $PO - O$ and $PO - O$ contains no point of $\bar{R}_1 + \bar{R}_2 + G_2\,OH_2$. Thus the supposition that O is in $\bar{R}$ leads to a contradiction.

Case II. Suppose that A is in $\bar{R}$. Then J lies in $\bar{R}$. But O_1 is a point of J and PO_1 contains no point of $\bar{J}$. Hence P is in $\bar{R}$. Similarly every other point of Y is in $\bar{R}$. But Y contains an infinite set of points that has no limit point. Hence, by Theorem 13, Y is not a subset of $\bar{R}$. Thus the supposition that A is in $\bar{R}$ leads to a contradiction.

Case III. Suppose that neither O nor A is in $\bar{R}$. Then no point of J, or of ADB is in $\bar{R}$. Hence $\bar{R}$ is a subset of $R + R_1 + R_2 + Y$. But $\bar{R}$ is connected and no one of the point-sets R_1, R_2, R, and Y contains a limit point of one of the others. Hence $\bar{R}$ cannot contain a point of one of these point-sets and also a point of one of the others. Therefore $\bar{R}$ must be a subset of either R, R_1, R_2, or Y. But P_1 and P_2 are both limit points of $\bar{R}$ and moreover P_1 is a point of R_1 and P_2 is a point of R_2. Hence either (1) R_1 contains a limit point of R, R_2, or Y or (2) R_2 contains a limit point of R_1. Thus the supposition that $\bar{R}$ contains neither O nor A leads to a contradiction.

It follows that R, R_1, and R_2 are not mutually exclusive.

THEOREM 27. *If the points A and B separate the points C and D on the closed curve J and AXB is an arc such that $\underline{AXB}$ is without J then* (1) *either D is without $AXBCA$ or C is without $AXBDA$,* (2) *If D is without $AXBCA$ then C is within $AXBDA$ and the interior of $AXBDA$ = $\underline{ACB}$ + the interior of $ACBDA$ + the interior of $AXBCA$.*

* Cf. proof of Theorem 25.

Proof. (1) Suppose D is not without $AXBCA$. Then $\underline{ADB}$ is within $AXBCA$. Hence, by Theorem 24, C is without $AXBDA$.

(2) Suppose that D is without $AXBCA$. Then $\underline{ADB}$ is without $AXBCA$. Let R and R_1 denote the interiors of $ACBDA$ and $AXBCA$ respectively. Suppose that R and R_1 have a point in common. Then $R_1 = M + M_1$ where M is a subset of R but no point of M_1 belongs to R'. Neither of the point-sets M and M_1 can contain a limit point of the other one. Thus Axiom 2 is contradicted. It follows that if D is without $AXBCA$ then the interiors of $AXBCA$ and $ACBDA$ can contain no point in common. If at the same time C were without $AXBDA$ it would follow that the interiors of $ACBDA$, $AXBDA$, and $AXBCA$ are mutually exclusive. But this would be contrary to Theorem 26. It is thus established that if D is without $AXBCA$ then C is within $AXBDA$. It follows by Theorem 25 that in this case the interior of $AXBDA = \underline{ACB} +$ the interior of $AXBCA +$ the interior of $ACBDA$.

THEOREM 28. *If O is a point on the closed curve J and $\bar{R}$ is a region about O then if M denotes either the interior or the exterior of J, there exists a simple continuous arc AXB such that* (1) *A and B are on J,* (2) *$\underline{AXB}$ is common to M and $\bar{R}$,* (3) *of the two arcs into which A and B divide J that one which contains O lies in $\bar{R}$.*

Proof. By Axioms 6 and 7 there exist in $\bar{R}$ regions L and $\bar{L}$ such that L contains O, $\bar{L}$ lies in M and all those points of J which lie in L belong to the boundary of $\bar{L}$. There exist on J two points A_1 and B_1 such that the arc $A_1 OB_1$ (on J) lies in L. Since $A_1 OB_1$ is in L and on J it must be a part of the boundary of $\bar{L}$. There exist on $A_1 OB_1$ points A_2 and B_2 in the order $A_1 A_2 OB_2 B_1$. There exist regions R_A and R_B about A_2 and B_2 respectively such that R_A contains no point of $A_1 B_1 O$ and R_B contains no point of $B_1 A_1 O$, where $A_1 B_1 O$ and $B_1 A_1 O$ are intervals of J. There exist in $\bar{L}$ three distinct points P, C, and D such that C is in R_A and D is in R_B. There exist in $\bar{L}$ arcs PC and PD. In R_A and R_B respectively there exist arcs CA_2 and DB_2. The point-set $A_2 C + CP + PD + DB_2$ contains as a subset an arc AXB satisfying conditions (1), (2), and (3).

THEOREM 29. *If on the closed Jordan curve J the points A and B separate the points C and D and AXB and CYD are arcs such that $\underline{AXB}$ and $\underline{CYD}$ are either both within or both without J then $\underline{AXB}$ and $\underline{CYD}$ have at least one point in common.*

Proof. Case I. Suppose that $\underline{AXB}$ and $\underline{CYD}$ are both within J. By Theorem 25, $R = R_1 + R_2 + \underline{AXB}$, where R, R_1, and R_2 are the interiors of J, $AXBCA$, and $AXBDA$ respectively. There exists about C a region K containing no point of R_2'. Since C is a limit point of $\underline{CYD}$, K contains a point P belonging to $\underline{CYD}$. The point P is in R_1 while D is without R_1. Hence the interval $\underline{PD}$ of the arc CYD contains a point on the boundary of R_1. Every such point must be on $\underline{AXB}$.

Case II. Suppose that AXB and CYD are both without J. Then, by Theorem 27, either C is without $AXBDA$ or D is without $AXBCA$. Suppose that C is without $AXBDA$. Then $R_1 = R + ADB + R_2$ where R, R_1, and R_2 are the interiors of J, $AXBCA$, and $AXBDA$ respectively. The point D is a limit point of CYD. Hence R_1 contains a point P belonging to CYD. But CYD contains no point of R'. Hence P belongs to R_2. But C is without R_2. Therefore the interval CP of the arc CYD contains a point in common with the boundary of R_2. Every such point must be on AXB.

THEOREM 30. *Under the same hypothesis as in Theorem* 29, *the interiors of* $AXBDA$ *and* $CYDBC$ *have at least one point in common.*

THEOREM 31. *If* CD *is an arc,* B *is a point of* CD, *and* BX *is an arc which has no point except* B *in common with* CD *and* R *is a region containing* B *then there exists in the region* R *an arc* EFG *such that* (1) *the points* E *and* G *are on* CD *in the order* $CEBGD$, (2) *the segment* EBG *of* CD *lies in* R, (3) EFG *has no point in common with* CD, (4) *there exists on* BX *a point* H *such that the segment* BH *of* BX *is* (*except for the point* B) *entirely within the closed curve* $EFGBE$.

Proof. About B (Fig. 3) there exists a region $\bar{R}$ which contains neither C nor D. There exists an arc CYD which lies entirely without R. There exist points C_1 and D_1 such that C_1 is the last point that CYD has in common with CB and D_1 is the first point following C_1 that CYD has in common with BD. The arc $C_1 X_1 D_1$ (on CYD) has no point except C_1 in common

