ON THE EXPRESSIBILITY OF A UNIFORM FUNCTION OF SEVERAL COMPLEX VARIABLES AS THE QUOTIENT OF TWO FUNCTIONS OF ENTIRE CHARACTER*

BY

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1. Introduction

It is a classical fact in the theory of functions of one complex variable that any meromorphic function may be expressed as the quotient of two entire functions without common zeros. When $f(x)$ is a uniform function with essential singularities at finite distance, this theorem may be extended, as was shown by Weierstrass† for a finite number of essential singularities, and by Mittag-Leffler in the general case: $f(x)$ is expressible as the quotient of two functions of entire character (that is, uniform and without poles, but generally both having the same essential singularities as $f(x)$) without common zeros.

Before taking up the corresponding question for several variables, it is convenient to recall the following definitions:

The complex variables $x_1, x_2, \cdots, x_n$ are interior to the region $(S_1, S_2, \cdots, S_n)$ when x_1 is interior to the region S_1 in the x_1-plane, $\cdots$, x_n interior to the region S_n in the x_n-plane; the regions $S_1, \cdots, S_n$ may be simply or multiply connected.

A *uniform* function $f(x_1, x_2, \cdots, x_n)$ of the complex variables $x_1, x_2, \cdots, x_n$ is *meromorphic* in $(S_1, S_2, \cdots, S_n)$ when, in the vicinity of every point $a_1, a_2, \cdots, a_n$ interior to $(S_1, S_2, \cdots, S_n)$, we have

$$f(x_1, x_2, \cdots, x_n) = \frac{P_1(x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n)}{P_0(x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n)},$$

where P_0 and P_1 are power series in $x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n$. A *uniform* function $G(x_1, x_2, \cdots, x_n)$ is *of entire character* in $(S_1, S_2, \cdots, S_n)$

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† K. Weierstrass, *Zur Theorie der eindeutigen analytischen Functionen*, *Mathematische Werke*, vol. 2 (Berlin, 1895), pp. 77–124. G. Mittag-Leffler, *Sur la représentation analytique des fonctions monogènes uniformes d'une variable indépendante*, Acta Mathematica, vol. 4 (1884), pp. 1–79.

when holomorphic at every point $a_1, a_2, \cdots, a_n$ interior to $(S_1, S_2, \cdots, S_n)$. Two functions *of entire character* $G_0(x_1, x_2, \cdots, x_n)$ and $G_1(x_1, x_2, \cdots, x_n)$ have *a common divisor* when there exists a point $a_1, a_2, \cdots, a_n$ interior to $(S_1, S_2, \cdots, S_n)$ such that in its vicinity

$$G_0(x_1, x_2, \cdots, x_n) = P(x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n) \cdot P_0(x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n),$$

$$G_1(x_1, x_2, \cdots, x_n) = P(x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n) \cdot P_1(x_1 - a_1, x_2 - a_2, \cdots, x_n - a_n),$$

with $P(0, 0, \cdots, 0) = 0$. Two functions *of entire character* are relatively prime when they have no common divisor.

Poincaré has shown,* by the theory of harmonic functions of four real variables, that when $n = 2$ and S_1 and S_2 contain all points at finite distance in the x_1- and x_2-planes respectively, every meromorphic function is expressible as the quotient of two entire functions without common divisor. In a later paper,† he has modified this method and extended it to n variables.

The Cauchy integral was used by Cousin‡ to prove Poincaré's result and extend it to more general regions. His most general results are the following, of which **A** may be regarded as the extension to several variables of Mittag-Leffler's theorem, while **B** generalizes Weierstrass's theorem on the existence of uniform functions with given zeros:

A. When for every point $a_1, a_2, \cdots, a_n$ interior to $(S_1, S_2, \cdots, S_n)$ there are given

(1) a region $\Gamma_{a_1, a_2, \cdots, a_n}$ consisting of n circles $|x_\nu - a_\nu| < r_\nu$ $(\nu = 1, 2, \cdots, n)$, each of these circles being interior to the corresponding region S_ν;

(2) a function $f_{a_1, a_2, \cdots, a_n}(x_1, x_2, \cdots, x_n)$ uniform in $\Gamma_{a_1, a_2, \cdots, a_n}$ and such that when two regions $\Gamma_{a_1, a_2, \cdots, a_n}$ and $\Gamma_{a_1', a_2', \cdots, a_n'}$ have a region in common, the difference

$$f_{a_1, a_2, \cdots, a_n}(x_1, x_2, \cdots, x_n) - f_{a_1', a_2', \cdots, a_n'}(x_1, x_2, \cdots, x_n)$$

is holomorphic in the common region;

Then there exists a function $F(x_1, x_2, \cdots, x_n)$ uniform in $(S_1, S_2, \cdots, S_n)$ and such that for every interior point $a_1, a_2, \cdots, a_n$ the difference

$$F(x_1, x_2, \cdots, x_n) - f_{a_1, a_2, \cdots, a_n}(x_1, x_2, \cdots, x_n)$$

is holomorphic in $\Gamma_{a_1, a_2, \cdots, a_n}$.

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* H. Poincaré, *Sur les fonctions de deux variables*, Acta Mathematica, vol. 2 (1883), pp. 97–113.

† H. Poincaré, *Sur les propriétés du potentiel et sur les fonctions Abéliennes*, Acta Mathematica, vol. 22 (1899), pp. 89–178.

‡ P. Cousin, *Sur les fonctions de n variables complexes*, Acta Mathematica, vol. 19 (1895), pp. 1–62.

