JACOBI'S CONDITION FOR PROBLEMS OF THE CALCULUS OF VARIATIONS IN PARAMETRIC FORM*

BY

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There are two well-known methods of deducing Jacobi's necessary condition in the calculus of variations. One is geometric in character, depending upon a property of an envelope of a one-parameter family of extremals through a fixed point, the cases when the envelope has a singular point being usually excluded.† The second proof involves complicated manipulations of the second variation. For the problem in parametric form in the plane the reduction of the second variation was devised by Weierstrass and is a remarkable piece of analysis.‡ It is, however, very artificial and not easily extensible to problems in more than two dimensions. For problems in parametric form in higher spaces a discussion of the second variation has been made by von Escherich§ by methods in part quite unsymmetrical. The lack of symmetry is due to the division of an arc

$$y_i = y_i(t) \qquad (t_1 \leqq t \leqq t_2;\ i = 1, 2, \cdots, n)$$

into a finite number of pieces on each of which one at least of the derivatives $y_i'(t)$ is different from zero, a device which leads to inelegant complications, though his results are symmetric in form.

In the present paper a proof of Jacobi's condition is given which applies with equal simplicity to the parametric case in the plane or in higher spaces, and which when suitably modified can be used with advantage for other problems of the calculus of variations also.‖ It makes use of the second variation without complicated reductions of any sort, is symmetric in all the variables, and includes the exceptional cases of the geometric proof mentioned above.

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† Serious complications are introduced when it is attempted to cover all possibilities. See a remark by Bolza, *Vorlesungen über Variationsrechnung*, p. 634.

‡ See Bolza, loc. cit., p. 224.

§ Sitzungsberichte der kaiserlichen Akademie der Wissenschaften in Wien, vol. 110 (1901), Abtheilung IIa, p. 1355.

‖ See M. B. White, *The dependence of focal points upon curvature for problems of the calculus of variations in space*, these Transactions, vol. 13 (1912), p. 189.

1. NOTATIONS AND PRELIMINARY THEOREMS

The integral to be minimized has the form

$$I = \int f(y_1, \cdots, y_n, y'_1, \cdots, y'_n)\, dt = \int f(y, y')\, dt,$$

where y, y' are symbols for the multipartite numbers

$$y = (y_1, \cdots, y_n), \qquad y' = (y'_1, y'_2, \cdots, y'_n).$$

Every symbol used below represents either a multipartite number or a matrix, with the exception of

$$I, f, t, \kappa, \epsilon, \Omega, \rho, \lambda, \gamma, \delta, T,$$

which are scalars standing for single elements. Thus the expressions f_y, $f_{y'y}$ stand, respectively, for the row of first partial derivatives $\partial f/\partial y_i$ and the matrix of second derivatives $\partial^2 f/\partial y'_i\, \partial y_k$. The products of a scalar by a multipartite number, of two multipartite numbers, or of a matrix with a multipartite number, are illustrated by the three examples

$$\kappa y' = (\kappa y'_1, \kappa y'_2, \cdots, \kappa y'_n), \qquad f_y \eta = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i} \eta_i,$$
$$f_{y'y}\, \eta\eta' = \sum_{i,\, k=1}^{n} \frac{\partial^2 f}{\partial y'_i\, \partial y_k} \eta_k \eta'_i = f_{yy'}\, \eta'\, \eta,$$

from which the meanings of the other products used below will be readily inferred. Two multipartite numbers or matrices are equal if and only if their corresponding elements are equal.*

With these agreements as to notations, let R be a region of $2n$-dimensional points (y, y') with $y' \neq 0$, and having the property that if (y, y') is interior to R so are all of the points $(y, \kappa y')$ for $\kappa > 0$. In order to carry through the following analytical developments and insure the invariance of the integral I under a change of parametric representation, it will be assumed that f is of class C''' † and satisfies the usual homogeneity relation

$$f(y, \kappa y') = \kappa f(y, y') \qquad (\kappa > 0) \tag{1}$$

at all points of R.