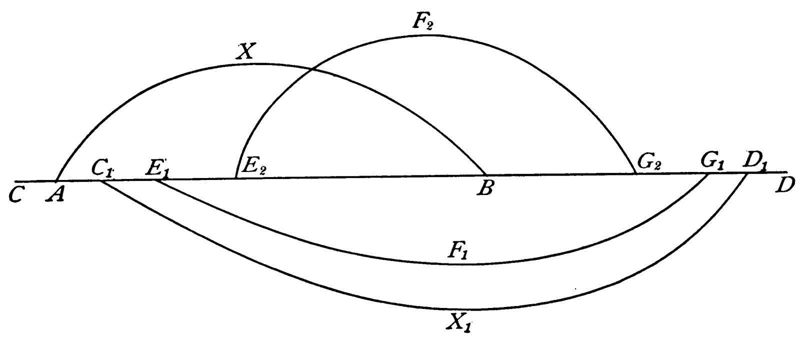

FIG. 3

with CD. About B there is a region K which contains no point of the closed point-set $CC_1 + C_1 X_1 D_1 + D_1 D$. The regions K and R contain in common a region R_1 which contains B. By Theorem 28 there exists an arc $E_1 F_1 G_1$ such that (1) the points E_1 and G_1 are on CD in the order $C_1 E_1 B G_1 D_1$ (2) the closed curve $E_1 F_1 G_1 B E_1$ is a subset of R_1, (3) $E_1 F_1 G_1$ is a subset of the interior of the closed curve $C_1 X_1 D_1 B C_1$. About B there is a region H containing no point of the closed point-set $CE_1 + E_1 F_1 G_1 + G_1 D$. The regions H and R contain in common a region R_2 that contains B. By Theorem 28 there exists an arc $E_2 F_2 G_2$ such that (1) the points E_2 and G_2 are on CD in the order $C_1 E_1 E_2 BG_2 G_1 D_1$, (2) the closed curve $E_2 F_2 G_2 BE_2$ is a subset of R_2, (3) $E_2 F_2 G_2$ is a subset of the exterior of the closed curve $E_1 F_1 G_1 BE_1$. The curve $E_1 E_2 F_2 G_2 G_1 F_1 E_1$ encloses a Jordan region R_3. By Theorem 25, $R_3 = R_1 + R_2 + \overline{E_2 BG_2}$ where R_1 and R_2 are the interiors of $E_1 F_1 G_1 BE_1$ and $E_2 F_2 G_2 BE_2$ respectively. Since the connected point-set $\overline{BXA}$ contains no point of $E_2 BG_2$ therefore it cannot contain a point in R_1 and also a point in R_2. But since B is a limit point of $\overline{BX}$ the latter must contain points in either R_1 or R_2. It follows that there exists on $\overline{BX}$ a point H such that the point-set $BH - B$ is a subset of R_1 or a subset of R_2.

THEOREM 32. *If, in a domain H, AB is an arc, A_1, B_1, and D are three points on AB in the order $AA_1 DB_1 B$ and $A_1 X_1 B_1$ is an arc which has no point in common with AB except A_1 and B_1, and finally C is a point in R, the interior of the closed curve $A_1 X_1 B_1 DA_1$, then there exists,* in H, *an arc CB which has no point except B in common with AB.*

Proof. There exists an infinite sequence of regions, R_1, R_2, R_3, $\cdots$ satisfying conditions (1), (2), and (3) of Theorem 5 except that P is replaced by B. A point P belonging to AB is said to be in Class I if there exists, in H, a system of points $A_2, A_3, \cdots, A_n, B_2, B_3, \cdots, B_n$ on AB and arcs $A_2 X_2 B_2$, $\cdots$, $A_n X_n B_n$ such that

(1) A_i and B_i separate A_{i+1} from B_{i+1} $(1 \leqq i < n)$;

(2) P lies between A_n and B_n;

(3) $A_i X_i B_i$ has no point in common with AB $(1 < i \leqq n)$;

(4) the interior of the closed curve $A_i X_i B_i B_{i-1} A_i$ $(1 < i \leqq n)$ does not contain either of the points C and B;

(5) there exists on $B_i X_i A_i$ a point H_i such that $B_i H_i - B_i$ lies in R_{i+1}, the interior of $A_{i+1} B_i B_{i+1} X_{i+1} A_{i+1}$ $(1 \leqq i < n)$.

Under these conditions the arc DP is said to be covered by the set of arcs $A_1 X_1 B_1$, $A_2 X_2 B_2$, $\cdots$, $A_n X_n B_n$.

Suppose there exist on $\overline{AB}$ points that do not belong to Class I. Let Class II denote the set of all such points. Then there exists on AB a point O which, in the order from A to B on AB, is either the last point in Class I or the first point in Class II. About O there is a region $\bar{R}$ that lies in H and containing neither C nor B nor any point of the interval AB_1 of the arc AB. There exist* points $\bar{A}_1$, $\bar{A}_2$, $\bar{B}_1$, $\bar{B}_2$ on AB in the order $A\bar{A}_1 \bar{A}_2 O\bar{B}_2 \bar{B}_1 B$ and arcs $\bar{A}_1 \bar{X}_1 \bar{B}_1$, $\bar{A}_2 \bar{X}_2 \bar{B}_2$, having no point in common with AB except their endpoints, such that the closed curves $\bar{A}_1 \bar{X}_1 \bar{B}_1 O\bar{A}_1$ and $\bar{A}_2 \bar{X}_2 \bar{B}_2 O\bar{A}_2$ lie in $\bar{R}$ and such that the interiors of these closed curves have no point in common. The arc $D\bar{A}_2$ can be covered by a finite set, G, of arcs. It follows†

* Cf. proof of Theorem 31.

† Cf. proof of Theorem 31.

that DO may be covered by a finite set, $\bar{G}$, of arcs composed of a subset of G together with either $\bar{A}_1 \bar{X}_1 \bar{B}_1$ or $\bar{A}_2 \bar{X}_2 \bar{B}_2$ and possibly* one additional arc. But if $\bar{G}$ covers DO it must also cover DP_2 where P_2 is some point that follows O and therefore belongs to Class II. Thus the supposition that Class II exists leads to a contradiction. Hence if P is any point of $\underline{AB}$, DP can be covered by a finite set of arcs. It easily follows that there exists an infinite set of arcs $A_2 X_2 B_2$, $A_3 X_3 B_3$, $\cdots$ which satisfy conditions (1), (2), (3), (4), and (5) (∞ being substituted for n) and are such that, if m is a positive integer, there exists δ such that if $n > \delta$ then $A_n X_n B_n B_{n-1} A_n$ is a subset of R_m. Let $\bar{R}_i$ denote the interior of the closed curve $A_i X_i B_i B_{i-1} A_i$. For every i, $\bar{R}_i$ has a point in common with $\bar{R}_{i+1}$. By Zermelo's postulate there exists a sequence of points $P_1, P_2, P_3 \cdots$ such that P_i is common to $\bar{R}_i$ and $\bar{R}_{i+1}$. There exists in $\bar{R}_1$ an arc CP_1 and in $\bar{R}_i$ $(i > 1)$ an arc $P_{i-1} P_i$. There exists a point $\bar{P}_2$ which is the first point that CP_1 has in common with $P_1 P_2$. The arc $C\bar{P}_2$ (on CP_1) and the arc $\bar{P}_2 P_2$ (on $P_1 P_2$) constitute together an arc CP_2. Similarly there exists an arc CP_3 formed by the arc $C\bar{P}_3$ (on CP_2) together with the arc $\bar{P}_3 P_3$ (on $P_2 P_3$) where $\bar{P}_3$ is the first point that CP_2 has in common with $P_2 P_3$. In general CP_{i+1} denotes $C\bar{P}_{i+1} + \bar{P}_{i+1} P_{i+1}$ where $\bar{P}_{i+1}$ is the first point that CP_i has in common with $P_i P_{i+1}$. For a given $\bar{n}$ there exists $n_{\bar{n}}$ such that if $n > n_{\bar{n}}$ then CP_n contains $CP_{\bar{n}}$. It follows that for a given n there exists an arc CX_n which is common to all the arcs CP_n, CP_{n+1}, CP_{n+2}, $\cdots$, and contains every other arc which is common to the arcs of this sequence and has C for one end point. For every m, CX_{m+1} contains CX_m and there exists r such that CX_m is a proper subset of CX_{m+r}. Let τ denote the point-set $B + CX_1 + CX_2 + \cdots$. It will be shown that τ satisfies Lennes' definition of a Jordan arc from C to B.

First τ is closed. For suppose that P is a limit point of τ that does not belong to τ. Then clearly P is not a limit point of any single CX_n. But if R is a region about B then there exist m and δ such that (1) R'_m is a subset of R, (2) for every n greater than δ, $\tau - CX_n$ is a subset of R'_m. But P is a limit point of $\tau - CX_n$. Hence P is in R. It follows that P is identical with B. Thus the supposition that P does not belong to τ leads to a contradiction. It follows that τ is closed. It is easily seen that B is a limit point of $CX_1 + CX_2 + \cdots$ and that τ is connected and contains no proper connected subset that contains B and C. Thus τ is a simple continuous arc from C to B. Clearly it contains no point of AB except B.