B. When for every point $a_1, a_2, \cdots, a_n$ interior to $(S_1, S_2, \cdots, S_n)$ there are given

(1) a region $\Gamma_{a_1, a_2, \ldots, a_n}$ as in **A**;

(2) a function $u_{a_1, a_2, \ldots, a_n}(x_1, x_2, \cdots, x_n)$ of entire character in $\Gamma_{a_1, a_2, \ldots, a_n}$ and such that when two regions $\Gamma_{a_1, a_2, \ldots, a_n}$ and $\Gamma_{a_1', a_2', \ldots, a'}$ have a region in common, the quotient

$$u_{a_1, a_2, \ldots, a_n}(x_1, x_2, \cdots, x_n)/u_{a_1', a_2', \ldots, a_n'}(x_1, x_2, \cdots, x_n)$$

is holomorphic and different from zero in the common region;

Then there exists a function $G(x_1, x_2, \cdots, x_n)$ of entire character in $(S_1, S_2, \cdots, S_n)$ such that for every interior point $a_1, a_2, \cdots, a_n$ the quotient $G(x_1, x_2, \cdots, x_n)/u_{a_1, a_2, \ldots, a_n}(x_1, x_2, \cdots, x_n)$ is holomorphic and different from zero in $\Gamma_{a_1, a_2, \ldots, a_n}$.

C. When a function $f(x_1, x_2, \cdots, x_n)$ is meromorphic in $(S_1, S_2, \cdots, S_n)$, it may be expressed as the quotient of two relatively prime functions of entire character* in $(S_1, S_2, \cdots, S_n)$:

$$f(x_1, x_2, \cdots, x_n) = \frac{G_1(x_1, x_2, \cdots, x_n)}{G_0(x_1, x_2, \cdots, x_n)}.$$

Cousin establishes Theorem **A** in its various stages in an entirely rigorous manner, but his proofs of Theorem **B** (and hence of Theorem **C**, which is a quite elementary consequence of **B**—see Cousin, l. c., §§ 15, 19, and 25) contain a gap (at stages α and β) which considerably restricts the regions $(S_1, S_2, \cdots, S_n)$ in which they are applicable.

In § 2, the nature of this gap is explained, and Cousin's proofs of **B** are shown to be valid when all, or all but one, of the n regions $S_1, S_2, \cdots, S_n$ are simply connected. On the other hand, it is established by an example that Cousin's construction of $G(x_1, x_2, \cdots, x_n)$ does not always yield a uniform function when two of the regions $S_1, S_2, \cdots, S_n$ are multiply connected.

The question now arises as to the validity of Theorems **B** and **C** in the cases where Cousin's proofs do not apply. In § 3 it is shown by an example that Theorem **C** is false (and consequently Theorem **B**, since **C** would follow from **B**) when two of the regions $S_1, S_2, \cdots, S_n$ are multiply connected, that is, in the very cases where Cousin's proofs fail.

* In his proofs, Cousin proceeds by four stages: first the theorems are derived for any region $(s_1, s_2, \cdots, s_n)$ interior to $(S_1, S_2, \cdots, S_n)$, and this separately for $n = 2$ (stage α) and n general (stage β). Second, a limiting process is used to extend the region of validity of the theorems from $(s_1, s_2, \cdots, s_n)$ to $(S_1, S_2, \cdots, S_n)$, and this separately when all S_ν are circles (stage γ) and when S_ν are quite general (stage δ). For convenient reference, the numbers of Cousin's theorems corresponding to Theorems **A**, **B**, and **C** of the text at the various stages are given below:

	α	β	γ	δ
A	I	IV	VII, p. 33	XI
B	III	VI	IX	XII
C	—	VII, p. 32	X	XIV

Thus the results of the present paper may be summarized in the statement that

Theorems **B** *and* **C** *are valid when, and only when,* $n - 1$ *of the* n *regions* $S_1, S_2, \cdots, S_n$ *are simply connected; the remaining region may be simply or multiply connected.*

The author wishes to acknowledge his indebtedness to Professor Osgood, to whom he communicated the example of § 3 in June, 1913, for material assistance in locating the gap in Cousin's proofs.

2. The domain of validity of Cousin's proofs of Theorem B

To abridge the notation, we shall write x for the system of $n - 1$ variables $x_1, x_2, \cdots, x_{n-1}$ and S for $(S_1, S_2, \cdots, S_{n-1})$; x_n will be denoted by y and S_n by S'. A simply connected part Σ of S we define as a system of regions $(\Sigma_1, \Sigma_2, \cdots, \Sigma_{n-1})$ where, for $\nu = 1, 2, \cdots, n - 1$, every interior or boundary point of the simply connected region Σ_ν is interior to or on the boundary of S_ν. The boundaries of $S_1, S_2, \cdots, S_{n-1}, \Sigma_1, \Sigma_2, \cdots, \Sigma_{n-1}$, and S' are assumed to be regular, that is, each is to consist of a finite number of pieces of analytic curves without singular points.

We now assume S' to be subdivided, by a finite number of pieces of regular curves, into a finite number of simply connected regions $R_1, R_2, \cdots, R_p, \cdots$. When R_n and R_p are adjacent regions, we denote by l_{np} their common boundary, or, should this consist of several pieces, any one of these. If any l_{np} is a closed curve, we cut it at three points, thus obtaining three pieces such that no two of them taken together form a closed curve. The direction of l_{np} is that which leaves the interior of the region R_n to the left, so that l_{np} and l_{pn} are the same curve described in opposite directions. Finally, let T_{np} consist of all points in the y-plane interior to at least one circle with center on l_{np} and sufficiently small radius r, this r being constant not only for different points on l_{np}, but also for all the various curves l_{np}.