The class $\mathfrak{M}$ of admissible arcs is the totality of arcs expressible in the form

$$y = y(t) \qquad (t_1 \leqq t \leqq t_2),$$

* It is not possible to explain here in detail the use of single symbols for multipartite numbers or matrices. The notions used are very simple, however, and are of great assistance in simplifying the equations. For the purposes of the present paper the reader is referred to Bliss, *The solutions of differential equations of the first order as functions of their initial values*, Annals of Mathematics, 2d Series, vol. 6 (1905), p. 58.

† The class of a function is defined as in Bolza, loc. cit., p. 13.

joining two fixed points 1 and 2, continuous, and each consisting of a finite number of arcs of class C' with elements $[y(t), y'(t)]$ all interior to R. The integral I is then well-defined and independent of the parametric representation along every arc of $\mathfrak{M}$. In particular E is an admissible arc

$$(E) \qquad y = e(t) \qquad (t_1 \leqq t \leqq t_2),$$

which minimizes the integral I and whose properties are to be investigated. It will always be understood to be of class C' unless expressly stated otherwise.

Consider the one-parameter family of curves

$$(V) \qquad y = e(t) + \epsilon\eta(t) = v(t, \epsilon) \qquad (t_1 \leqq t \leqq t_2),$$

where $\eta(t)$ represents a set of admissible variations satisfying the conditions

$$(2) \qquad \eta(t) \text{ of class } D' \text{ for } t_1 \leqq t \leqq t_2, \qquad \eta(t_1) = \eta(t_2) = 0.$$

Then the function

$$I(\epsilon) = \int_{t_1}^{t_2} f(v, v')\,dt$$

has the derivatives

$$I'(0) = \int_{t_1}^{t_2} (f_y\,\eta + f_{y'}\,\eta')\,dt,$$

$$(3) \qquad I''(0) = \int_{t_1}^{t_2} (f_{yy}\,\eta\eta + 2f_{y'y}\,\eta\eta' + f_{y'y'}\,\eta'\,\eta')\,dt = \int_{t_1}^{t_2} \Omega(\eta, \eta')\,dt,$$

where the arguments of the derivatives of f are $(y, y') = (e, e')$. It is then easy to argue as usual that the conditions $I'(0) = 0$, $I''(0) \geqq 0$ must be satisfied for every set of admissible variations $\eta(t)$.

From the condition $I'(0) = 0$ it follows by the method of Du Bois Reymond* that there exists a set of constants c such that the equations

$$(4) \qquad f_{y'} = \int_{t_1}^{t} f_y\,dt + c$$

hold at every point of E, even if E has corner points. From these equations we can prove without difficulty the following three theorems:

THEOREM 1. *Along a minimizing arc E the functions $f_{y'}$ are of class C' and the Euler differential equations*

$$(5) \qquad f_y - \frac{d}{dt} f_{y'} = 0$$

are satisfied.

* Bolza, loc. cit., p. 27.

THEOREM 2. *At a corner point 3 of a minimizing arc E, the relations*

$$f_{y'}(t_3 - 0) = f_{y'}(t_3 + 0)$$

must hold.

THEOREM 3. *If E is expressed in terms of the length of arc as parameter, then near every point where the matrix $f_{y'y'}$ is of rank $n-1$ the arc E is of class C''' at least and satisfies the Euler equations* (5) *when differentiated out.**

The relation (1) has a number of important consequences, three of which are expressed by the formulas

$$f = f_{y'}y', \qquad f_y = f_{yy'}y' = y'f_{y'y}, \qquad f_{y'y'}y' = 0. \tag{6}$$

The first of these is found from (1) by differentiating with respect to κ and setting $\kappa = 1$; the last two are found from the first by differentiating with respect to the elements of y and y'.