THEOREM 33. *If A is a point on the boundary of a region R, B is a point in R, and AB is an arc such that $\underline{AB}$ is in R, then $R - AB - A)$ is connected.*

Proof (on the basis of Axioms 2–5, 8 and Theorems 5, 15, and 32) for the

* An additional arc may be necessary in case $\bar{A}_1$ or $\bar{A}_2$ coincides with an endpoint of some arc of the set G.

case where the boundary of R is a closed curve J. On J (Fig. 4) there is a point O different from A. About O there is a region $\bar{R}$ which contains no point of AB. The region $\bar{R}$ contains a point C in common with R. There exists an arc BC such that $\underline{BC}$ is in R and an arc CO lying in $\bar{R}$. The point-set $BC + CO$ contains as a subset an arc DE where D is a point of AB, E is a point in $\bar{R}$ and on J and $\underline{DE}$ is a subset of $R - (AB - A)$. The arc ADE divides R into two regions of which one contains $DB - D$ and the other one contains no point of AB. Let L denote the latter region and let F denote a point in L. By Theorem 32 there exists an arc FB

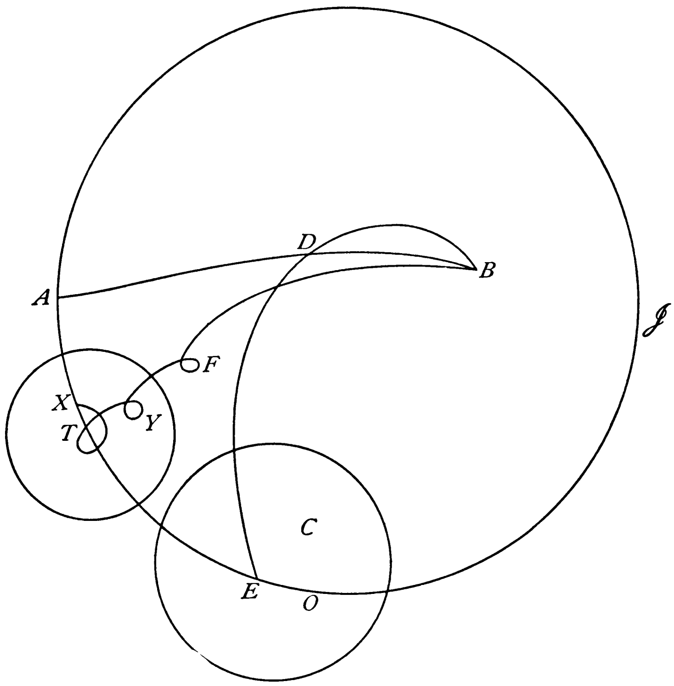

FIG. 4

which has no point in common with AB. Let AXE denote that arc from A to E on J which forms a part of the boundary of L. About X there is a region K which contains no point of $AD + DE$. The regions K and L have a point Y in common. There exists in L an arc FY and in K an arc YX. The point-set $BF + FY + YX$ contains as a subset an arc BT where T is a point on J and $\underline{BT}$ is a subset of $R - (AB - A)$. The points A and T divide J into two arcs AMT and ANT. By Theorem 25, $R = R_1 + R_2 + AB + BT$, where R_1 and R_2 are the interiors of $ABTMA$ and $ABTNA$ respectively. Suppose now that P_1 and P_2 are two points in $R - (AB - A)$. If P_1 and P_2 are both in R_1 or both in R_2 there exists an arc $P_1 P_2$ lying entirely in R_1 or entirely in R_2. Suppose P_1 is in R_1 and P_2 is in R_2. There exists on $\overline{BT}$ a point Z. About Z there is a region R_3 containing no point of AB. There exist arcs $P_1 Z_1$ and $P_2 Z_2$ such that $\overline{P_1 Z_1}$ is in R_1, $\overline{P_2 Z_2}$ is in R_2 and the points Z_1 and Z_2 are in R_3 and on the boundaries of R_1 and R_2 respectively and therefore on BT. If $Z_1 Z_2$ denotes that arc from Z_1 to Z_2 which is a subset of BT then $P_1 Z_1 + Z_1 Z_2 + Z_2 P_2$ is an arc from P_1 to P_2 lying in $R - (AB - A)$. The case where one or both of the points P_1 and P_2 is on DE can be easily treated. It follows that every two points in $R - (AB - A)$ can be joined by an arc lying entirely in $R - (AB - A)$. Hence $R - (AB - A)$ is connected.

THEOREM 34. *If J is a Jordan arc and AB is an arc lying wholly in R, the interior of J, then $R - AB$ is connected.*

Proof (on the basis of Axioms 2–5, 8 and Theorems 5, 15, and 32) for the case where the boundary of R is a closed curve J. There exists an arc $A_1 X_1 B_1$ such that A_1 and B_1 are on AB in the order $AA_1 B_1 B$ but $\overline{A_1 X_1 B_1}$ has no point in common with AB. There exists a point X in the interior of the closed curve formed by the arc $A_1 X_1 B_1$ and the interval $A_1 OB_1$ of the arc AB. By Theorem 32 there exist arcs XA and XB such that neither $\overline{XA}$ nor $\overline{XB}$ has a point in common with AB. The point-set $XA + XB$ contains as a subset an arc AYB. About Y there is a region $\bar{R}$ which contains no point of AB. On J there is a point P. There exists an arc PE such that $\overline{PE}$ is without $AYBOA$ and such that E is on $AYBOA$ and in $\bar{R}$ (and therefore not on AB). The arc PE, the arc from E to A on AYB and the arc AB constitute together an arc $PEAB$. By Theorem 33, $R - (PEAB - P)$ is connected. It easily follows that $R - AB$ is connected.

Proof of Theorems 33 and 34 (on the basis of Axioms 2–5, 8 and Theorems 5, 15, 32, and 31) for the case where R is any region.* Suppose C and D are two distinct points in R but not on the arc AB. There exists, in R, an arc CA. Let X denote the first point that CA has in common with AB. Suppose X is distinct from A. By Theorems 31 and 32, some point F on XA can be joined to A by an arc AF such that $AF - A$ is a subset of $(R + B) - AB$. It follows that there exists, in R, an arc CA that has no point except A in common with AB. Likewise there exists in R an arc DC having no point except C in common with the arc CAB.

7. CONSEQUENCES OF AXIOMS 2–5, 8 AND THEOREMS 5, 15, 20, 33, AND 34

THEOREM 35. *Every region contains at least one simple closed curve.*

Proof. Suppose R is a region. By Theorems 4 and 16, there exists in R a simple continuous arc AB. On AB there is a point X distinct from A and

* This proof is a modification of a proof due to Mr. J. R. Kline, one of my students.

from B. By Theorem 23, R contains a point C which is not on AB. By Theorems 34 and 15 there is, in R, a simple continuous arc AC which has no point in common with XB. Let O denote the last point that AB has in common with AC. There exists an arc XOC consisting of XO (on BA) together with OC (on AC). On $\overline{OC}$ there is a point T. By Theorem 34, there is in R an arc BC which has no point in common with XOT. There exist E and F such that E is the last point that BC has in common with BA and F is the first point that OC has in common with EC. Clearly EO (on BA) + OF (on OC) + FE (on CB) is a closed curve lying wholly in R.