The proof of Theorem **B** now depends on the following lemma:

Let a function $u_p(x, y)$ *be given for every region* R_p, *uniform and holomorphic in* (S, R_p), *boundaries included, and such that for any two adjacent regions* R_n *and* R_p, *the quotient*

$$\frac{u_p(x, y)}{u_n(x, y)} = g_{np}(x, y)$$

is holomorphic and different from zero in (S, T_{np}). *Then there exists a function* $G(x, y)$ *holomorphic in* (S, S'), *uniform in* (Σ, S'), *where* Σ *is any simply connected part of* S, *and such that in* (S, R_p) *(boundaries included, except those* y *which are end points of an* l_{np} *and lie on the boundary of* S'*) the* *quotient*

$$\frac{G(x, y)}{u_p(x, y)}$$

is holomorphic and different from zero.

When S is simply connected, we may evidently let Σ coincide with S. In his formulation of the lemma (l. c., § 7; proof in § 6) Cousin makes no distinction between Σ and S, so that, when S is multiply connected (that is, one at least of $S_1, S_2, \cdots, S_{n-1}$ is multiply connected) he tacitly assumes the function $G(x, y)$ to be uniform in (S, S'), while the uniformity is proved only in (Σ, S').

This constitutes the gap in Cousin's proofs referred to in the introduction. It might also be objected to his proof of the lemma (l. c., § 6) that he operates throughout with the multiform functions $\log u_p(x, y)$ and their differences $\log u_p(x, y) - \log u_n(x, y)$, and that it is not quite clear what branches of these functions are meant at the various points of (S, S'); but this objection is met by a modification of Cousin's argument due to Osgood.*

Since $u_p(x, y)$ and $u_n(x, y)$ are uniform in (S, T_{np}) by hypothesis, and their quotient $g_{np}(x, y)$ is holomorphic and different from zero in the same region, it follows that writing

$$G_{np}(x, y) = \log g_{np}(x, y),$$

where that branch of $\log g_{np}(x, y)$ is taken which assumes its principal value at some point x_0, y_0 interior to (Σ, T_{np}), the function $G_{np}(x, y)$ is holomorphic in (S, T_{np}) and uniform in (Σ, T_{np}). Next let

$$I_{np}(x, y) = \frac{1}{2\pi i}\int_{l_{np}} \frac{G_{np}(x, z)\, dz}{z - y},$$

the integral being taken in the positive direction of l_{np}. This function is holomorphic for all y at finite or infinite distance, except those on the curve l_{np}, and for any x in S, and uniform for the same y and any x in Σ. Moreover, as shown in Cousin §§ 2–3,

$$I_{np}(x, y) = H(x, y) + G_{np}(x, y)\lambda_{np}(y),$$

$$\lambda_{np}(y) = \frac{1}{2\pi i}\log\frac{y - b}{y - a}, \qquad \lambda_{np}(\infty) = 0,$$

where a and b are the end points of l_{np}, $\log[(y - b)/(y - a)]$ is that branch of the logarithm which vanishes for $y = \infty$, so that $\lambda_{np}(y)$ is uniform and holomorphic in the whole y-plane except on the curve l_{np}, and finally $H(x, y)$

* Letter to the author, July 7, 1913. This modified proof is reproduced here with the permission of Professor Osgood.

is holomorphic in (S, T_{np}) and uniform in (Σ, T_{np}). Now write

$$\Phi(x, y) = \sum I_{np}(x, y),$$

where the summation is extended over all the curves l_{np} which are common to the boundaries of two regions R (each curve taken once, and not in the two subscript combinations l_{np} and l_{pn}), and define

$$\phi_n(x, y) = \Phi(x, y) \text{ in } (S, R_n).$$

Then $\phi_n(x, y)$ is holomorphic in (S, R_n) and uniform in (Σ, R_n), boundaries included except the end points of the various l_{np} belonging to the boundary of R_n. Denoting by $\phi_n(x, y)_p$ the analytic continuation of $\phi_n(x, y)$ when x describes any path in S and y a path in T_{np} starting at a point inside R_n and ending at a point inside R_p, but not passing through an end point of l_{np}, we have (Cousin, l. c., §§ 2–3)

$$\phi_n(x, y)_p = \phi_p(x, y) + G_{np}(x, y). \tag{1}$$

A point $y = b$ interior to S' is called a *vertex* when it is an end point of any l_{np}.

Now make

$$\overline{G}_n(x, y) = u_n(x, y)\, e^{\phi_n(x, y)} \text{ in } (S, R_n);$$

then it follows from (1) that $\overline{G}_p(x, y)$ is the analytic continuation of $\overline{G}_n(x, y)$ across l_{np} (the path in the y-plane leading from R_n into R_p not crossing l_{np} at a vertex), and consequently the continuation of $\overline{G}_n(x, y)$ along a closed path in the y-plane not passing through any vertex brings us back to $\overline{G}_n(x, y)$. We may therefore define a single function $\overline{G}(x, y)$ by the consistent conditions $\overline{G}(x, y) = \overline{G}_n(x, y)$ in (S, R_n), and this $\overline{G}(x, y)$ is visibly uniform in (Σ, S'). Moreover, the quotient $\overline{G}(x, y)/u_p(x, y)$ is holomorphic and different from zero in (S, R_p), boundaries included, except when y coincides with an end point of an l_{np} while x takes any value inside or on the boundary of S.