The third of the above equations shows that the determinant of the matrix $f_{y'y'}$ vanishes identically in R. If this matrix is of rank $n-1$, then the matrices

$$\begin{vmatrix} f_{y'y'} \\ y' \end{vmatrix}, \qquad \begin{vmatrix} f_{y'y'} & y' \\ y' & 0 \end{vmatrix} \tag{7}$$

have ranks n and $n+1$ respectively. The former is found from $f_{y'y'}$ by adjoining the elements of y' as an $(n+1)$th row, while the latter is $f_{y'y'}$ bordered by a row and column as indicated by the notation. Consider the linear equations whose coefficients are the rows of the first matrix. From the third of equations (6) the unique ratios of the solutions of the first n of those equations are the ratios of the elements of y', which clearly cannot satisfy the $(n+1)$th equation since the elements of y' are not all zero. Hence the matrix is necessarily of rank n. Similarly the first n linear equations of the second matrix have solutions with the unique ratios of the $n+1$ elements y', 0, and these cannot satisfy the $(n+1)$th equation. Hence the second matrix is of rank $n+1$. These properties of the matrices (7) are of service in the succeeding pages, but are important also in the determination of the character of the solutions of Euler's equations and in the proof of the Theorem 3.†

* The first two theorems are immediate consequences of equations (4), and they hold quite independently of the homogeneity property (1). The proof of the third may be made as in Mason and Bliss, *The properties of a curve in space which minimizes a definite integral*, these Transactions, vol. 9 (1908), p. 244. The proof there given shows that E is necessarily of class C'', but when f is of class C''' the differentiation may be carried one step farther at least.

†See Mason and Bliss, loc. cit., p. 244.

2. The proof of Jacobi's condition

In order to prove the Jacobi condition it will be assumed, as is customary, that the matrix $f_{y'y'}$ is of rank $n - 1$ at every point of the minimizing arc E,* so that from Theorems 1 and 3 of § 1 the arc E must be a solution of Euler's equations of class C''' at least.

The expression for the second variation $I''(0)$ is an integral involving η, η' just as I involves the variables y and y'. Consequently the condition that $I''(0)$ must be positive suggests at once a new problem of the calculus of variations associated with the integral (3) in the $(n+1)$-dimensional $t\eta$-space. If an arc $\eta(t)$ can be found which satisfies the conditions (2), gives the integral $I''(0)$ its minimum value zero, and which nevertheless does not satisfy the necessary condition analogous to that described in Theorem 2, then it is clear that $I''(0)$ cannot always be positive and E cannot be a minimizing arc for the original integral I.

The Euler differential equations for the new problem in the $t\eta$-space are the n equations

$$J(\eta) = \Omega_\eta - \frac{d}{dt}\Omega_{\eta'} = 0. \tag{8}$$

They are linear and of the second order in η, and are called the Jacobi equations for the arc E of the original problem in the y-space. A *normal solution* of the Jacobi equations is a solution which satisfies identically the relation $y'\eta = 0$ along the arc E.

Lemma 1. *If a normal solution vanishes with its derivatives η' at a particular value of t, then the elements of η are identically zero.*

For every normal solution satisfies with $\lambda \equiv 0$ the $n+1$ equations

$$J(\eta) + y'\lambda'' = 0, \qquad y'''\eta + 2y''\eta' + y'\eta'' = 0, \tag{9}$$

the latter of which is found by differentiating the relation $y'\eta = 0$ twice with respect to t. But these equations are linear in the derivatives η'', λ'' and the determinant of the coefficients of these variables is the second of the expressions (7), different from zero along E by hypothesis. Hence the equations may be put into the normal form by solving for η'', λ''. From the existence theorems for such differential equations it is known that a solution is uniquely determined by the initial values of η, λ and their derivatives at a particular point. If these initial values are zero the only solutions corresponding to them is clearly $\eta = 0$, $\lambda = 0$.

* The condition that $f_{y'y'}$ is of rank $n - 1$ is equivalent to the condition $f_1 \neq 0$, where f_1 is the function defined by the author in the paper, *The Weierstrass E-function for problems of the calculus of variations in space*, these Transactions, vol. 15 (1914), p. 378. See also Bliss, *A note on symmetric matrices*, Annals of Mathematics, vol. 16 (1914), p. 43.

THEOREM 4. *If E is a minimizing arc as described above, then no normal solution $u \not\equiv 0$ of its Jacobi equations can exist satisfying the relations*

$$u(t_1) = u(t_3) = 0$$

for a value t_3 between t_1 and t_2.