THEOREM 36. *If O is a point in a region R there exists a simple closed curve which lies in R and encloses O.*

Proof. By Theorem 4 there exists in R a point X distinct from O. By Axiom 1 there exists about X a region which lies in R and does not contain O. By Theorem 35 this region contains a closed curve J (Fig. 5). By Theorem 21, O is without J. On J there is a point A_1. In R there is an arc $A_1 O$. There exists a point A which is the last point that $A_1 O$ has in common with J. The

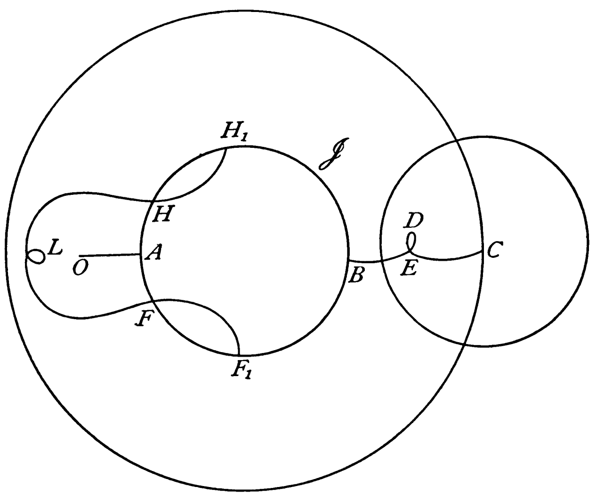

FIG. 5

arc AO has no point except A in common with J. By Theorem 19 there exists a point C on the boundary of R. There exists about C a region $\bar{R}$ containing no point of the closed point-set $OA + J + I$, where I is the interior of J. The region $\bar{R}$ contains at least one point D of R. On J there is a point B_1 distinct from A. By Theorem 34 there exists, in $R - AO$, an arc $B_1 D$. Let B denote the last point that $B_1 D$ has in common with J. There exists in $\bar{R}$ an arc DC. The arc DC can contain no point of $OA + J + I$. Let E be the last point which DC has in common with BD. Then BE (on BD) + EC (on DC) is an arc BC every point of which, except C, is in R and no point of which, except B, belongs to $OA + J + I$. The points A and B divide the curve J into two arcs AF_1B and AH_1B. There exists, about F_1, a region R_1 which contains no point of the closed point-set $OA + AH_1B + BC + (S - R)$. By Theorem 20, R_1 contains a point L not in $J + I$. There exists in R an arc LF_1. There exists a point F which is the first point that LF_1 has in common with J. By Theorem 33 there is, in R, an arc LH_1 which has no point in common with $OA + AF_1B + BC$. There is a point H which is the first point that LH_1 has in common with J. The point-set LF (on LF_1) $+ LH$ (on LH_1) contains as a subset an arc FZH which, except for its endpoints, is a subset of $R - (J + I)$. By Theorem 21 the interior of the closed curve $ZHAFZ$ is in R. Hence C is without $ZHAFZ$. Therefore if B were within $ZHAFZ$, $\underline{BC}$ would have a point in common with $ZHAFZ$ and therefore a point in common with LF_1 or LH_1 or J. It follows that B is without $ZHAFZ$. Hence, by Theorem 27, A is within the closed curve $ZHBFZ$. But the arc OA contains no point of $ZHBFZ$. Hence O is within $ZHBFZ$.

THEOREM 37. *If J and C are simple closed curves, O is a point on J but not on C, A_1 and A_2 are distinct points common to C and J and A_1XA_2 is an arc on C such that $\underline{A_1XA_2}$ lies within J then there exist two points O_1 and O_2 distinct from O such that*

(1) *O_1 and O_2 lie on the arcs* A_1O and A_2O respectively,*

(2) *there is on the curve C an arc O_1YO_2 such that $\underline{O_1YO_2}$ is within J,*

(3) *if B_1 and B_2 are points on the arcs* O_1O and O_2O respectively such that there exists on C, from B_1 to B_2, an arc which, except for its endpoints, lies entirely within J, then $B_1 = O_1$ and $B_2 = O_2$.*

Proof. Let M_1 be the set of all points $[P_1]$ on A_1O such that P_1 can be joined to some point of A_2O by an arc of C which, except for its endpoints, lies within J. Let M_2 be the set of all points $[P_2]$ on A_2O such that P_2 can be joined to some point X of M_1 by an arc of C which, except for its endpoints, lies within J, but such that no point between P_2 and O on A_2O can be so joined to the same point X. Suppose that Z_1 and W_1 are two distinct points of M_1, Z_2 and W_2 are points of M_2 and Z_1Z_2 and W_1W_2 are arcs such that $\underline{Z_1Z_2}$ and $\underline{W_1W_2}$ are within J. The arcs Z_1Z_2 and W_1W_2 can have no point in common.† Hence, in view of Theorem 29, it is clear that if W_1 is between Z_1 and O on the arc A_1O then W_2 must be between Z_2 and O on the arc A_2O.

If Theorem 37 is false there exists a point L_1 on A_1O and a point L_2 on A_2O such that L_1 is a limit point of M_1, L_2 is a limit point of M_2, but $\underline{L_1OL_2}$

* Here A_1O, A_2O, O_1O, and O_2O denote intervals of the arc A_1OA_2 on J.

† Two arcs of a closed curve can have no point in common unless one of them contains an endpoint of the other one.

contains no point either of M_1 or of M_2. On the curve C there exists a point E such that, on C, A_1 and L_2 separate L_1 from E. On the C-arc $A_1 L_1 L_2 E$ there exist points F and G in the order $L_1 FGL_2$. Since L_1 and L_2 are limit points of M_1 and M_2 respectively therefore M_1 contains two distinct points P_1 and $\bar{P}_1$ and M_2 contains two distinct points P_2 and $\bar{P}_2$ such that (1) on the arc $A_1 L_1 L_2 E$, P_1 and $\bar{P}_1$ lie between A_1 and F while P_2 and $\bar{P}_2$ lie between E and G, (2) there exist, on C, arcs $P_1 P_2$ and $\bar{P}_1 \bar{P}_2$ such that $\underline{P_1 P_2}$ and $\underline{\bar{P}_1 \bar{P}_2}$ lie entirely within J.

Since A_1 is on J, neither $P_1 P_2$ nor $\bar{P}_1 \bar{P}_2$ can contain A_1. Hence both of these arcs contain F and G. But it has been previously established that $P_1 P_2$ and $\bar{P}_1 \bar{P}_2$ have no point in common. Thus the supposition that Theorem 37 is false leads to a contradiction.

THEOREM 38. *If J and C are closed curves, O is a point on J but not on C, A_1 and A_2 are distinct points common to C and J, and $A_1 XA_2$ is an arc on C such that the interior of the closed curve $A_1 XA_2 OA_1$ is a subset of the exterior of J then there exist two points O_1 and O_2, distinct from O, such that* (1), *on the interval $O_1 OO_2$ of the curve J, O is between O_1 and A_2 and between O_2 and A_1,* (2) *there is on the curve C an arc $O_1 YO_2$ such that $\underline{O_1 YO_2}$ and the interior of the closed curve $O_1 YO_2 OO_1$ are subsets of the exterior of J,* (3) *if the exterior of J contains $\underline{B_1 ZB_2}$ and the interior of the closed curve $B_1 ZB_2 OB_1$ where B_1 and B_2 are points on that arc of J from O_1 to O_2 which does not contain O, and $B_1 ZB_2$ is an arc of C, then $B_1 = O_1$ and $B_2 = O_2$.*

THEOREM 39. *If O is a point on a closed curve J then every point not on J can be joined to O by an arc having no point except O in common with J.*

Proof. It may be easily shown with the help of Theorems 5 and 36 that there exists a sequence of closed Jordan curves $J_1, J_2, \cdots$, such that (1) the interior of J_1 does not contain I, the interior of J, (2) for each n, J_{n+1} lies in I_n, the interior of J_n, (3) O is the only point that the regions $I_1, I_2, \cdots$ have in common, (4) if Q is a region about O there exists n such that I'_n lies in Q.

By Theorem 20, I_n contains at least one point of E, the exterior of J. Hence, by Theorem 18, $E = E_n + \bar{E}_n$ where E_n is a subset of I_n but no point of $\bar{E}_n$ belongs to I_n. By Axiom 3, $\bar{E}_n$ must contain a point P which is a limit point of E_n. Every such point P must belong to J_n. With the use of (1) it may be shown that I contains a point of J_n. Thus J_n contains a point within J as well as a point without J and therefore it must contain at least two points in common with J.