We shall now modify $\overline{G}(x, y)$ so as to remove the last restriction for those end points of an l_{np} which are vertices. Let b be a vertex, and suppose that, for instance, $R_1, R_2, \cdots, R_m$ are those regions R which are adjacent to this vertex. Let $1 \leqq \nu \leqq m$ and denote by R'_ν that part of R_ν which lies within or on the circle $|y - b| = r'$, where r' is less than the radius r of the circles used in defining all T_{np}. Then we have in (S, R'_ν)

$$\phi_\nu(x, y) = \Phi(x, y) = A(x, y) + G_{12}(x, y)\lambda_{12}(y) + G_{23}(x, y)\lambda_{23}(y)$$
$$+ \cdots + G_{m-1, m}(x, y)\lambda_{m-1, m}(y) + G_{m1}(x, y)\lambda_{m1}(y),$$

$A(x, y)$ being holomorphic in $(S, |y - b| \leqq r')$ and uniform in $(\Sigma, |y - b| \leqq r')$. Make

$$L_\nu(y - b) = \frac{1}{2\pi i}\log(y - b),$$

where any branch of the logarithm is chosen and rendered uniform by a cut issuing from $y = b$, but having no other point in common with R'_ν or its boundary. None of the l_{np} abutting at b being closed, we may continue $\lambda_{np}(y)$ analytically from $y = \infty$ to a point inside R'_ν along a curve intersecting none of these l_{np}, and in the relation

$$\lambda_{np}(y) - L_\nu(y - b) = -\frac{1}{2\pi i}\log(y - a),$$

where now $\log(y - a)$ is a definite branch of the logarithm, for y in R'_ν, the right-hand member is holomorphic in the entire region $|y - b| \leqq r'$. Hence we have, *for y interior to R'_ν*,

$$\phi_\nu(x, y) = B_\nu(x, y) + [G_{12}(x, y) + G_{23}(x, y) + \cdots + G_{m-1,m}(x, y) + G_{m1}(x, y)] L_\nu(y - b),$$

where $B_\nu(x, y)$ is holomorphic in $(S,\ |y - b| \leqq r')$ and uniform in $(\Sigma,\ |y - b| \leqq r')$. On the other hand, the sum in brackets equals

$$\log\frac{u_2(x, y)}{u_1(x, y)} + \log\frac{u_3(x, y)}{u_2(x, y)} + \cdots + \log\frac{u_m(x, y)}{u_{m-1}(x, y)} + \log\frac{u_1(x, y)}{u_m(x, y)},$$

where each log refers to a definite branch of the function—the branch chosen at the beginning, and this sum therefore equals a definite value of $\log 1$, which we denote by $2\pi i K_b$, the integer K_b being evidently independent of ν. Consequently, for y interior to R'_ν,

$$(y - b)^{-K_b}\,\bar{G}(x, y) = u_\nu(x, y)\, e^{\phi_\nu(x, y) - 2\pi i K_b L_\nu(y - b)},$$

or

$$(y - b)^{-K_b}\,\bar{G}(x, y) = u_\nu(x, y)\, e^{B_\nu(x, y)};$$

but the expression to the right being holomorphic in $(S,\ |y - b| \leqq r')$ and uniform in $(\Sigma,\ |y - b| \leqq r')$, it follows by analytic continuation that the same is true of the left-hand member, and that the quotient of the latter by $u_\nu(x, y)$, which equals $e^{B_\nu(x, y)}$ in $(S,\ R'_\nu)$, is holomorphic and different from zero in that region.

Finally determine the integer K_b for each vertex b and write

$$G(x, y) = \bar{G}(x, y)\prod_b (y - b)^{-K_b},$$

the product extending over all vertices. It then follows immediately from the preceding argument that $G(x, y)$ has all the properties mentioned in the lemma.

As already stated, Cousin tacitly assumes that from the proven uniformity of $G(x, y)$ in $(\Sigma,\ S')$ it follows that $G(x, y)$ is also uniform in $(S,')\ S$ when S is multiply connected.

I shall now show by an example that this conclusion is not legitimate; it is evidently sufficient to assume $n = 2$, so that now x stands for a single variable, and S for a region in the x-plane. This example, as well as the one in § 3, is based on the simplest properties of Theta functions of two variables. It is well known that, given the constants τ_{11}, τ_{12}, τ_{22} such that the real part of $2\pi i(\tau_{11} n_1^2 + 2\tau_{12} n_1 n_2 + \tau_{22} n_2^2)$ is a definite negative quadratic form in n_1 and n_2, the two expressions*

$$\text{(2)} \quad \begin{aligned} \phi_\nu(v_1, v_2) = \sum_{n_1, n_2=-\infty}^{+\infty} \text{Exp}\Bigg[\left(n_1 - \frac{\nu}{2}\right)^2 \tau_{11} + 2\left(n_1 - \frac{\nu}{2}\right) n_2 \tau_{12} \\ + n_2^2 \tau_{22} - 2\left(n_1 - \frac{\nu}{2}\right) v_1 - 2n_2 v_2\Bigg], \end{aligned}$$

where $\nu = 0$ or 1, define entire functions of v_1 and v_2 with the properties

$$\begin{aligned} \phi_\nu(v_1 + 1, v_2) &= \phi_\nu(v_1, v_2), \\ \phi_\nu(v_1, v_2 + \tfrac{1}{2}) &= \phi_\nu(v_1, v_2), \\ \phi_\nu(v_1 + \tau_{11}, v_2 + \tau_{12}) &= \text{Exp}(-2v_1 - \tau_{11}) \cdot \phi_\nu(v_1, v_2), \\ \phi_\nu(v_1 + \tau_{12}, v_2 + \tau_{22}) &= \text{Exp}(-2v_2 - \tau_{22}) \cdot \phi_\nu(v_1, v_2) \end{aligned} \qquad (\nu = 0, 1).$$