To prove this suppose $u(t)$ to be a normal solution with elements vanishing at t_1 and a value t_3 between t_1 and t_2, and select a function η as follows:

$$\eta = u \quad \text{for} \quad t_1 \leqq t \leqq t_3,$$

$$\eta = 0 \quad \text{for} \quad t_3 \leqq t \leqq t_2.$$

Then $\eta(t)$ is an admissible variation giving $I''(0)$ the value

$$I''(0) = \int_{t_1}^{t_3} \Omega(u, u')\,dt = 0.$$

For from the relation

$$u\Omega_u + u'\,\Omega_{u'} = 2\Omega \tag{10}$$

due to the fact that Ω is a homogeneous quadratic form in u, u', it follows from (8) that

$$I''(0) = u\Omega_{u'}\Big|_{t_1}^{t_3} + \int_{t_1}^{t_3} u\left(\Omega_u - \frac{d}{dt}\Omega_{u'}\right)dt = 0.$$

The curve $\eta(t)$ cannot, however, minimize $I''(0)$. For at its corner point t_3 the relation

$$y''\,\eta + y'\,\eta'\,\big|^{t_3-0} = y'\,u'\,\big|^{t_3-0} = 0 \tag{11}$$

holds because u is a normal solution. The corner conditions

$$\Omega_{\eta'}(t_3 - 0) = \Omega_{\eta'}(t_3 + 0)$$

imply the relations

$$\Omega_{\eta'}(t_3 - 0) = f_{y'y'}\,u'\,\big|^{t_3-0} = 0 \tag{12}$$

on account of the definition of η. But the last equations could be satisfied only if u' were zero at t_3, since the matrix of coefficients of the elements of u' in (11) and (12) is exactly the first of the expressions (7), one of whose determinants is different from zero. By Lemma 3, therefore, the normal solution u would in that case vanish identically, which is contrary to the hypothesis of the theorem.

DEFINITION. *A point 3 conjugate to 1 on E is a point for which there exists a normal solution $u \not\equiv 0$ of Jacobi's equations such that $u(t_1) = u(t_3) = 0$.*

The Jacobi condition may now be stated in the form usually given:

JACOBI'S NECESSARY CONDITION. *If E is a minimizing arc of class C' at every point of which the matrix $f_{y'y'}$ is of rank $n - 1$, then no point 3 conjugate to 1 can lie between 1 and 2 on E.*

3. Properties of solutions of Jacobi's equations

With the help of two fundamental properties of quadratic forms, which when applied to Ω are expressible by (10) and the equation

$$\eta\Omega_u + \eta'\,\Omega_{u'} = u\Omega_\eta + u'\,\Omega_{\eta'}, \tag{13}$$

it is possible to deduce a number of important properties of the Jacobi equations and their solutions. It is understood that the arguments in the derivatives of f in the coefficients of these equations are always the values y, y' belonging to a solution E of class C''' of Euler's equations along which the matrix $f_{y'y'}$ is of rank $n-1$.

Lemma 2. *The Jacobi equations satisfy the relation*

$$y'\,J(\eta) = 0 \tag{14}$$

identically in t, η, η', η'', and hence are not independent.

For from the definition of Ω

$$\Omega_\eta = 2\,(f_{yy}\,\eta + f_{yy'}\,\eta'), \qquad \Omega_{\eta'} = 2\,(f_{y'y}\,\eta + f_{y'y'}\,\eta'), \tag{15}$$

and with the help of (6), and (5), when $\eta = y'$, $\eta' = y''$,

$$\Omega_{y'} = 2\,\frac{d}{dt}f_y, \qquad \Omega_{y''} = 2\,\frac{d}{dt}f_{y'} = 2f_y, \qquad y'\,\Omega_{\eta'} = 2f_y\,\eta. \tag{16}$$