It follows that there exist, on J, two points A and B (Fig. 6), belonging to J_n, such that no point of J_n, except A and B, is on the arc AOB. If X is a point of J_n that lies in I then J_n and J have in common two points $\bar{A}_x$ and $\bar{B}_x$ such that the J_n-arc $\bar{A}_x X\bar{B}_x$ lies (except for its endpoints) in I. By Theorem 37 there exists a J_n-arc $A_x B_x$ such that

(1) A_x is $\bar{A}_x$ or is on J in the order $\bar{B}_x \bar{A}_x A_x O$,

(2) B_x is $\bar{B}_x$ or is on J in the order $\bar{A}_x \bar{B}_x B_x O$,

(3) $A_x B_x$ is in I,

(4) if Y and Z are points on J on the arcs* $A_x O$ and $B_x O$ respectively such that an arc of J_n from Y to Z lies (except for its endpoints) in I, then $Y = A_x$ and $Z = B_x$. It is clear that, for a given n, the arc $A_x B_x$ is completely determined by the point X. For every X on J_n that lies in I construct the corresponding $A_x B_x$. Let $\bar{J}_n$ denote the point-set which is composed of all

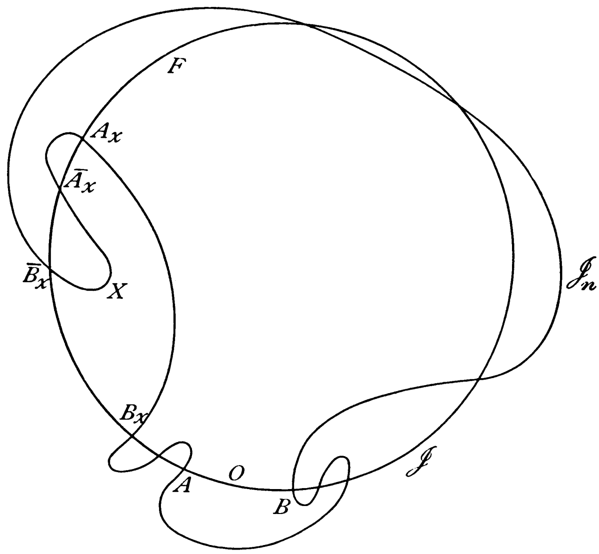

Fig. 6

the $A_x B_x$'s so constructed together with every point F on J which has the property that for no X is F separated from O by A_x and B_x. It may be easily proved that $\bar{J}_n$ is a closed Jordan curve. By Theorem 21, $\bar{I}_n$ (the interior of $\bar{J}_n$) is a subset of I. That $\bar{I}_n$ is a subset of I_n may be proved as follows.

Let F be a point of $\bar{J}_n$ that does not belong to J_n. The curves J_n and J have in common two points C and D such that the J-arc CFD contains no point of J_n. About F and O respectively there exist regions R_1 and R_2 neither of which contains a point of the closed point-set $J_n + (J - \overline{AOB} - \overline{CFD})$. Since F and O are on its boundary the region $\bar{I}_n$ contains a point P_1 in R_1 and a point P_2 in R_2. There exists an arc $P_1 P_2$ lying entirely in $\bar{I}_n$ and therefore having no point in common with J_n. It easily follows† that there exists an arc $T_1 T_2$ such that (1) T_1 is on J and in R_1 and therefore on $\overline{CFD}$, (2) T_2 is on J and in R_2 and therefore on $\overline{AOB}$, (3) $\overline{T_1 T_2}$ is in $\bar{I}_n$. But neither

* Here $A_x O$ and $B_x O$ denote intervals of $\bar{B}_x \bar{A}_x O$ and $\bar{A}_x \bar{B}_x O$ respectively.

† Cf. proof of Theorem 25.

$\underline{AOB}$ nor $\underline{CFD}$ contains a point of J_n. It follows that F can be joined to O by an arc FT_1† + $T_1T_2 + T_2O$* that has no point in common with J_n. But O is within J_n. Hence F also is within J_n. Thus every point of $\bar{J}_n$ is either on or within J_n. Hence $\bar{I}_n$ is a subset of I_n.

For every n, $\bar{I}_{n+1}$ is a proper subset of $\bar{I}_n$. In $\bar{I}_1$ there is a point P_1 not belonging to $\bar{I}_2$. In $\bar{I}_2$ there is a point P_2. There exists an arc P_1P_2 lying in $\bar{I}_1$. Since the regions $\bar{I}_1, \bar{I}_2, \cdots$ have no point in common there exists n_2 such that P_1P_2 contains a point P_{n_2} lying in I_{n_2} but contains no point in I_{n_2+1}. In I_{n_2+1} there is a point P_{n_2+1}. There exists an arc $P_{n_2}P_{n_2+1}$ lying in I_{n_2}. Let $\bar{P}_{n_2}$ denote the first point that P_1P_2 has in common with $P_{n_2}P_{n_2+1}$. Similarly there exists n_3 such that $P_{n_2}P_{n_2+1}$ contains a point P_{n_3} in I_{n_3} but no point in I_{n_3+1}. There exists in I_{n_3+1} a point P_{n_3+1} and in I_{n_3} an arc $P_{n_3}P_{n_3+1}$. Let $\bar{P}_{n_3}$ denote the first point that $P_{n_2}P_{n_2+1}$ has in common with $P_{n_3}P_{n_3+1}$. Continue this process. In general, if $k > 1$, there exists n_{k+1} such that $P_{n_k}P_{n_k+1}$ contains a point $P_{n_{k+1}}$ in $I_{n_{k+1}}$ but no point in $I_{n_{k+1}+1}$. There exists in $I_{n_{k+1}+1}$ a point $P_{n_{k+1}+1}$ and in $I_{n_{k+1}}$ an arc $P_{n_{k+1}}P_{n_{k+1}+1}$. The first point that $P_{n_k}P_{n_k+1}$ has in common with $P_{n_{k+1}}P_{n_{k+1}+1}$ is denoted by $\bar{P}_{n_{k+1}}$. It may be easily proved that the point-set $O + P_1\bar{P}_{n_2}$ (on P_1P_2) $+ \bar{P}_{n_2}\bar{P}_{n_3}$ (on $P_{n_2}P_{n_2+1}$) $+ \bar{P}_{n_3}\bar{P}_{n_4}$ (on $P_{n_3}P_{n_3+1}$) $+ \cdots$ is an arc from P_1 to O. This arc P_1O lies in I and has no point except O in common with J. If P is any point in I other than P_1 there exists an arc PP_1 lying in I and the point-set $PP_1 + P_1O$ contains as a subset an arc PO which has no point except O in common with J.

That a point without J can be so joined to any point of J may be proved in a similar manner.

THEOREM 40. *If A and B are two points on a Jordan curve J, A and B can be joined by Jordan arcs AXB and AYB such that $\underline{AXB}$ is within J and $\underline{AYB}$ is without J.*

THEOREM 41. *If J_1 and J_2 are two closed Jordan curves whose interiors I_1 and I_2 have a point O in common then there exists a Jordan curve J, every point of which belongs to either J_1 or J_2, such that $I_1 + I_2$ is a subset of I, the interior of J.*

Proof. Case I. Suppose J_1 and J_2 have not more than one point in common. Then, since I_1 and I_2 have a point in common, it easily follows that either I_1 is in I_2 or I_2 is in I_1. In the former case J is J_2. In the latter case it is J_1.