Assume $\tau_{12} \neq 0$, introduce new variables w_1 and w_2 by the relations

$$\tau_{12} w_1 = -2\tau_{22} v_1 + 2\tau_{12} v_2, \qquad \tau_{12} w_2 = v_1,$$

and write $\phi_\nu(v_1, v_2) = \psi_\nu(w_1, w_2)$; then $\psi_\nu(w_1, w_2)$ are entire functions of w_1 and w_2 with the properties

$$\text{(3)} \quad \begin{aligned} \psi_\nu(w_1 + 1, w_2) &= \psi_\nu(w_1, w_2), \\ \psi_\nu(w_1, w_2 + 1) &= \text{Exp}(-w_1 - 2\tau_{22} w_2 - \tau_{22}) \cdot \psi_\nu(w_1, w_2), \\ \psi_\nu\left(w_1 - \frac{2\tau_{22}}{\tau_{12}}, w_2 + \frac{1}{\tau_{12}}\right) &= \psi_\nu(w_1, w_2), \qquad (\nu = 0, 1). \\ \psi_\nu\left(w_1 + \frac{2\tau_{12}^2 - 2\tau_{11}\tau_{22}}{\tau_{12}}, w_2 + \frac{\tau_{11}}{\tau_{12}}\right) &= \text{Exp}(-2\tau_{12} w_2 - \tau_{11}) \cdot \psi_\nu(w_1, w_2) \end{aligned}$$

Finally write $\psi(w_1, w_2) = \text{Exp}(\tau_{22} w_2^2) \cdot \psi_0(w_1, w_2)$; then the entire function $\psi(w_1, w_2)$ has the properties

$$\text{(4)} \quad \begin{aligned} \psi(w_1 + 1, w_2) &= \psi(w_1, w_2), \\ \psi(w_1, w_2 + 1) &= \text{Exp}(-w_1) \cdot \psi(w_1, w_2). \end{aligned}$$

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* To simplify the typography, we shall use the notation $e^{2\pi i x} = \text{Exp}(x)$.

Once more we introduce new variables by the equations

$$(5) \qquad x = \operatorname{Exp}(w_1), \qquad y = \operatorname{Exp}(w_2)$$

and write

$$(6) \qquad u(x, y) = \psi(w_1, w_2) = \psi\left(\frac{1}{2\pi i}\log x, \frac{1}{2\pi i}\log y\right);$$

then $u(x, y)$ is holomorphic for all x, y at finite distance, except $x = 0$, $y = y$ and $x = x$, $y = 0$. Starting with some definite branches of $\log x$ and $\log y$, say those that equal zero for $x = 1$ and $y = 1$ respectively, it follows from (4) that $u(x, y)$ is uniform in respect to x, while the analytic continuation along a path winding about $y = 0$ once in the positive sense transforms the initial branch $u(x, y)$ into a new branch $\bar{u}(x, y)$ such that

$$(7) \qquad \bar{u}(x, y) = \frac{1}{x} u(x, y).$$

Now let us construct the function $G(x, y)$ of the lemma from the following data:

S: the circular ring $\frac{1}{2} < |x| < 2$;
S': the circular ring $\frac{1}{2} < |y| < 2$;
R_1: the part of S' to the right of the imaginary axis;
R_2: the part of S' to the left of the imaginary axis;
l_{12}: the straight line segment from $y = 2i$ to $y = \frac{1}{2}i$;
l'_{12}: the straight line segment from $y = -\frac{1}{2}i$ to $y = -2i$, so that the common part of the boundaries of R_1 and R_2 consists of l_{12} and l'_{12};
$u_1(x, y)$: the initial branch of $u(x, y)$ defined above;
$u_2(x, y)$: the analytic continuation of $u_1(x, y)$ across the line l_{12}.

Then $u_1(x, y)$ and $u_2(x, y)$ are uniform and holomorphic in (S, R_1) and (S, R_2) respectively, boundaries included. On l_{12},

$$g_{12}(x, y) = \frac{u_2(x, y)}{u_1(x, y)} = 1,$$

while on l'_{12} we have

$$g'_{12}(x, y) = \frac{u_2(x, y)}{u_1(x, y)} = \frac{1}{x}$$

according to (7). We now make

$$G_{12}(x, y) = \log 1 = 0, \qquad G'_{12}(x, y) = -\log x,$$

where that branch of the logarithm is taken which vanishes at $x = 1$; since there are no vertices and therefore no integers K_b to be determined, we may proceed at once to write down $\Phi(x, y)$:

$$\Phi(x, y) = \frac{1}{2\pi i}\int_{-\frac{1}{2}i}^{-2i} \frac{-\log x dz}{z - y} = \frac{1}{2\pi i}\log x \cdot \log\frac{y + \frac{1}{2}i}{y + 2i},$$

where the last logarithm is the branch that vanishes for y infinite. Finally we obtain

$$(8)\qquad G(x, y) = u_p(x, y)\ \mathrm{Exp}\left(\frac{1}{2\pi i}\cdot\frac{1}{2\pi i}\log x\cdot\log\frac{y+\frac{1}{2}i}{y+2i}\right)$$

in (S, R_p) for $p = 1, 2$. This $G(x, y)$ now has all the properties indicated in the lemma (as is also readily verified directly in this particular case). Nevertheless, $G(x, y)$ is not uniform in (S, S'), for letting x describe a closed path in S starting and ending at $x = 1$, and winding about $x = 0$ once in the positive sense, while y describes a closed path interior to R_1, $\log x$ increases by $2\pi i$, while $\log (y + \frac{1}{2}i)/(y + 2i)$ and $u_1(x, y)$ remain unchanged, and we arrive at a branch $\bar{G}(x, y)$ connected with the initial branch $G(x, y)$ by the relation

$$\bar{G}(x, y) = \frac{y+\frac{1}{2}i}{y+2i}G(x, y).$$

Hence Cousin's lemma, and with it his proofs of Theorem **B**, are valid when, and only when, not more than one of the regions $S_1, S_2, \cdots, S_n$ is multiply connected.