Hence by (8), (13), and (16),

$$\begin{aligned} y'\,J(\eta) &= y'\,\Omega_\eta + y''\,\Omega_{\eta'} - \frac{d}{dt}\,y'\,\Omega_{\eta'} \\ &= \eta\Omega_{y'} + \eta'\,\Omega_{y''} - \frac{d}{dt}\,y'\,\Omega_{\eta'} \\ &= 2\,\frac{d}{dt}\,\eta f_y - 2\,\frac{d}{dt}\,f_y\,\eta = 0. \end{aligned}$$

Lemma 3. *The Jacobi equations are satisfied identically by the functions $\eta = \rho y'$, where ρ is an arbitrarily selected function of t of class C'.*

For from the expressions (15) and equations (6) and (8),

$$\Omega_\eta\,(\rho y',\, \rho y'' + \rho'\,y') = 2\,(\rho f_{yy}\,y' + \rho f_{yy'}\,y'' + \rho'\,f_{yy'}\,y') = 2\,\frac{d}{dt}\,\rho f_y,$$

$$\Omega_{\eta'}(\rho y',\, \rho y'' + \rho'\,y') = 2\,(\rho f_{y'y}\,y' + \rho f_{y'y'}\,y'' + \rho'\,f_{y'y'}\,y') = 2\rho\,\frac{d}{dt}\,f_{y'} = 2\rho f_y.$$

Lemma 4. *To every solution η of the Jacobi equations there corresponds a unique normal solution of the form $\eta - \rho y'$.*

Every such expression is a solution of the equations, by Lemma 1, and since the equations are linear. It will be a normal solution when ρ is determined, as it is uniquely, by the condition

$$y'(\eta - \rho y') = y'\eta - \rho y' y' = 0.$$

Let U and V be matrices with n rows and $n-1$ columns, each of the latter being a solution of the Jacobi equations, and consider the determinant of the $2n$th order

$$D(t) = \begin{vmatrix} U & V & y' & 0 \\ U' & V' & y'' & y' \end{vmatrix} \tag{17}$$

where U' and V' are the matrices of the derivatives of the elements of U and V.

LEMMA 5. *The determinant $D(t)$ is either identically zero or else everywhere different from zero. In the latter case the solutions U, V are called a fundamental system of solutions of the Jacobi equations.*

Every one of the first $2n-2$ columns of D may be made into a normal solution by multiplying the $(2n-1)$th column by a suitably selected function $\rho(t)$, multiplying the last column by $\rho'(t)$, and adding the two columns of elements so formed to the column of solutions in question. Suppose that this alteration has been made and that the determinant is zero at a value t_1. Choose multipartite constants α, β each with $n-1$ elements, and two single constants γ, δ, satisfying the $2n$ equations whose coefficients are the rows of $D(t_1)$. By multiplying the first n of these equations by the elements of $y'(t_1)$ and adding, it appears that $\gamma = 0$. Similarly when the $2n$ equations are multiplied by the elements of $y''(t_1)$, $y'(t_1)$ and added, it turns out that $\delta = 0$. The n functions

$$U(t)\alpha + V(t)\beta$$

now constitute a normal solution with elements and derivatives vanishing at t_1. Hence they are identically zero, and the same is true of $D(t)$.

COROLLARY. *A necessary and sufficient condition that $2n-2$ normal solutions U, V form a fundamental system is that they be linearly independent.*

LEMMA 6. *If $2n-2$ normal solutions U, V are linearly independent then every other normal solution η is expressible in terms of them in the form*

$$\eta(t) = U(t)\alpha + V(t)\beta.$$

For the $2n$ linear equations expressed by the notation

$$[u(t_1), u'(t_1)] = D(t_1)[\alpha, \beta, \gamma, \delta]$$

are solvable for the constants α, β, γ, δ, and it can be shown as above that $\gamma = \delta = 0$. The normal solution $u(t) - U(t)\alpha - V(t)\beta$ then vanishes with its derivatives at t_1 and hence is identically zero.