Case II. Suppose J_1 and J_2 (Fig. 7) have at least two points in common. If every point of J_1 is within or on J_2 then J is J_2. Suppose that at least one point P of J_1 is without J_2 and at least one point of J_2 is without J_1. Then there exist two points $\bar{A}$ and $\bar{B}$ common to J_1 and J_2 such that the arc $\bar{A}P\bar{B}$

* Here FT_1 and T_2O denote intervals of CFD and AOB respectively.

of J_1 lies (except for its endpoints) entirely without J_2. The points $\bar{A}$ and $\bar{B}$ divide J_2 into two arcs. By Theorem 27, one of these arcs, $\bar{A}C\bar{B}$, is such that the curve $P\bar{B}C\bar{A}$ contains I_2. Let $\bar{A}D\bar{B}$ denote the other arc of J_2 from $\bar{A}$ to $\bar{B}$. By Theorem 38 there exist, on the arc $\bar{B}C\bar{A}$, points A_P and B_P such that $\bar{A}$ and A_P do not separate $\bar{B}$ from B_P and such that (1) there exists

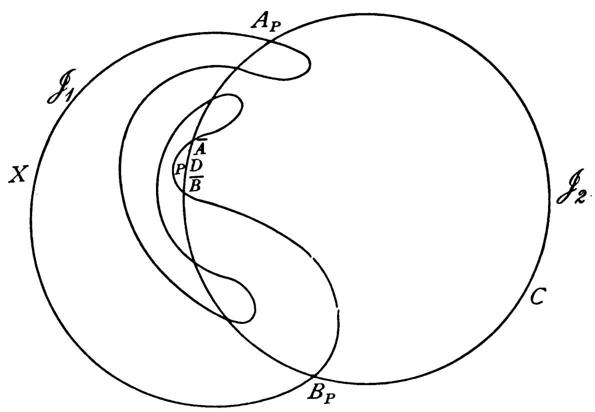

FIG. 7

an interval $A_P XB_P$ of J_1 lying entirely without J_2 except for its endpoints and such that if $\bar{C}$ is a point on J_2 that is separated from D by A_P and B_P then $A_P XB_P \bar{C} A_P$ includes I_2 within it, (2) if Z and W are points on $A_P \bar{C}$ and $B_P \bar{C}$ respectively and there is, on J_1, an arc ZVW lying entirely without J_2 except for its endpoints and such that $ZVWCZ$ includes I_2 within it then $Z = A_P$ and $W = B_P$. The curve $A_P XB_P \bar{C} A_P$ contains within it or on it every point of the curve $P\bar{A}A_P \bar{C} B_P \bar{B} P$ and every point of the curve J_2. For every point P which is on J_1 but without J_2 construct the corresponding arc $A_P XB_P$. Call this arc t_P and let k_P denote that interval of J_2 from A_P to B_P which has the property that the interior of the closed curve $t_P + k_P$ is entirely without J_2. Let J denote the point-set composed of all such arcs t_P together with the set of all points $[Y]$ on J_2 such that for no P is Y on the arc k_P. It may be proved that J is a closed curve satisfying the requirements of Theorem 41.

THEOREM 42. *If the sum of the interiors of a finite set G of closed curves is a connected point-set then there exists a closed curve J such that* (1) *every point of J belongs to some curve of the set G*, (2) *the interior of J contains the interiors of all the curves of the set G.*

THEOREM 43. *If the point O of the closed curve J is within every curve of the finite set of closed curves G and the interior of J is not a subset of the interior of any curve of the set G then there exist curves J_1 and J_2 containing O such that* (1) *every point on J_1 or on J_2 belongs to J or to some curve of the set G*, (2) *every* *point within* J_1 *is without* J *while every point within* J_2 *is within* J, (3) *every point within* J_1 *or within* J_2 *is within every curve of the set* G.

THEOREM 44. *If the point* O *of the closed curve* J *is without every curve of the finite set of closed curves* G *and every curve of* G *has at least one point in common with the* $\left\{\begin{matrix}\textit{interior}\\ \textit{exterior}\end{matrix}\right\}$ *of* J *then there exists a closed curve* $\bar{J}$ *containing* O *and such that* (1) *every point of* $\bar{J}$ *belongs either to* J *or to some curve of the set* G, (2) *every point* $\left\{\begin{matrix}\textit{without}\\ \textit{within}\end{matrix}\right\}$ $\bar{J}$ *is* $\left\{\begin{matrix}\textit{without}\\ \textit{within}\end{matrix}\right\}$ J *and without every curve of the set* G.

THEOREM 45. *If* O *is a point on a closed curve* J *and* $\bar{R}$ *is a region containing* O *then there exists an arc* AXB *such that* (1) A *and* B *are on* J, (2) *the segment* AOB *of the curve* J *is in* $\bar{R}$, (3) $\overline{AXB}$ *and the interior of* $AOBXA$ *are in* $\bar{R}$ *and without* J.

Proof. On J there is a point P different from O. There exists about O and within $\bar{R}$ a region $\bar{K}$ that does not contain P. By Theorem 36, there is in $\bar{K}$ a closed curve $\bar{J}$ enclosing O. There exist on J two points A and B such that the segment AOB of the curve J is within $\bar{J}$. By Theorem 40 there exists an arc AYB such that $\overline{AYB}$ is without J. It easily follows with the aid of Theorem 27 that either the interior or the exterior of $AYBOA$ is a subset of the exterior of J. Hence by Theorems 21 and 43 there exists an arc AXB such that $\overline{AXB}$ and the interior of $AXBOA$ are in $\bar{R}$ and without J.

THEOREM 46. *If the closed curve* J *has no point in common with the closed set of points* K *then there exist closed curves* J_1 *and* J_2 *such that* J *lies between** J_1 *and* J_2 *but no point of* K *lies between* J_1 *and* J_2.

Proof. If P is a point of J there exists, about P, a region $\bar{R}$ containing no point of the closed point-set K. By Theorem 45 there exist two points A and B and an arc AXB such that (1) A and B are on J, (2) the segment APB of the curve J is in $\bar{R}$, (3) $\overline{AXB}$ and the interior of $APBXA$ are in $\bar{R}$ but without J. It easily follows with the aid of the Heine-Borel Theorem (as applied to a set of segments covering a closed curve) that there exist a finite set of J − arcs $A_1 B_1$, $A_2 B_2$, $\cdots$ $A_n B_n$, and associated closed curves $\bar{J}_1$, $\bar{J}_2$, $\bar{J}_3$, $\cdots$ $\bar{J}_n$, such that

(1) $A_i B_i$ contains a segment in common with $A_{i-1} B_{i-1}$† and a segment in common with $A_{i+1} B_{i+1}$, but contains no point in common with any other $A_j B_j$,

(2) the curves $\bar{J}_k$ and J contain only the arc $A_k B_k$ in common,

(3) the interior of $\bar{J}_k$ is without J,

(4) neither $\bar{J}_k$ nor its interior contains any point of the set K.

* A point-set M is said to lie between two closed curves if one of these curves lies within the other one and M lies without the first one and within the second one.

† It is understood throughout this argument that the subscripts of the A's and B's are to be reduced modulo n.

Let $\bar{G}$ denote the set of curves $\bar{J}_1, \bar{J}_2, \bar{J}_3, \cdots \bar{J}_n$ and let U denote the set of arcs $A_1 B_1, A_2 B_2, \cdots A_n B_n$. The arc $A_k B_k$ contains a point O_k which belongs to no other arc of the set U. By Theorem 44 there exists a closed curve C_k, containing O_k, such that every point of C_k belongs to some curve of the set $\bar{G}$ and such that every point within C_k is within $\bar{J}_k$ but without every other curve of the set $\bar{G}$. The arcs $A_k B_k$ and $A_{k+1} B_{k+1}$ have in common an arc $A_{k+1} P_k B_k$. By Theorem 40 there exists an arc $A_k \bar{P}_k B_k$ such that $A_k \bar{P}_k B_k$ is a subset of I (the interior of J). The point-set $(\bar{J}_k - \overline{A_k O_k B_k}) + A_k \bar{P}_k B_k$ is a closed curve containing in its interior the point P_k of the closed curve $\bar{J}_{k+1}$. With the assistance first of Theorem 43 and afterwards of Theorem 44 it may be easily proved that there exists a closed curve $\bar{C}_k$, containing P_k, such that (1) every point of $\bar{C}_k$ belongs to some curve of the set $\bar{G}$, (2) every point within $\bar{C}_k$ is within $\bar{J}_k$ and within $\bar{J}_{k+1}$ but without every other curve of the set $\bar{G}$. It is clear that the curves C_k and J have only the arc $B_{k-1} O_k A_{k+1}$ in common while the curves $\bar{C}_k$ and J have only the arc $A_{k+1} P_k B_k$ in common. It is clear furthermore that C_k and $\bar{C}_k$ have in common an arc $D_k A_{k+1}$ which has no point except A_{k+1} in common with J. The curves C_k and $\bar{C}_{k-1}$ have in common an arc $E_k B_{k-1}$ which has no point except B_{k-1} in common with J. By Theorem 40 there exist arcs $E_k D_k$ and $D_k E_{k+1}$ such that $\underline{E_k D_k}$ lies within C_k and $\underline{D_k E_{k+1}}$ lies within $\bar{C}_k$. Let J_1 denote the point-set

$$E_1 D_1 + D_1 E_2 + E_2 D_2 + D_2 E_3 + \cdots + E_n D_n + D_n E_1 .$$

It can be proved with the help of Theorem 27 that J_1 is a closed curve enclosing J and such that no point of K is between J_1 and J.