3. Example of a function of two variables, meromorphic in a region (S, S'), which cannot be expressed as the quotient of two relatively prime functions of entire character

From (3) it is evident that the quotient

$$\frac{\psi_1(w_1, w_2)}{\psi_0(w_1, w_2)} = \frac{\phi_1(v_1, v_2)}{\phi_0(v_1, v_2)}$$

is a meromorphic quadruply periodic function of w_1 and w_2 with the periods

$$\begin{array}{lllll} 1, & 0, & -\dfrac{2\tau_{22}}{\tau_{12}}, & \dfrac{2\tau_{12}^2 - 2\tau_{11}\tau_{22}}{\tau_{12}} & \text{in } w_1, \\[2ex] 0, & 1, & \dfrac{1}{\tau_{12}}, & \dfrac{\tau_{11}}{\tau_{12}} & \text{in } w_2. \end{array}$$

By (2), $\phi_0(v_1, v_2)$ contains only even, and $\phi_1(v_1, v_2)$ only odd, powers of Exp. (v_1); hence these two functions are linearly independent, and the quotient considered is not a constant. Introducing the variables x and y by (5) and writing

$$f(x, y) = \frac{\psi_1(w_1, w_2)}{\psi_0(w_1, w_2)},$$

$f(x, y)$ is a non-constant, uniform function of x and y, meromorphic in the region (S, S'), where S consists of all points at finite distance in the x-plane, the point $x = 0$ excepted, and S' is defined similarly in the y-plane. This function has the properties

$$f(hx, ky) = f(x, y),$$
(9)
$$f(lx, my) = f(x, y),$$

where

$$h = \operatorname{Exp}\left(-\frac{2\tau_{22}}{\tau_{12}}\right), \qquad k = \operatorname{Exp}\left(\frac{1}{\tau_{12}}\right),$$
(10)
$$l = \operatorname{Exp}\left(\frac{2\tau_{12}^2 - 2\tau_{11}\tau_{22}}{\tau_{12}}\right), \qquad m = \operatorname{Exp}\left(\frac{\tau_{11}}{\tau_{12}}\right).$$

Now let us subject $\tau_{11}, \tau_{12}, \tau_{22}$ to the further condition that

(11) $$l^a m^b \neq h^c k^d$$

for any integers a, b, c, and d which are not all equal to zero. By (10), this is equivalent to the condition that the equation

(12) $$b\tau_{11} + n\tau_{12} + 2c\tau_{22} + 2a(\tau_{12}^2 - \tau_{11}\tau_{22}) - d = 0$$

shall have no solution in integers a, b, c, d, n which are not all equal to zero.* Then $f(x, y)$ cannot be expressed as the quotient of two relatively prime functions of entire character in (S, S'). For the purpose of an example, it is sufficient to carry out the proof in a special case, giving numerical values to $\tau_{11}, \tau_{12}, \tau_{22}$.† Let us make

$$\tau_{11} = i, \qquad \tau_{12} = \frac{1}{\sqrt[4]{2}}, \qquad \tau_{22} = i\sqrt{2};$$

then the real part of $2\pi i(\tau_{11}n_1^2 + 2\tau_{12}n_1n_2 + \tau_{22}n_2^2)$ is $-2\pi(n_1^2 + \sqrt{2}n_2^2)$, a definite negative quadratic form in n_1 and n_2. Furthermore $\tau_{12} \neq 0$, and (12) gives upon separation of the real and imaginary parts

$$b + 2c\sqrt{2} = 0, \qquad n + 3a\sqrt[4]{8} - d\sqrt[4]{2} = 0,$$

whence

$$b = c = 0, \qquad n^2 + 12ad - (18a^2 + d^2)\sqrt{2} = 0, \qquad a = d = n = 0.$$

Hence (11) is satisfied, and in particular we have for any integers λ and μ, except $\lambda = \mu = 0$,

(13) $$h^\lambda k^\mu - 1 \neq 0, \qquad l^\lambda m^\mu - 1 \neq 0.$$

---

* In the theory of Theta functions, this condition expresses the fact that the period system $\tau_{11}, \tau_{12}, \tau_{22}$ is non-singular.

† This has the advantage of simplifying the convergence proof for the series (19).

Now assume that $f(x, y)$ can be expressed in the form*

$$f(x, y) = \frac{G_1(x, y)}{G_0(x, y)}, \tag{14}$$

where $G_0(x, y)$ and $G_1(x, y)$ are of entire character and relatively prime in (S, S'); we shall show that this leads to a contradiction. From (9) and (14) it follows that

$$\frac{G_0(hx, ky)}{G_0(x, y)} = \frac{G_1(hx, ky)}{G_1(x, y)}, \qquad \frac{G_0(lx, my)}{G_0(x, y)} = \frac{G_1(lx, my)}{G_1(x, y)},$$

and since $G_0(x, y)$ and $G_1(x, y)$ are relatively prime, we conclude that both these quotients, which are evidently uniform, are holomorphic and different from zero in (S, S').† Let us denote them by $g(x, y)$ and $g'(x, y)$ respectively; then

$$G_\nu(hx, ky) = g(x, y) G_\nu(x, y), \qquad G_\nu(lx, my) = g'(x, y) G_\nu(x, y) \qquad (\nu = 0, 1). \tag{15}$$