LEMMA 7. *If U is a matrix of linearly independent normal solutions with elements vanishing at t_1, then every other normal solution $u(t)$ with elements vanishing at t_1 is expressible in the form*

$$u(t) = U(t)\alpha .$$

In the first place the determinants of order $n - 1$ of the matrix U' cannot all vanish at t_1. Otherwise constants α could be determined satisfying the equations $U'(t_1)\alpha = 0$, in which case the normal solution $U(t)\alpha$ would vanish with its derivatives at t_1 and be identically zero. This would imply that the columns of U are not linearly independent.

The hypotheses that U and u are composed of normal systems imply the identities

$$y' U = 0, \qquad y'' U + y' U' = 0, \qquad y' u = 0, \qquad y'' u + y' u' = 0,$$

where the elements of $y' U$, for example, are the sums of the products of the elements of y' by those of a column of U. But at the point t_1 the elements of U and u vanish, and the second and fourth of these relations show therefore that the determinant $|u' \; U'|$ vanishes. Hence the equations

$$u'(t_1) - U'(t_1)\alpha = 0$$

can be solved for the constants α, and the normal solution $u - U\alpha$, vanishing with its derivatives at t_1, must be identically zero.

4. CRITERIA FOR CONJUGATE POINTS

Let U and V be two matrices of solutions of Jacobi's equations for the arc E, as described in § 3, and denote by $\Theta(t, t_1)$, the $2n$-rowed determinant

$$\Theta(t, t_1) = \begin{vmatrix} U(t) & V(t) & y'(t) & 0 \\ U(t_1) & V(t_1) & 0 & y'(t_1) \end{vmatrix} .$$

THEOREM 5. *If $\Theta(t, t_1)$ is not identically zero, then its zeros determine the points 3 conjugate to 1 on the extremal arc E.*

For in the first place let t_3 be a zero of $\Theta(t, t_1)$, and suppose the determinant altered, in a manner similar to that described in the proof of Lemma 5, so that its columns are normal solutions. Then the equations

$$U(t_3)\alpha + V(t_3)\beta + y'(t_3)\gamma = 0,$$
$$U(t_1)\alpha + V(t_1)\beta + y'(t_1)\delta = 0,$$

have solutions not all zero for the $2n$ constants $\alpha, \beta, \gamma, \delta$. If the first n equations are multiplied by the elements of $y'(t_3)$ and added, it follows that

$\gamma = 0$, since U and V contain only normal solutions; and a similar argument shows that the same is true of δ. The n functions

$$u = U(t)\alpha + V(t)\beta$$

form a normal solution of the Jacobi equations vanishing at t_1 and t_3, and not identically zero since the same is true of Θ. Hence t_3 defines a point conjugate to 1.

On the other hand let t_3 define a point conjugate to 1, and let $u(t)$ be the normal solution vanishing at t_1 and t_3 as described in the definition of § 2. The $2n - 2$ columns of U and V are linearly independent since Θ does not vanish identically. Hence, by Lemma 6, the normal solution $u(t)$ is expressible in the form

$$u(t) = U(t)\alpha + V(t)\beta$$

and it follows readily that Θ has a zero at t_3.

THEOREM 6. *If the matrix U is such that the determinant $D(t, t_1) = |Uy'|$ has at t_1 the rank 1 but is not identically zero, then the conjugate points to 1 on the arc E are determined by the zeros of $D(t, t_1)$.*

At the value $t = t_1$ every column of D has elements proportional to those of y', since the rank of the determinant is unity. From this property it can be readily seen that if the determinant is altered so that the columns of U are normal solutions, then all of the elements of U vanish at t_1. Let $t_3 \neq t_1$ be a zero of the determinant. Then there exists a normal solution

$$u(t) = U(t)\alpha \tag{18}$$

vanishing at t_1 and t_3, and not identically zero since D does not vanish identically. On the other hand, every normal solution with elements vanishing at t_1 is, by Lemma 7, expressible in the form (18), since if the columns of U were linearly dependent the determinant D would vanish identically. Hence every other point t_3 where the elements of $u(t)$ all vanish must be a zero of the determinant.

So far no remark has been made relative to the existence of matrices U and V satisfying the hypotheses of the last two theorems. It can be shown, however, by an application of the usual existence theorems for differential equations to the equations (9), that such matrices do exist. For the equation (14) shows that every solution η, λ of the equations (9) has $\lambda'' \equiv 0$, and hence that the functions η satisfy the Jacobi equations (8).