Similarly there exists a closed curve J_2 lying within J and such that no point of K is between J and J_2.

THEOREM 47. *If the closed curve J_1 encloses the closed curve J_2 then the set of all points between J_1 and J_2 is connected.*

Proof. Suppose A and B are two points between J_1 and J_2. Let R_2 denote the interior of J_2. By Theorem 46 there exists within J_1 a closed curve J such that there is no point of the closed point-set $A + B + J_2 + R_2$ between J_1 and J. By Theorem 16 there exists an arc AXB in the exterior of J_2. The point-set $J + AXB$ contains as a subset an arc AYB that lies between J_1 and J_2. It follows that the set of all points between J_1 and J_2 is connected.

THEOREM 48.* *Suppose that K is a closed, bounded set of points and that $S - K = S_1 + S_2$ where S_1 and S_2 are point-sets such that* (1) *every two points of S_i $(i = 1, 2)$ can be joined by an arc lying entirely in S_i,* (2) *every arc joining*

* Cf. A. Schoenflies, *Ueber einen grundlegenden Satz der Analysis Situs*, Nachrichten der Göttinger Gesellschaft der Wissenschaften, 1902, p. 185.

a point of S_1 to a point of S_2 contains a point of K, (3) *if O is a point of K and P is a point not belonging to K then P can be joined to O by an arc that has no point except O in common with K. Every point-set K that satisfies these conditions is a simple closed curve.*

Theorem 48 can be proved on the basis of the preceding theorems by an argument in large part similar to that employed by Lennes.* He makes use of straight lines and polygons but an argument that is in large part similar can be carried through with the use of arcs and closed curves. Schoenflies uses metrical properties.

DEFINITION. An *open curve* is a closed, connected set of points M such that if P is a point of M then $M-P$ is the sum of two mutually exclusive connected point-sets, neither of which contains a limit point of the other one.†

It is easy to see that every open curve l satisfies Axioms 1–4, 5′, 6, and 8 of my papers *The Linear Continuum in Terms of Point and Limit*‡ and *On the Linear Continuum*.§ It follows that Theorems 1–11 of I and Theorem E of II hold true on l.

THEOREM 49. *If A and B are distinct points on the open curve l, then the point-set t composed of A, B, and the segment*|| *AB of l is a simple continuous arc from A to B.*

Proof. Clearly t is closed and connected and contains no connected proper subset that contains A and B. Suppose t is not bounded. Then l contains two points X and Y such that if Z is any point on the ray XY then the segment XZ is not bounded. There exists a closed curve J enclosing X. There exists on the ray XY a countably infinite sequence of points $P_1, P_2, P_3, \cdots$ that has X as a sequential limit point. The segment XP_n contains a point without J. It follows that it has a point $\bar{P}_n$ in common with J. There exists, on J, a point O which is a limit point of $\bar{P}_1 + \bar{P}_2 + \bar{P}_3 + \cdots$. But it is easy to see that X is the only limit point of $\bar{P}_1 + \bar{P}_2 + \bar{P}_3 + \cdots$. Thus the supposition that t is not bounded leads to a contradiction. Hence t satisfies all the requirements of Lennes' definition of a continuous simple arc from A to B.

THEOREM 50. *Every ray of an open curve contains a set of points that has no limit point.*

Proof. Suppose that every infinite subset of the ray OA of the open curve l

* Loc. cit., § 5.

† If P is a point of the open curve M and $M - P$ is the sum of two connected point-sets M_1 and M_2, then M_1 and M_2 are called *rays*.

‡ Annals of Mathematics, vol. 16 (1915), pp. 123–133. This paper will be referred to as I.

§ Bulletin of the American Mathematical Society, vol. 22 (1915), pp. 117–122. This paper will be referred to as II.

|| For definition of segment see II, p. 120.

has a limit point. Then the closed set of points $OA + O$ possesses the Heine-Borel property. There exists on l a point D in the order AOD. If P is a point of the ray OA there exists a point B in the order OPB. There exists about P a region R containing no point of l that is not on the segment DB. It follows that $OA+O$ is covered by a finite set of regions $R_1, R_2, R_3, \cdots R_n$. But for each R_n there is a point B_n such that every point that R_n has in common with l lies between D and B_n. It follows that there exists on OA a point E such that no region of the set $R_1, R_2, \cdots R_n$ contains any point of l that does not lie between O and E. But OA contains points that do not lie between O and E. Thus the supposition that l is compact leads to a contradiction.

Theorem 51. *If l is an open curve then $S - l = S_1 + S_2$ where S_1 and S_2 are connected point-sets such that every arc from a point of S_1 to a point of S_2 contains at least one point of l.*

Proof. There exists on l an arc $\bar{A}O\bar{B}$. There exists about O a region $\bar{R}$ containing neither $\bar{A}$ nor $\bar{C}$. There exists an arc $\bar{A}\bar{X}\bar{B}$ that contains no point of $\bar{R}$. There exist points A and B such that A is the last point that $\bar{A}\bar{X}\bar{B}$ has in common with the ray $O\bar{A}$ while B is the first point following A that $\bar{A}\bar{X}\bar{B}$ has in common with the ray OB. The arc $\bar{A}\bar{X}\bar{B}$ contains a segment AXB. This segment has no point except A and B in common with l.

On l there exist points $\bar{C}$ and $\bar{D}$ in the order $\bar{C}AOB\bar{D}$.

$$l = \underline{AOB} + \text{ray } B\bar{D} + \text{ray } A\bar{C}.$$

The ray $A\bar{C}$ is not bounded. Hence it contains at least one point without the curve $AOBXA$. Hence if the ray $A\bar{C}$ contains a point within $AOBXA$ it must contain a point on $AOBXA$. But this is contrary to hypothesis. It follows that the interior of $AOBXA$ contains no point of l. About O there is a region which contains no point of AXB + ray $A\bar{C}$ + ray $B\bar{D}$. In this region there is an arc CYD such that C is on $\underline{OA}$, D is on $\underline{OB}$, and $\underline{CYD}$ is entirely without $AOBXA$. By Theorem 40 there exists an arc CZD such that $\overline{CZD}$ is within $AOBXA$. The closed curve $CYDZC$ contains no point of l except points of $\underline{COD}$. The interior of $CYDZC = \underline{COD}$ + the interior of $CODZC$ + the interior of $CODYC$. Let O_1 and O_2 denote definite points within $CODZC$ and within $CODYC$ respectively. Let S_1 denote the set of all points $[P_1]$ such that P_1 can be joined to O_1 by an arc containing no point of l. Let S_2 denote the set of all points $[P_2]$ such that P_2 can be so joined to O_2. Clearly S_1 and S_2 are connected. It remains to show (1) that every arc joining a point of S_1 to a point of S_2 contains at least one point of l, (2) that every point of $S - l$ belongs either to S_1 or to S_2.

(1) If a point of S_1 can be joined to a point of S_2 by an arc that contains no point of l then O_1 can be joined to O_2 by such an arc.

Suppose $O_1 W O_2$ is an arc from O_1 to O_2. Then there exist points $\overline{O}_1$ and $\overline{O}_2$ such that $\overline{O}_1$ is the last point that $O_1 W O_2$ has in common with $\overline{CZD}$ while $\overline{O}_2$ is the first point following $\overline{O}_1$ that $O_1 W O_2$ has in common with $\overline{CYD}$. The arc $O_1 W O_2$ contains a segment $\overline{O}_1 \overline{W} \overline{O}_2$ which lies entirely without $CYDZC$. By Theorem 25 the interior of $CYDZC$ lies either within $\overline{O}_1 \overline{W} \overline{O}_2 C \overline{O}_1$ or within $\overline{O}_1 \overline{W} \overline{O}_2 D \overline{O}_1$. Suppose it lies within the former. Then the ray OD contains a point D within $\overline{O}_1 \overline{W} \overline{O}_2 C \overline{O}_1$. Hence it contains a point on $\overline{O}_1 \overline{W} \overline{O}_2 C \overline{O}_1$. But the arc $\overline{O}_2 C \overline{O}_1$ contains no point of the ray OD. Hence $\overline{O}_1 \overline{W} \overline{O}_2$ must contain a point of this ray.