Since $g(x, y)$ is of entire character and different from zero in (S, S'), we may expand its logarithmic derivatives in Laurent's series‡

$$\frac{\partial \log g(x, y)}{\partial x} = \sum_{\lambda, \mu = -\infty}^{+\infty} a_{\lambda\mu} x^\lambda y^\mu, \qquad \frac{\partial \log g(x, y)}{\partial y} = \sum_{\lambda, \mu = -\infty}^{+\infty} b_{\lambda\mu} x^\lambda y^\mu,$$

both series being absolutely and uniformly convergent for $\epsilon \leqq |x| \leqq 1/\epsilon$, $\epsilon \leqq |y| \leqq 1/\epsilon$, where ϵ is as small as we please. From

$$\frac{\partial^2 \log g(x, y)}{\partial y \partial x} = \frac{\partial^2 \log g(x, y)}{\partial x \partial y}$$

it follows that

$$\sum \mu a_{\lambda\mu} x^\lambda y^{\mu-1} = \sum \lambda b_{\lambda\mu} x^{\lambda-1} y^\mu,$$

so that in particular $\mu a_{-1, \mu} = 0, \lambda b_{\lambda, -1} = 0$, whence

$$a_{-1, 0} = a, \qquad a_{-1, \mu} = 0 \qquad (\mu \neq 0),$$

$$b_{0, -1} = b, \qquad b_{\lambda, -1} = 0 \qquad (\lambda \neq 0).$$

---

* The following investigation is closely related to one made by Appell to an entirely different purpose in his paper *Sur les fonctions périodiques de deux variables*, Journal de Mathématiques, ser. 4, vol. 7 (1891), pp. 157–219. See pp. 185–201.

† This is a simple consequence of Weierstrass' preparation theorem; compare Cousin, l. c., § 15, and Appell, l. c., pp. 182–185.

‡ K. Weierstrass, *Einige auf die Theorie der analytischen Funktionen mehrerer Veränderlichen sich beziehende Sätze, Mathematische Werke*, vol. 2 (Berlin, 1895), pp. 135–188. See pp. 183–188.

Treating $g'(x, y)$ in the same way, and integrating, we finally obtain

$$
(16) \qquad \begin{aligned} g(x, y) &= x^a y^b \operatorname{Exp}\left(\sum_{\lambda, \mu = -\infty}^{+\infty} A_{\lambda\mu} x^\lambda y^\mu \right), \\ g'(x, y) &= x^c y^d \operatorname{Exp}\left(\sum_{\lambda, \mu = -\infty}^{+\infty} B_{\lambda\mu} x^\lambda y^\mu \right), \end{aligned}
$$

the series being absolutely and uniformly convergent as before, and from the uniformity of $g(x, y)$ and $g'(x, y)$ it is evident that a, b, c, d are all integers. We arrive at a relation between $g(x, y)$ and $g'(x, y)$ by observing that according to (15)

$$
\frac{G_\nu(hlx, kmy)}{G_\nu(x, y)} = \frac{G_\nu(hlx, kmy)}{G_\nu(lx, my)} \cdot \frac{G_\nu(lx, my)}{G_\nu(x, y)} = g(lx, my) g'(x, y),
$$

$$
\frac{G_\nu(lhx, mky)}{G_\nu(x, y)} = \frac{G_\nu(lhx, mky)}{G_\nu(hx, ky)} \cdot \frac{G_\nu(hx, ky)}{G_\nu(x, y)} = g'(hx, ky) g(x, y),
$$

whence

$$
g(lx, my) g'(x, y) = g'(hx, ky) g(x, y).
$$

Introducing the expressions (16) into this relation, we obtain

$$
l^a m^b \operatorname{Exp}\left[\sum (A_{\lambda\mu} l^\lambda m^\mu + B_{\lambda\mu}) x^\lambda y^\mu \right] = h^c k^d \operatorname{Exp}\left[\sum (B_{\lambda\mu} h^\lambda k^\mu + A_{\lambda\mu}) x^\lambda y^\mu \right],
$$

which evidently gives

$$
(17) \qquad A_{\lambda\mu}(l^\lambda m^\mu - 1) = B_{\lambda\mu}(h^\lambda k^\mu - 1)
$$

and $l^a m^b = h^c k^d$. But in the last relation it follows from (11)—and this is the main point of the proof—that the integers a, b, c, and d are all equal to zero. Moreover, (13) shows that we may write (17) in the form

$$
(18) \qquad \frac{A_{\lambda\mu}}{h^\lambda k^\mu - 1} = \frac{B_{\lambda\mu}}{l^\lambda m^\mu - 1}, \qquad \text{except for } \lambda = \mu = 0.
$$

Denote by $\sum'$ a series from which the combination $\lambda = \mu = 0$ is excluded, and write

$$
(19) \qquad G(x, y) = \sum_{\lambda, \mu = -\infty}^{+\infty}{}' \frac{A_{\lambda\mu}}{h^\lambda k^\mu - 1} x^\lambda y^\mu = \sum_{\lambda, \mu = -\infty}^{+\infty}{}' \frac{B_{\lambda\mu}}{l^\lambda m^\mu - 1} x^\lambda y^\mu ;
$$

then (18) shows that the two definitions of $G(x, y)$ are formally consistent. For the convergence proof, separate the terms where $\lambda \neq 0$ from those with $\lambda = 0$; we obtain with the aid of (18)

$$
G(x, y) = \sum_{\mu = -\infty}^{+\infty} \sum_{\lambda \neq 0} \frac{A_{\lambda\mu}}{h^\lambda k^\mu - 1} x^\lambda y^\mu + \sum_{\mu \neq 0} \frac{B_{0\mu}}{m^\mu - 1} y^\mu .
$$

Introducing the numerical values of τ_{11}, τ_{12}, τ_{22} in (10), we find

$$h = e^{4\pi\sqrt[4]{2}}, \qquad k = e^{2\pi i\sqrt[4]{2}}, \qquad m = e^{-2\pi\sqrt[4]{2}},$$

and consequently

$$|h^\lambda k^\mu - 1| \geqq \big| |h|^\lambda |k|^\mu - 1 \big| = |e^{4\pi\sqrt[4]{2}\cdot\lambda} - 1|;$$

the last expression being greater than $e - 1$ or $1 - e^{-1}$ according as λ is a positive or negative integer, we have $|h^\lambda k^\mu - 1| > \frac{1}{2}$ for $\lambda \neq 0$, and similarly $|m^\mu - 1| > \frac{1}{2}$ for $\mu \neq 0$. Therefore (19) converges absolutely and uniformly in the same region as (16), that is, for $\epsilon \leqq |x| \leqq 1/\epsilon$, $\epsilon \leqq |y| \leqq 1/\epsilon$. Evidently $G(x, y)$ satisfies the relations

$$\begin{aligned} G(hx, ky) - G(x, y) &= \textstyle\sum' A_{\lambda\mu}\, x^\lambda y^\mu, \\ G(lx, my) - G(x, y) &= \textstyle\sum' B_{\lambda\mu}\, x^\lambda y^\mu. \end{aligned} \tag{20}$$