The general solutions of Euler's equations (5) depend upon $2n - 2$ constants and may be represented in the form

$$y = \phi(t, a, b) \tag{19}$$

where the constants of integration a, b are multipartite numbers with $n-1$ elements. It may be shown in the usual manner that the n functions found by differentiating the elements of ϕ with respect to one of the constants of integration, form a system of solutions of Jacobi's equations. The matrices $U = \phi_a$, $V = \phi_b$ are then of the form required in Theorem 5, provided that $\Theta(t, t_1)$ is not identically zero. The existence theorems applied to Euler's equations provide a family of solutions for which this hypothesis is satisfied.

The family ot extremals through the point 1 has the form

$$y = \phi(t, a) \tag{20}$$

and satisfies relations of the form

$$y_1 = \phi(t_1, a).$$

Here y_1 is a symbol for the multipartite number $(y_{11}, y_{21}, \cdots, y_{n1})$ whose elements are the coördinates of the point 1, and $t_1 = T(a)$ is a function of the $n-1$ constants a. When the last equations are differentiated for the constants of integration, the relations

$$0 = \phi_t T_a + \phi_a$$

show that the determinant D for the matrix $U = \phi_a$ has the rank prescribed in Theorem 6. The existence theorems again justify the existence of a family (20) for which the determinant D does not vanish identically.

In the plane the equations (19) have the form

$$x = \phi(t, a, b), \qquad y = \psi(t, a, b)$$

where now none ot the symbols are multipartite numbers. The determinant Θ has the form

$$\Theta(t, t_1) = \begin{vmatrix} \phi_a(t) & \phi_b(t) & \phi_t(t) & 0 \\ \psi_a(t) & \psi_b(t) & \psi_t(t) & 0 \\ \phi_a(t_1) & \phi_b(t_1) & 0 & \phi_t(t_1) \\ \psi_a(t_1) & \psi_b(t_1) & 0 & \psi_t(t_1) \end{vmatrix}$$

which except for sign is the Weierstrassian function*

$$\Theta(t, t_1) = \vartheta_1(t)\,\vartheta_2(t_1) - \vartheta_1(t_1)\,\vartheta_2(t)$$

with

$$\vartheta_1(t) = \psi_t\,\phi_a - \phi_t\,\psi_a, \qquad \vartheta_2(t) = \psi_t\,\phi_b - \phi_t\,\psi_b.$$

According to the definition given in § 2 it would seem that there might be points conjugate to 1 in every neighborhood of the latter point. That there is always a first conjugate point with $t_3 > t_1$ is a consequence of the following theorem:

* See for example, Bolza, loc. cit., p. 233.

THEOREM 7. *If the columns of U and V are a fundamental system of solutions of Jacobi's equations, then the determinant $\Theta(t, t_1)$ is expressible in the form*

$$\Theta(t, t_1) = (t - t_1)^{n-1}\lambda(t, t_1),$$

where $\lambda(t_1, t_1) \neq 0$.

For the determinant Θ may be written in the form

$$\Theta(t, t_1) = \begin{vmatrix} U(t) - U(t_1) & V(t) - V(t_1) & y'(t) & y'(t) - y'(t_1) \\ U(t_1) & V(t_1) & 0 & y'(t_1) \end{vmatrix},$$

where the elements of the matrix $U(t) - U(t_1)$, for example, are simply the differences of the elements of $U(t)$ and $U(t_1)$. After applying Taylor's formula with the integral form of the remainder term* to the differences in the first n rows, dividing these rows each by $t - t_1$, and multiplying the $(2n - 1)$th column by the same difference, a factor $(t - t_1)^{n-1}$ is obtained. The remaining factor $\lambda(t, t_1)$ reduces to the value of the determinant (17) when $t = t_1$, with the exception possibly of a change in sign.

* See for example, Jordan, *Cours d'Analyse*, 2d ed., vol. 1, p. 247.

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