(2) Suppose P is a point not lying on l. There exists an arc PO. Let G denote the first point that PO has in common with l. There exist (Fig. 8) arcs KLM and KNM having no point in common except K and M and such that K and M are the only points that the closed curve $KLMNK$ has in common with l while the interior of $KLMNK = \overline{KGM} +$ the interior of $KGMLK +$ the interior of $GKMNK$. It can* be proved that there exist

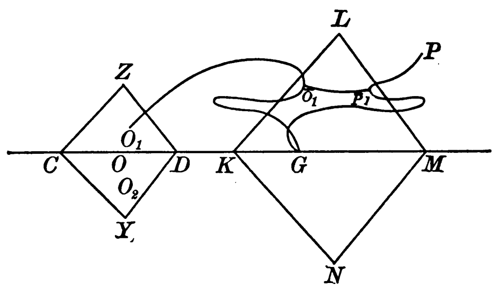

FIG. 8

arcs $O_1 G$ and $O_2 G$ neither of which contains any point except G in common with l. The arc $O_1 G$ contains either a point within $KGMLK$ or a point within $KGMNK$. Suppose it contains a point $\overline{O}_1$ within $KGMLK$. Then $O_2 G$ cannot contain a point within $KGMLK$. For if it did then O_1 and O_2 could be joined by an arc containing no point of l. Hence $O_2 G$ must contain a point $\overline{O}_2$ within $KGMNK$. But PG must contain a point within $KGMLK$ or a point within $KGMNK$. Suppose it contains a point $\overline{P}_1$ within $KGMLK$. Then there exists an arc $\overline{P}_1 \overline{O}_1$ lying entirely within $KGMLK$ and therefore containing no point of l. The point-set $PG + \overline{P}_1 \overline{O}_1 + O_1 G$ contains as a subset an arc PO_1 which contains no point of l. Hence in this case P belongs to S_1. If PG contains a point within $KGMNK$ then P belongs to S_2. Thus every point of $S - l$ belongs either to S_1 or to S_2.

THEOREM 52. *If there exists in S a system of open curves such that through every two points of S there is one and only one curve of this system then there is a one-to-one continuous correspondence between S and an ordinary number plane.*

* Cf. proofs of Theorems 46 and 32.

Theorem 52 may be proved with the assistance of Theorem 51 of the present paper together with results obtained in my paper *On a set of postulates which suffice to define a number-plane.**

8. Independence examples

Let Σ_1 denote the set of Axioms 1–8. In the following, E_i† denotes an example of a system in which Axiom i is false but all the other axioms of the set Σ_1 are true. In each example E_i use is made of a well-defined space δ_i. In every case the *points* of E_i are the ordinary points of S_i but the *regions* of the various E_i's are defined in various ways. For every i, except 1 and 8, S_i is an ordinary euclidean space ot two dimensions.

E_1. S_1 is the S_1 described in example E_{1567} of my paper *The linear continuum in terms of point and limit.*‡ The point $(x_1, y_1; x_2, y_2)$ is a *limit point* of the point-set M it and only if Condition I§ is fulfilled. *Starting with this definition of limit point,* one may define the terms *arc* and *closed curve* as in § 4. It can be shown that every such closed curve J divides S_1 into two subsets I and E such that every infinite set of points in I has at least one limit point. The point set I is called the *interior* of J. Finally a *region* is defined as the interior of a closed curve.

E_2. A set ot points M is a *region* if and only if M is either an ordinary Jordan region or the sum of two ordinary Jordan regions R_1 and R_2 such that R_1' and R_2' have no point in common.

E_3. A set of points M is a *region* it and only if M is either an ordinary Jordan region or the set of all points lying between two Jordan curves one of which encloses the other one.

E_3'. S_3' is the set of all points on an ordinary straight line. A *region* is a segment.

E_4. A set of points M is a *region* if and only it M is either a half-plane or an ordinary Jordan region.

E_5. S_5 is an ordinary sphere. The point-set M is a *region* if and only if M is one of the two parts into which S_5 is divided by an ordinary closed curve.

E_6. Choose a set of rectangular axes OX and OY. Let k_1 denote the closed interval from $(0, -1)$ to $(0, 1)$ together with the point $(1/\pi, 0)$ and all points of $y = \sin 1/x$ that lie between $x = 0$ and $x = 1/\pi$. Let k_2 denote some definite arc that joins the point $(1/\pi, 0)$ to the point $(0, 1)$, contains no point of $y = \sin 1/x$, except the point $(1/\pi, 0)$, and lies, except for its endpoints, entirely in Quadrant I. A point-set M is a *region* if and only if M

* These Transactions, vol. 16 (1915), pp. 27–32.

† For Axiom 3 I give two independence examples, E_3 and E_3'.

‡ Loc. cit., page 131.

§ Loc. cit.

is either an ordinary Jordan region or the set of points enclosed by $k_1 + k_2$.

E_7. This example is the same as E_6 except that k_2 is replaced by an arc from $(1/\pi, 0)$ to $(0, 1)$ that contains no point of $y = \sin 1/x$, except the point $(1/\pi, 0)$, and furthermore contains no point in Quadrant I.

E_8. S_8 is an ordinary euclidean space of three dimensions. A set of points is a *region* if and only if it is the interior of a cube.

9. Concerning Σ_1, Σ_2, and Σ_3

Let Σ_2 denote the set of Axioms 1–5, 6′, 7′, 8, where 6′ and 7′ are as follows:

Axiom 6′.* *If R is a region and AB is an arc such that $AB - A$ is a subset of R then $(R + A) - AB$ is connected.*

Axiom 7′.† *Every boundary point of a region is a limit point of the exterior of that region.*

Theorem A. *In a space satisfying Σ_1 every region is the interior of a closed curve.*

Theorem A may be proved with the assistance, in particular, of Axioms 6 and 7 and Theorem 48.

Theorems 1–45 *are consequences of Σ_2 as well as of Σ_1.* It is not true however that in every space satisfying Σ_2 every region is the interior of a simple closed curve. Indeed Σ_2 is satisfied if in an ordinary euclidean plane the term region is applied to every bounded, connected set of points R, of connected exterior, such that every point of R is interior to some triangle that lies in R. It is easy however to show that *though Σ_1 and Σ_2 are not absolutely equivalent, they are equivalent with respect to*‡ *point and limit point of a point-set as defined in* § 2.

Let Σ_3 denote the set of Axioms 1′, 2′, 3, 4, 5, 6′, 7′, and 8, where 1′ and 2′ are as follows:

Axiom 1′. *If P is a point, there exists an infinite sequence of regions R_1, R_2, R_3, $\cdots$ such that* (1) *P is the only point they have in common,* (2) *for every n, R_{n+1} is a proper subset of R_n,* (3) *if R is a region about P then there exists n such that R'_n is a subset of R.*

Axiom 2′. *Every two points of a region are the extremities of at least one simple continuous arc that lies wholly in that region.*

* Cf. Theorems 31 and 32.

† Cf. Theorem 20.

‡ The statement that Σ_1 and Σ_2 are equivalent with respect to point and limit point as defined in § 2 signifies that every statement in terms of point and limit point of a point-set that follows from Σ_1 (together with the above mentioned definition of limit point of a point-set) follows also from Σ_2 (together with that definition) and conversely. The statement that Σ_1 and Σ_2 are not absolutely equivalent signifies that they are not equivalent with respect to point and region, the undefined symbols in terms of which they are both stated.

Theorems 1–52 are all consequences of Σ_3. Nevertheless there exist spaces (see for instance Example E_1 of § 8) that satisfy Σ_3 but are neither metrical, descriptive, nor separable. If however there be added to Σ_3 the axiom that there exists a system of open curves such that through every two points there is one and only one curve of this system, the resulting set of axioms is potentially metrical and, indeed, is categorical with respect to *point* and *limit point of a point-set*. See Theorem 52.

UNIVERSITY OF PENNSYLVANIA