If we now write

$$G'_\nu(x, y) = \operatorname{Exp}[-G(x, y)] \cdot G_\nu(x, y) \qquad (\nu = 0, 1),$$

$G'_0(x, y)$ and $G'_1(x, y)$ are of entire character (and relatively prime) in (S, S'), and by (14)

$$f(x, y) = \frac{G'_1(x, y)}{G'_0(x, y)}. \tag{21}$$

From (15), (16), and (20) we find, bearing in mind that $a = b = c = d = 0$,

$$\begin{aligned} G'_\nu(hx, ky) &= \operatorname{Exp}(A_{00}) \cdot G'_\nu(x, y), \\ G'_\nu(lx, my) &= \operatorname{Exp}(B_{00}) \cdot G'_\nu(x, y) \end{aligned} \qquad (\nu = 0, 1). \tag{22}$$

Expanding $G'_0(x, y)$ and $G'_1(x, y)$ in Laurent's series

$$G'_0(x, y) = \sum_{\lambda,\,\mu=-\infty}^{+\infty} C_{\lambda\mu}\, x^\lambda y^\mu, \qquad G'_1(x, y) = \sum_{\lambda,\,\mu=-\infty}^{+\infty} D_{\lambda\mu}\, x^\lambda y^\mu,$$

the first equation (22) gives

$$C_{\lambda\mu}[h^\lambda k^\mu - \operatorname{Exp}(A_{00})] = D_{\lambda\mu}[h^\lambda k^\mu - \operatorname{Exp}(A_{00})] = 0.$$

Since $G'_0(x, y)$ is not identically zero, one $C_{\lambda\mu}$ at least must be different from zero, say $C_{\rho\sigma}$, so that $h^\rho k^\sigma - \operatorname{Exp}(A_{00}) = 0$. If $h^\lambda k^\mu - \operatorname{Exp}(A_{00}) = 0$, it follows that $h^{\lambda-\rho} k^{\mu-\sigma} - 1 = 0$, whence $\lambda = \rho$, $\mu = \sigma$ by (13). Therefore $h^\lambda k^\mu - \operatorname{Exp}(A_{00}) \neq 0$, and $C_{\lambda\mu} = D_{\lambda\mu} = 0$ except for $\lambda = \rho$, $\mu = \sigma$, and (21) gives

$$f(x, y) = \frac{D_{\rho\sigma}\, x^\rho y^\sigma}{C_{\rho\sigma}\, x^\rho y^\sigma} = \text{const.}$$

But we have seen from the definition of $f(x, y)$ that this function is not a constant, and this contradiction shows that Theorem **C** (and consequently

Theorem **B**, since **B** implies **C**) is not valid when two of the regions $S_1, S_2, \cdots, S_n$ are multiply connected.

It is possible however to express our function $f(x, y)$ as the quotient of two functions $G_1(x, y)$ and $G_0(x, y)$ of entire character in (S, S'), if we remove the condition that these two functions shall be relatively prime. To prove this, let $\rho = 0$ or 1 and write

$$\psi_2(w_1, w_2) = \text{Exp}\,(2\tau_{22}\, w_2^2)\cdot\psi_\rho(w_1, -w_2);$$

it then follows from (3) that

$$\psi_2(w_1+1, w_2) = \psi_2(w_1, w_2),$$
$$\psi_2(w_1, w_2+1) = \text{Exp}\,(w_1 + 2\tau_{22}\, w_2 + \tau_{22})\psi_2(w_1, w_2),$$

so that

$$\left.\begin{aligned}\psi_2(w_1+1, w_2)\psi_\nu(w_1+1, w_2) &= \psi_2(w_1, w_2)\psi_\nu(w_1, w_2),\\ \psi_2(w_1, w_2+1)\psi_\nu(w_1, w_2+1) &= \psi_2(w_1, w_2)\psi_\nu(w_1, w_2)\end{aligned}\right. \qquad (\nu = 0, 1),$$

and consequently, writing

$$G_\nu(x, y) = \psi_2(w_1, w_2)\psi_\nu(w_1, w_2) \qquad (\nu = 0, 1),$$

$G_0(x, y)$ and $G_1(x, y)$ are both *uniform* functions of x and y, holomorphic in (S, S'). Since $f(x, y) = \psi_1(w_1, w_2)/\psi_0(w_1, w_2)$, we have in

$$f(x, y) = \frac{G_1(x, y)}{G_0(x, y)}$$

a representation of $f(x, y)$ of the required character. Evidently $G_0(x, y)$ and $G_1(x, y)$ have here the common manifold of zeros defined by

$$\psi_2(w_1, w_2) = 0,$$

and from what we have proved before regarding $f(x, y)$, it follows that the common divisor cannot be removed without destroying the uniformity of $G_0(x, y)$ and $G_1(x, y)$.

In a subsequent paper, it will be shown that this representation as the quotient of two functions of entire character with common divisor is possible for any function $f(x, y)$, meromorphic everywhere at finite distance except at the points defined by $G(x, y) = 0$, where $G(x, y)$ is an entire function. The common divisor cannot in general be removed except when $G(x, y)$ is irreducible.